Chapter 4

Automorphic Properties

This chapter deals with the mathematical aspects of the full GW potential in product form, and is essentially mathematical – though the first three sections present results without derivations. The evaluation of an integral over the fundamental plane [HM-95] yields the logarithm of an infinite product, of which Borcherds [B-95] had already predicted the automorphic properties (though obvious in this approach). The generalisation to arbitrary GW potentials is tried with Borcherds’ lifting of Jacobi forms to automorphic forms, but remains inconclusive. Prerequisites are chapter 2 and familiarity with infinite products à-la Borcherds; results will not be used in later chapters.

4.1 Torus as Target Space

If we replace our three-dimensional Calabi-Yau space by a one-dimensional one, i.e. by a mundane elliptic curve, then the Gromov-Witten problem boils down to the Hurwitz problem of counting covers of a Riemann surface. The free energies $F_g$ have been similarly defined, and an explicit expression for the partition function $Z := \exp \sum_1 \lambda^{2g-2} F_g$ was given in [Dou-93]. It involves a generalised theta function, thereby ensuring modular properties of the $F_g$’s. For example, $F_1 = -\log \eta(q)$, and $F_2$ is a linear combination of Eisenstein series [Ru-94]. This is to be compared with the GW potential $F_1 = -\sum_d n_d^1 \log \eta(q^d)$ where we have an additional sum over the homology class of the image curve. Of course, in the case of covers of an elliptic curve, there is no degree to keep track of, and $t$ is the Kähler modulus of the flat torus. Mirror symmetry relates $t$ to the complex modulus $\tau$ of the mirror elliptic curve, which accounts for its modular covariance under $PSL(2,\mathbb{Z})$. More generally, $F_g$ is a quasi-modular form of weight $6g - 6$. One might then wonder whether our present GW potentials enjoy similar modular properties.

However, the occurrence of the $n_d^\tau$ in (2.3.4) and the fact that $d$ is a tuple
(and not just an integer) spoil the following naive hope of modular properties:

\[ F_g \sim \sum_{d>0} \text{Li}_{3-2g}(q^d) = \sum_n n^{2g-3} \frac{q^n}{1-q^n} = E_{2g-2} \]

yielding the Eisenstein series of weight \(2g-2\). So one wonders if a favourable choice for the integers \(n_d\) and for the summation over \(d\) would keep the modular properties. This was indeed the case for part of the results of [HM-95], to which we now turn.

### 4.2 Harvey-Moore and the Theta Transforms

In that work, the prepotential for heterotic string theory, which coincides with \(F_0\), was computed by hand. By equating the Wilsonian coupling with the one-loop coupling renormalisation (5.1.2), one obtains a differential equation for the prepotential, involving a class of integrals over the fundamental domain. The integrand can be replaced by the "new susy index", as in (5.2.1), which can be explicitly computed and yields a generalised theta function (or lattice function for the lattice \(\Gamma_{n+2,2}\)) times a modular function, see for instance (5.4.2) or (5.4.4). Thus these are integrals of the form of a theta transform, i.e. of an integral over the fundamental domain of a Siegel theta function times an almost holomorphic modular function of weight \(-s/2\) with Fourier expansion

\[ F(q) = \sum c(n,k) \, q^n \tau^{-k}, \quad q = e^{2\pi i \tau}, \]

with summation running over \(n \geq -n_0\) and \(k = 0, 1, \ldots, k_0\) for some non-negative integers \(n_0\) and \(k_0\):

\[ \Phi_{s+2,2}(y) := \int_{\mathcal{F}} \frac{d^2 \tau}{T_2} \, \Theta(\tau, y) \, F(\tau). \]

This is roughly the Howe correspondence, sending automorphic functions \(F\) for \(SL(2,\mathbb{Z})\) to automorphic functions \(\Phi\) for \(O(s + 2, 2; \mathbb{Z}) := O_{s+2,2}(\mathbb{Z}) := \text{Aut}(\Gamma_{s+2,2})\). In general, \(F\) is allowed to be modular covariant at level \(N\) (or, equivalently, vector-valued at level 1) and up to a character, to have poles at cusps and even to have rational powers of \(q\) in its expansion. For instance, the function \(1/\tau_2\) is modular of weight \((1,1)\). The theta function of weight \((s/2 + 1,1)\) is defined for an even self-dual lattice \(\Gamma_{s+2,2} \cong \Gamma_s \oplus (\Gamma_1,1)^2\) (with \(8|s\)):

\[ \Theta(\tau, y) = \sum_{p \in \Gamma_{s+2,2}} q^{p_L^2/2} \, \bar{q}^{p_R^2/2} \]

where \(y\) lies in the Grassmanian \(G(s + 2, 2) = O(s + 2, 2)/O(s + 2) \times O(2)\), also called the generalised upper half plane \(\mathcal{H}^{s+1,1} \cong \mathbb{R}^{s+1,1} + i \, C_{+}^{s+1,1}\) for the positive light cone \(C_{+}^{s+1,1}\). Then \(y\) corresponds to a choice of a positive definite \(s + 2\) dimensional subspace of \(\mathbb{R}^{s+2,2} \cong \Gamma_{s+2,2} \otimes \mathbb{R}\), so that every lattice vector \(p\) can be projected onto left and right (or positive and negative definite) subspaces: \(p = (p_L, p_R)\) with \(p^2 = p_L^2 - p_R^2\) and \(p_L^2, p_R^2 \geq 0\). Note that even though \(y \in \mathcal{H}\), the function \(\Theta - \) and hence \(\Phi -\) is actually automorphic under
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| Aut(Γs+2,2) = O_{s+2,2}(Z). In a more general treatment [B-96], Θ also depends on a homogeneous polynomial and its defining lattice need not be self-dual (in which case the modular properties are recovered by considering vector-valued theta functions). |

4.3. The Result for the Prepotential

In the easier case where \( F(q) =: \sum_{n \geq -n_0} c(n) q^n \) is a meromorphic modular form, [HM-95] have computed the above integral by a generalisation of the technique of [DKL-91] via “unfolding the fundamental domain”. It yields the logarithm of an automorphic product à la Borcherds for the lattice \( \Gamma_{s+1,1} \):

\[
\Phi_{s+2,2}(y) = -2 \log \left| e^{-2\pi \rho \cdot y} \prod_{r > 0} (1 - e^{-2\pi r \cdot y})^c(-r^2/2) \right|^2 + c(0)(-\log[-(\Re y)^2] - \mathcal{K}),
\]

with \( r \) the “positive points” of the lattice \( \Gamma_{s+1,1} \), \( \rho \) the so-called Weyl vector of the lattice, and \( \mathcal{K} \) some insignificant constant. Explicitly, for \( r = (\vec{r}, k, l) \in \Gamma_{s+1,1} \cong \Gamma_s \oplus \Gamma_{1,1} \), the meaning of \( r > 0 \) is: \( k > 0 \), or \( k = 0 \) and \( l > 0 \), or \( k = l = 0 \) and \( \vec{r} \) in some chosen Weyl chamber.

According to Borcherds\(^1\) [B-95], if \( F(q) \) has weight \(-s/2\) and the \( c(n) \) are integers (with \( 24|c(0) \) if \( s = 0 \)), then such a product can be analytically continued to a meromorphic automorphic form of weight \( c(0)/2 \) on \( O_{s+2,2}(Z) \); its zeroes and poles lie on rational quadratic divisors \( ay^2 + y \cdot c = 0 \) (\( a, c \in \mathbb{Z} \)). Note that inside the radius of convergence, there are no poles and all zeroes lie on linear divisors \( r \cdot y + c = 0 \) (\( c \in \mathbb{Z} \)).

A second type of integral, \( \Phi_{s+2,2}(y) \), in which \( F(q) \) is replaced by \( F(q)(E_2(q) - \frac{3}{\pi^2}) \), was also computed in [HM-95] and yielded a similar result containing the above

\[
\log \prod_{r > 0} (1 - e^{-2\pi r \cdot y})^c(-r^2/2) = \sum_{r > 0} c(-\frac{r^2}{2}) \log(1 - e^{-2\pi r \cdot y})
\]

\[= - \sum_{r > 0} c(-\frac{r^2}{2}) \text{Li}_1(e^{-2\pi r \cdot y}), \text{ together with further polylogs } \text{Li}_2 \text{ and } \text{Li}_3.\]

Hence for the linear differential equation where these two integrals occur, we also expect its solution to share the automorphic properties. This solution is the one-loop prepotential; explicitly:

\[
F_0(y) := h^{(1)}(y) = \frac{1}{384\pi^2} \tilde{d}_{ijk} y^i y^j y^k - \frac{1}{2(2\pi)^4} c(0) \zeta(3) - \frac{1}{(2\pi)^4} \sum_{r > 0} c(-\frac{r^2}{2}) \text{Li}_3(e^{-2\pi r \cdot y})
\]

(4.3.1)

Here \( y = (\vec{y}, T, U) \in H^{s+1,1} \) and \( y^2 = \vec{y}^2 + 2TU \). The symmetric tensor \( \tilde{d}_{ijk} \) depends on the Weyl vector and the particular algebra at hand; it is irrelevant for us. This prepotential was obtained in the context of heterotic compactifications on \( K3 \times T^2 \), with standard embedding yielding a gauge group \( E_7 \times SU(2) \times E_8 \times U(1)^4 \) with 388 vector multiplets, 388 being the dimension (of the adjoint) of the gauge group.

\(^1\)Note our exponent of \(-2\pi r \cdot y\) as opposed to Borcherds’ \(2\pi r \cdot y\).
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In the particular $\mathbb{Z}_2$ orbifold limit of the $K3$, the massless matter spectrum transforms in the following representation of the the gauge group:

$$(56, 2, 1) + 8(56, 1, 1) + 32(1, 2, 1) + 4(1, 1, 1) \in E_7 \times SU(2) \times E_8,$$

see [W-87] or the hint in the paragraph containing (5.3.9). These total of 628 hypermultiplets and 388 vector multiplets are too high to hope for an easily describable CY dual on which to compactify the type IIA theory. For the latter, we need few Kähler moduli, i.e. few vector multiplets on the heterotic side, i.e. a smaller gauge group.

This is achieved by introducing $s$ Wilson lines for the $E_8$ part of $E_7 \times SU(2) \times E_8$ and fully Higgsing the remainder (see [G2-04]). Here, $s$ is the same as in $\mathcal{H}^{s+1,1}$. [HM-95] considered the cases $s = 0$ and $8$ which yield residual gauge groups of $U(1)^4$ and $U(1)^{12}$ respectively, with 4 or 12 vector multiplets.

4.4 Two Cases and Counting Rational Curves

For the case $s = 0$, $y = (T, U)$, $c(n)$ are the Fourier coefficients of $F_{s=0} := E_4E_6/\eta^{24}$ (which is $\Gamma_8 = E_8$ times $F_{s=8}$ below) and the sum runs over all positive roots $r = (k, l)$ of the monster Lie algebra: $l > 0$; or $l = 0, k > 0$. The Yukawa coupling $\partial_U^3 h^{(1)}$ agrees with another expression [AFGNT-95]:

$$\partial_U^3 F_0(T, U) = -\frac{1}{2\pi} \left( 1 - \sum_{r > 0} c(kl) l^3 \frac{e^{-2\pi(iT+U)}}{1 - e^{-2\pi(iT+U)}} \right) = -\frac{1}{2\pi} \frac{E_4(iU)E_4(iT)E_6(iT)}{(J(iT) - J(iU)\eta(iT)^{24})}$$

This involves only modular forms in $T, U$ separately and is of weight $(-2, 4)$ in $(T, U)$. Note that the $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ symmetry is isomorphic to the symmetry group $SO(2, 2; \mathbb{Z})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ stands for the exchange of $T$ and $U$, and $SO(2, 2; \mathbb{Z})$ is the automorphic group of $\Gamma_{2,2}$. An interesting question is how much of these modular properties are left for the prepotential $h^{(1)}$ itself? The answer does not look bright a priori, as the usual partial derivative does not preserve modularity, rather one needs a covariant one, as in (C.0.60).

As announced, our gauge group has been broken from $E_7 \times SU(2) \times E_8 \times U(1)^4$ to $U(1)^4$. The Higgsing involved cost us $133 + 3 + 248 = 384$ scalars to give mass to the gauge fields in the adjoint (as outlined in the examples of [G2-04]). Thus we are left with $628 - 384 = 244$ hypers and 4 vectors, which begs for a CY threefold with $h^{2,1} = 243$ and $h^{1,1} = 3$.

Such a CY $X$ fortunately exists; it is a degree 24 hypersurface in $\mathbb{P}(1, 1, 2, 8, 12)$, i.e. a $K3$ fibration over $\mathbb{P}^1$ where the $K3$ is a degree 12 hypersurface in $\mathbb{P}(1, 1, 4, 6)$ [KLM-95]. In this type IIA theory on $X$, which is a model for topological string theory, we see that the prepotential $F_0$ in (4.3.1) has remarkably the shape desired for counting instantons - recall the $L_3$ from (2.3.4). Then we see that the rational holomorphic curves in the fibres of the CY are parametrised by the positive roots $r$ of the $E_{10}$ root lattice, and their number is given by $c(-r^2/2)$.

For the case $s = 8$, to which we turn for the remainder of this section, $c(n)$ are the Fourier coefficients of $F_{s=8} := E_6/\eta^{24}$ – see eqn (5.2.2) – and the sum...
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runs over the positive roots \( r = (f, k, l) \) of the \( E_{10} \) Lie algebra where \( f \) is itself a positive element of \( \Gamma_8 \) (the root lattice of \( E_8 \)).

As announced, our gauge group has been broken from \( E_7 \times SU(2) \times E_8 \times U(1)^4 \) to \( U(1)^{12} \). The Higgsing of \( E_7 \times SU(2) \) cost us 133 + 3 scalars. Thus we are left with 628 - 136 = 492 hypers and 12 vectors, which begs for a CY threefold with \( h^{2,1} = 491 \) and \( h^{1,1} = 11 \). The candidate CY \( X \) is a degree 84 hypersurface in \( \mathbb{P}(1,1,12,28,42) \), again a K3 fibration over \( \mathbb{P}^1 \) where the K3 is a degree 42 hypersurface in \( \mathbb{P}(1,6,14,21) \).

As before, we see that the rational holomorphic curves in the fibres of the CY are parametrised by the positive roots \( r \) of the \( E_{10} \) root lattice, and their number is given by \( c(-r^2/2) \). So the only question left is whether \( F_0 \) (or any derivative thereof) in this case enjoys similar modular properties as in the case \( s = 0 \).

Furthermore, one is driven to ponder on the following issues: What is the relation between the sum over \( d > 0 \) in (2.3.4) and the sum over \( r > 0 \) in (4.3.1)? Would the \( F_g \) (\( g > 0 \)) in (2.3.4) enjoy modular or automorphic properties? Could they also be expressed as a sum over \( \Gamma_{s+2,2} \) or over a root lattice of some algebra, rather than over \( H_2(X,\mathbb{Z}) = \mathbb{Z}^{h^{1,1}} \)? If so, what is the relation between the CY \( X \) and the lattice of which the positive roots govern the counting of holomorphic curves? Would the \( n_d^r \) be the (integer) coefficients of some nearly-holomorphic modular forms, like in (4.3.1)?

We conjecture that this should indeed be the case. The work of [KY-00] sheds some light in this direction. Let us turn to it.

4.5 Extension of the Moduli Space

The way to convert the general form of (2.3.4) into a modular product à la Borcherds is to generalise Borcherds' lifting of a Jacobi form \( \Phi_0(\tau, z) \) of weight zero for a positive definite lattice to a lifting of a form \( \Phi_0(\tau, z, \lambda) \) defined on a Lorentzian lattice \( \Gamma \). This extension comes along with the extension of the moduli space to include the string coupling constant \( \lambda \) next to the moduli \( \tau \): \( H^2(X,\mathbb{Z}) \oplus H_0^0(X,\mathbb{Z}) \). The crucial idea in [KY-00] is to rewrite the GW potential as a sum over Hecke operators \( V_l \) acting on \( \Phi_0 \):

\[
F = \sum_{g \geq 0} \lambda^{2g-2} F_g(q_i) = \sum_{g \geq 0} \lambda^{2g-2} F_g(p, \tau, z) = \sum_{l \geq 0} p^l \Phi_0|V_l| (\tau, z, \lambda),
\]

i.e. to absorb the parameter \( \lambda \) into the lattice and to express \( F_g \) using variables \( (p, \tau, z) \) instead of the tuple \( q_i = e^{t_i} \). The new lattice is defined to be \( \Gamma := Q^\vee(m) \oplus (-2) \) where \( Q^\vee \) is the coroot lattice of a simple Lie algebra \( g \) and \( Q^\vee(m) = (Q^\vee, m(\ ,\ )) \). The lattice \( \Gamma^* \) contains points \( (\gamma, j) \) which will be multiplied with the variables \( (z, \lambda) \in \Gamma_C \) to yield \( e^{2\pi i (\gamma \cdot z + j \lambda)} =: \zeta^\gamma y^j \) in the Fourier series to follow. Let also \( \eta := e^{2\pi i r} \).

To further understand the procedure, we give here concrete formulae: Let \( \phi_{-2,m} \) be a nearly-holomorphic Jacobi form of weight \(-2\) and index \( m \), invariant under the action of the Weyl group of \( g \). The function \( (\eta_1(\tau, \lambda)/\eta(\tau)^3)^2 \) is itself
a weak Jacobi form of weight \(-2\) and index \(1\). We define \(\Phi_0\) and its Fourier and Taylor coefficients as follows:

\[
\Phi_0(\tau, z, \lambda) := -\phi_{-2,m}(\tau, z) \frac{\eta(\tau)^6}{\vartheta_1(\tau, \lambda)^2} = \sum_{n \geq -n_0} D(n, \gamma, j) q^n \zeta^j y^j (\gamma, j) \in \Gamma^*
\]

\[
= -\sum_{g=0}^{\infty} \lambda^{2g-2} \varphi_{2g-2,m}(\tau, z)
\]

The \(\varphi_{2g-2,m}(\tau, z)\) are quasi Jacobi forms of weight \(2g-2\) since the \(\lambda\)-expansion of \(1/\vartheta_1^2\) has (quasi) modular forms as coefficients.

Then one can check that

\[
\Phi_0|_l V_i(\tau, z, \lambda) = -\sum_{g \geq 0} \lambda^{2g-2} \varphi_{2g-2,m}|_l V_i(\tau, z), \quad (\forall \ l \geq 0)
\]

which allows us to express \(\mathcal{F}_g\) directly in terms of the \(\varphi_{2g-2,m}\):

\[
\mathcal{F}_g(p, \tau, z) = \sum_{l \geq 0} p^l \varphi_{2g-2,m}|_l V_i(\tau, z)
\]

\[
= \ldots
\]

\[
= \frac{c_g(0,0)}{2} \zeta(3-2g) + \sum_{(l,n,\gamma) > 0} c_g(ln, \gamma) \text{Li}_{3-2g}(p^l q^n \zeta^j),
\]

where \((l, n, \gamma) > 0\) means \(l > 0\) or \(l = 0, n > 0\) or \(l = n = 0, \ z > 0\), and the \(c_g\) are the Fourier coefficients of \(\varphi_{2g-2,m}\):

\[
\varphi_{2g-2,m}(\tau, z) = \sum_{n, \gamma} c_g(n, \gamma) q^n \zeta^j.
\]

A similar action of the Hecke operators is valid on any Jacobi form \(\Phi_k\) of weight \(k\):

\[
\sum_{l=0}^{\infty} p^l \Phi_k|_l V_i(\tau, z, \lambda) = \sum_{(l,n,\gamma,j) > 0} D(ln, \gamma, j) \text{Li}_{k-1}(p^l q^n \zeta^j y^j),
\]

modulo constant terms. The benefit of having chosen \(\Phi_0\) of weight \(0\) is that the \(\text{Li}_1\) is but a logarithm, and the \textit{partition function} can thus be expressed as an infinite product:

\[
\mathcal{Z}(\sigma, \tau, z, \lambda) = e^{\mathcal{F}} = \exp \sum_{g \geq 0} \lambda^{2g-2} \mathcal{F}_g \sim \prod_{(l,n,\gamma,j) > 0} (1-p^l q^n \zeta^j y^j)^{D(ln, \gamma, j)}
\]

modulo the Weyl vector. Recall that \(p = e^{2\pi i \sigma}, q = e^{2\pi i \tau}, \ z = e^{2\pi i z}, y = e^{2\pi i \lambda}.\)
4.6 Mapping the Moduli \( q_i \) to \((u, p, q, \zeta)\)

The question arises as to how do we map the tuple \( q_i = e^{t_i} \) to the variables \((p, \tau, z)\). We now carry this out for the special case of a CY threefold \( X \) that can be realised both as \( K3 \) fibration over \( \mathbb{P}^1 \) and as an elliptic fibration over a surface \( W2 \). We choose our simple Lie algebra \( \mathfrak{g} \) with coroot lattice \( \mathbb{Q}^V \) to be of rank \( s = h^{1,1}(X) - 3 \) and such that the Picard lattice of a general \( K3 \) fibre is isomorphic to \( \Gamma_{1,1} \oplus \mathbb{Q}^V(-m) \) for the even self-dual Lorentzian lattice \( \Gamma_{1,1} \) of signature \((1,1)\). In other words, we need three more variables next to the lattice variables \( \zeta \) to correspond to the Kähler moduli \( t_1, \ldots, t_{h^{1,1}} \) of \( X \). Let those three be \( u, p, q \), where \( u := e^{t_1} \) is new here: it is related to \( q_1 = e^{t_1} \) and will become redundant in the limit of large base \( \mathbb{P}^1 \) (i.e. \( t_1 \to \infty \)). This is why \( u \) does not appear in the argument of \( \mathcal{F}_g = \mathcal{F}_g(p, \tau, z) \). We furthermore assume that \( \eta(\tau)^{2d} \phi_{-2,m}(\tau, z) \) is a Jacobi function of weight \( 10 \) and index \( m \), equal to \(-2E_4E_6 \) at \( z = 0 \), and such that \( c_0(0,0) = -\chi(X) \).

The precise mapping between the \( t_i \)'s and \((u, p, q, \zeta)\) was suggested by [KY-00] as

\[
\begin{align*}
t_1 &= \log u - \log q, \\
t_2 &= \log p - \log q, \\
t_3 &= \log q - (\gamma_0, \log \zeta), \\
t_{i+3} &= (\Lambda_i, \log \zeta), \quad (i = 1, \ldots, s),
\end{align*}
\]

for \( \gamma_0 \) some positive weight and \( \Lambda_i \) \((i = 1, \ldots, s)\) the fundamental weights of \( \mathfrak{g} \). Thus

\[
q^l = e^{t-d} = e^{t_1d_1}e^{t_2d_2}e^{t_3d_3}e^{\sum_i t_{i+3}d_{i+3}} \\
= (u^{d_1}q^{-d_1})(p^{d_2}q^{-d_2})(q^{d_3}\zeta^{-\gamma_0d_3})(\zeta^{\sum_i \Lambda_i d_{i+3}}) \\
= u^{d_1}p^{d_2}q^{-d_1-d_2+d_3}\zeta^{\gamma_0d_3+\sum \Lambda_i d_{i+3}},
\]

and as \( u \to 0 \), only \( d_1 = 0 \) contributes, and thus the identification is:

\[
p^{d_2}q^{d_3-d_2}\zeta^{\gamma_0d_3+\sum \Lambda_i d_{i+3}} \cong p^l q^n \zeta^{\gamma}.
\]

When summing over \( d > 0 \), i.e. over all \( d_2, \ldots, d_l \) not all zero, we see that the following three cases occur:

\[
l > 0, \text{ and } n, \gamma \text{ arbitrary}, \\
l = 0, n > 0, \gamma \text{ arbitrary}, \\
l = n = 0, \gamma > 0.
\]

These are just the conditions \((l, n, \gamma) > 0 \) that we needed, as in (4.5.1). In other words, we have attained our goal of rewriting \( \mathcal{F}_g \) in (2.3.4) as a sum over positive roots of a lattice \( \Gamma_{1,1} \oplus (\mathbb{Q}^V)^{\ast} \left( \frac{1}{m} \right) \). At least, this is what [KY-00] conjectured:

\[
\mathcal{F}_g = \mathcal{F}_g \quad (g \geq 2).
\]
(plus extra constant terms at $g = 0, 1$: $F_0^{\text{const}}, F_1^{\text{const}}$). This translates into their main conjecture for the partition function:

$$Z(q, \lambda) = Z(\sigma, \tau, z, \lambda) = \exp \left( x^{-2} F_0^{\text{const}} + F_1^{\text{const}} \right) \prod_{(l, n, \gamma, j) > 0} \left( 1 - p^l q^n \zeta^\gamma y^j \right)^{D(l, n, \gamma, j)} ,$$

with $(l, n) \in \Gamma_{1,1}$, $(\gamma, j) \in \Gamma^*$. This is a modular product à la Borcherds for the lattice $\Gamma_{1,1} \oplus \Gamma^*$, with the only difference that $\Gamma$ here is a Lorentzian lattice instead of a positive definite one. The term in the exponential is then the corresponding Weyl vector. The vector $v = (\sigma, \tau, z, \lambda)$, such that $e^{2\pi iv} := pq\zeta y$, would then be element of the Grassmanian corresponding to the modular product. Would this be $O(s + 2, 3)$?