Gromow-Witten Invariants and Elliptic Genera

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Chapter 5

Threshold Corrections for Heterotic Orbifolds

This crowning section is rather intricate, both mathematically and physically. It aims at bringing together the different objects touched upon in the two previous sections, as well as giving the details for some computations that we left out. Thus, we shall come back on the “new susy index” (3.3.2), explain our explicit values for the $K3$ elliptic genus $C_{K3}/\eta^3$ in (3.2.1) and for the functions $F(q) = \sum c(n)q^n$ in section 4.3. In fact, the latter are nothing but results that pop up in computations of threshold corrections.

These are upper-half-plane integrals and are presented in (5.1.2) of the first subsection, including a gauge theory factor which we shall neglect until the very last subsection.

An explicit expression, $\Gamma_{10,2}E_6/\eta^{24}$, for the integrand of the threshold correction (5.1.2) is given in (5.3.8), after a direct but cumbersome calculation via the orbifold partition functions of section 5.3 for $K3 \times T^2$ (we assume familiarity with partition functions for bosons/fermions on several compactification spaces). The same expression can be obtained after first re-writing the original integrand (5.1.2) as the “new susy index” (5.2.1) and then evaluating the latter in the particular case of $K3 \times T^2$ to obtain (5.2.2).

Thus our presentation can be summarised in the following commutative diagram:

\[
\begin{array}{ccc}
(5.1.2) & \longrightarrow & (5.2.1) \\
\downarrow & & \downarrow & \text{1-loop threshold correction} & \longrightarrow & \text{new susy index} \\
(5.3.8) & \longrightarrow & (5.2.2) & \text{het. orb. part. fct.} & \downarrow \begin{array}{c}
\Gamma_{10,2} E_6/\Delta
\end{array} & \downarrow \begin{array}{c}
\Gamma_{10,2} E_6/\Delta
\end{array}
\end{array}
\]

The last subsection 5.4 finally incorporates the gauge theory factor of the integrand of the threshold correction. The results there are just the functions $F(q)$ or $F(q)E_2$ used in the integrands of section 4.3 that yielded the infinite
5.1 Effective Field Theory and One-Loop Threshold Corrections

Threshold corrections to coupling constants are the differences between calculations in two separate frameworks: the fundamental theory (or field theory, FT) and the effective field theory (EFT). The latter takes only light states into account (the massless NS-NS fields $G_{\mu\nu}, B_{\mu\nu}, \phi$) and computes correctly in the low-energy range, i.e. up to energies neighbouring the masses of the heavy states (which occur at the scale of the string length). At tree-level, it computes the on-shell scattering amplitudes for light states up to terms that vanish by the equations of motion, i.e. it integrates out the heavy states. The FT on the other hand takes both light and heavy states into account; so the difference between the FT and EFT result is precisely the contribution from heavy states that the EFT misses out. This contribution should be added to the loop-expansion in the EFT and is called the threshold correction $\Delta_i$. The subscript $i$ refers to the gauge group for which we compute the gauge coupling.

At tree level, the gauge coupling constant is equal to the string coupling. Denoting the one-loop correction by $\Delta_i$, we have to one-loop order:

$$\frac{1}{g^2_i} = \frac{k_i}{g^2_{\text{str}}} + \Delta_i, \quad (5.1.1)$$

The constant $k_i$ is the central element for the left-moving algebra which generates the gauge group $G_i$. Since $G_i$ is non-abelian, we set $k_i = 1$. The threshold correction itself can be computed for an N=2 heterotic compactification on $K3 \times T^2$ to be (see [G2-04] for a sketch):

$$\Delta_i = \int_\mathcal{F} \frac{d^2 \tau}{\tau_2} \left( \frac{i}{\pi \bar{\eta}^4} \sum_{a+b \in \text{even}} (-1)^{a+b} \bar{\eta} \left( \frac{\partial_q^{[a]} [q]}{\bar{\eta}} \right) \right) \text{Tr}_{\text{int}} \left( Q_i^2 - \frac{k_i}{4\pi \tau_2} [q] - b_i \right). \quad (5.1.2)$$

The subscript $i$ refers to one of the several factors of the gauge group, while “even” spin structures means $(a, b) \neq (1, 1)$ $(a, b = 0, 1)$. The constant $b_i$ is called the beta function coefficient and represents the constant term of the $q$-expansion of the integrand (see also appendix E of [G2-04]); subtracting it renders the integral IR finite. Note that we have suppressed for convenience the factors of $q$ in the trace: the proper expression should contain also $C_{\text{int}}^{[a]} := \text{tr} (-1)^{b_0} q^{L_0-\frac{1}{2}} \bar{q}^{\bar{L}_0-\frac{1}{2}} [q]^{[a]}$ for the $(c, \bar{c}) = (22, 9)$ internal theory, where $J_0$ coincides with the right-moving fermion number $F_R$. The notation $[q]$ stands for the spin structure, see also (5.3.5).
5.2. Threshold via New Susy Index & K3 Elliptic Genus

An alternative way of writing the integrand (5.1.2) is as a "new susy index" (3.3.2) (see again [G2-04] for a sketch of this derivation)

\[
\frac{16\pi^2}{g_i^2} \left. \right|_{1\text{-loop}}^{\frac{d^2 \tau}{\tau_2}} = -\frac{1}{\eta^2} \left( \text{Tr}_{R, \text{int}} \left( \hat{F}(-1)^F q^{L_0 - \frac{1}{12} q^{L_0 - \frac{3}{2}}} \left[ Q_i^2 - \frac{k_i}{4\pi\tau_2} \right] \right) - b_i \right),
\]

with the usual group theory factor in square brackets. Indeed, this was the starting point in [HM-95].

The computation of this new susy index is quite different for the left- and right-moving sectors. In the right-moving sector, the algebra factorises into the direct sum of a \( c = 3 \) \( N=2 \) SCA and a \( a \) \( \tilde{c} = 6 \) \( N=4 \) SCA, which simplifies considerably the computation of the new susy index: the result is a mere constant, \(-2i\), thanks to the equal but opposite contributions of vector- and hypermultiplets towards \( J_0(-1)^F \) (see [HM-95] or section 10.5 of [G2-04]). In the left-moving sector, the new susy index enters in the form of the above elliptic genus \( \text{Tr}_R (-1)^{J_0 + J_0} q^\Delta \), to which we now turn.

For the left-moving sector, we have to evaluate the \( K3 \) elliptic genus and find the proper partition functions. In our compactification on \( T^2 \times K3 \), we shall use bosonic formulation for one of the \( E_8 \) gauge groups and fermionic formulation (with sixteen left-moving fermions) for the other. The gauge bundle is a direct sum of a bundle on \( T^2 \) and one on \( K3 \), with flat connection or a.s.d. connection respectively. We choose to couple 12 fermions with the former connection and 4 with the latter, so as to obtain the desired \( (c, \tilde{c}) = (6, 6) \) heterotic sigma model on \( K3 \).

The partition functions for the former fermions on \( T^2 \) are the familiar \( \vartheta_i/\eta \), \( i = 1, 2, 3, 4 \) for the NS/R sectors with/without \((-1)^F\) insertion. The partition functions for the latter fermions on \( K3 \) are the elliptic genera \( \Phi \) of (6.3.2). The bosonic realisation of the other \( E_8 \) factor yields the familiar theta function \( \Gamma_8/\eta^8 \), which we join with the \( \Gamma_{2,2} \) lattice of \( T^2 \) to obtain \( \Gamma_{10,2}/\eta^{12} \). Taking into account the \(-2i\) from the right-moving sector, and summing over all worldsheet boundary conditions of the left-moving sector, we obtain for the whole "new susy index" of the internal theory:

\[
\text{tr}_R \ J_0 e^{i\pi(J_0 - \tilde{J}_0)} q^{L_0 - c/24} \tilde{q}^{L_0 - \tilde{c}/24}
= -2i \frac{\Gamma_{10,2}}{\eta^{12}} \left( \left( \frac{\vartheta_3}{\eta} \right)^6 \Phi_A^+ + \left( \frac{\vartheta_4}{\eta} \right)^6 \Phi_A^- + \left( \frac{\vartheta_2}{\eta} \right)^6 \Phi_\sigma + \left( \frac{\vartheta_1}{\eta} \right)^6 \Phi_x \right)
= -2i \frac{\Gamma_{10,2}}{\eta^{12}} 2 \left( \vartheta_3^8 (\vartheta_2^4 - \vartheta_4^4) - \vartheta_4^8 (\vartheta_2^4 + \vartheta_3^4) + \vartheta_2^8 (\vartheta_3^4 + \vartheta_4^4) \right),
\]

\[\text{In general, } (16 - 2n) \text{ fermions on } T^2 \text{ plus } 2n \text{ on } K3 \text{ gives a } (c, \tilde{c}) = (4 + n, 6) \text{ model. Hence each fermion coupled to } K3 \text{ increases the left-moving central charge by } 1/2. \text{ We have } n = 2.\]
hence

\[
\tr_r \tilde{J}_0 e^{i \pi (J_0 - J_0') q^{L_0 - c/24} \tilde{q}^{L_0 - c/24}} = 8i \Gamma_{10.2} \frac{E_6}{\Delta} , \tag{5.2.2}
\]

where we noted that the last bracket is but \( \sum_{i \neq j} \partial_i^8 \partial_j^4 \) with a minus sign if \( 2i + j > 8 \), which is just \(-2E_6\) by (C.0.51).

### 5.3 Threshold via Heterotic Orbifold Partition Function

We will now derive the same result for the “new susy index”, but this time starting from the expression (5.1.2) for the partition function

\[
Z_{D=4}^{\text{het}} = \frac{1}{\tau_2 \eta^2 \bar{\eta}^2} \sum_{a,b=0}^{1} (-1)^{a+b+ab} \frac{\bar{q}[^{a}_{b}]}{\eta} C_{\text{int}}[^{a}_{b}]. \tag{5.3.1}
\]

with the appropriate internal contribution \( C_{\text{int}}[^{a}_{b}] = \tr_{\text{int}} (-1)^{b \tilde{J}_0} q^{\Delta} \bar{q}^{\Delta}[^{a}_{b}] \), but still without the gauge theory factor with the Casimir operator. It is encouraging to see the alternative result (5.3.8) agree with (5.2.2).

The internal contribution consists of the partition functions for the \( T^2 \) bosons \((\Gamma_{2,2}(T,U)/\eta^2 \bar{\eta}^2)\) and fermions \((\frac{1}{2} \sum_{a,b} (-1)^{a+b+ab} \bar{q}[^{a}_{b}]) / \eta\), for the \( K^3 \) bosons and fermions, as well as for the gauge bundle and its two \( E_6 \) factors which we take in the bosonic realisation \((\Gamma_e / \eta^8 = E_4 / \eta^8)\) and fermionic realisation \((\frac{1}{2} \sum_{\gamma, \delta} \partial^8 [\gamma] / \eta^8)\) respectively. Since the partition function (or the elliptic genus) is a topological object, it does not depend on the hypermultiplet moduli, so we will choose a limit for these moduli where the \( K^3 \) surface is described by a \( \mathbb{Z}_2 \) orbifold breaking the gauge group to \( E_8 \times E_7 \times SU(2) \). This has a well-known partition function for the bosonic (4,4) blocks:

\[
Z_{(4,4)}[^{0}_{0}] = Z(R) = \frac{\Gamma_{4,4}(G, B)}{\eta^4 \bar{\eta}^4}, \quad Z_{(4,4)}[^{h}_{g}] = 2^4 \frac{\eta^2 \bar{\eta}^2}{\eta^2 [1-h] \bar{\eta}^2 [1-g]} \quad (h, g) \neq (0, 0), \tag{5.3.2}
\]

where the lattice function depends on the metric \( G_{ij} \) and the B-field \( B_{ij} \):

\[
\frac{\Gamma_{4,4}(G, B)}{\eta^4 \bar{\eta}^4} := \sum_{m, n \in \mathbb{Z}^4} q^{\frac{m^2}{2}} \bar{q}^{\frac{n^2}{2}} / \eta^4 \bar{\eta}^4, \quad p_{L,R}^i := \frac{G_{ij}}{\sqrt{2}} (m_j + (B_{jk} \pm G_{jk}) n_k)
\]

\[
= \frac{\sqrt{\det \bar{G}}}{(\sqrt{\tau_2 \eta \bar{\eta})}^4} \sum_{m, n \in \mathbb{Z}^4} \exp \left( -\frac{\pi}{\tau_2} (G_{ij} + B_{ij})(m_i + n_i \tau)(m_j + n_j \bar{\tau}) \right) \tag{5.3.3}
\]

with \( p_{L,R}^i \) the inner product w.r.t. the metric, i.e. \( p_{L,R}^i G_{ij} p_{L,R}^j \). (Note that our metric \( G_{ij} \) has absorbed a factor of \( R_t \) (orbifold radii) compared to other conventions.)
Similarly, the compact $K3$ fermions ($\bar{\vartheta}^2/\eta^2$) are twisted in the $(4,4)$ blocks: $\bar{\vartheta}^{[a+b]}_{(b+g)} \bar{\vartheta}^{[a-h]}_{(b-g)} / \eta^2$. Additionally, the orbifold projection on the fermionic $E_8$ factor will correspond to a sign change for the doublet of the $SU(2)$ subgroup of $E_8 \supset E_7 \times SU(2)$. Note that the adjoint decomposes as

$$248 \rightarrow (133,1) + (1,3) + (56,2) \in E_7 \times SU(2),$$

and that the projection acts on the $SU(2)$ representations as $3 \rightarrow 3, \quad 2 \rightarrow -2$. This entails that two of our eight complex fermions are twisted by the projection and this part of the partition function reads then

$$\frac{1}{2} \sum_{\gamma, \delta=0} \frac{\bar{\vartheta}^{[\gamma+h]}_{\delta+g} \bar{\vartheta}^{[\gamma-h]}_{\delta-g} \vartheta^{[\gamma]}_{\delta}}{\eta^8}. \quad (5.3.4)$$

The other $E_8$ factor (the bosonic realisation with $\Gamma_8$) is not affected by the projection. The full partition function for the internal theory is the product of all the above-mentioned parts:

$$C_{\text{int}} = \frac{\Gamma_{2,2}}{\eta^2} \frac{\Gamma_{8}}{\eta^8} \frac{1}{2} \sum_{a,b} (-1)^{a+b+ab} \bar{\vartheta}^{[a]}_{b} \frac{1}{\eta} \frac{1}{2} \sum_{g,h=0} Z_{(4,4)}^{[g]} \frac{\bar{\vartheta}^{[a+h]}_{b+g} \bar{\vartheta}^{[a-h]}_{b-g}}{\eta^2} \times \frac{1}{2} \sum_{\gamma, \delta=0} \frac{\bar{\vartheta}^{[\gamma+h]}_{\delta+g} \bar{\vartheta}^{[\gamma-h]}_{\delta-g} \vartheta^{[\gamma]}_{\delta}}{\eta^8}. \quad (5.3.5)$$

In (5.3.1) or implicitly in (5.1.2), $C_{\text{int}}[\vartheta]_{\delta}$ is just the above with $C_{\text{int}} =: \sum_{a,b} (-1)^{a+b+ab} C_{\text{int}}[\vartheta].$

Note that the term in (5.1.2) with the $\bar{r}$-derivative includes the non-compact fermions. We will combine these with the $T^2$ and $K3$ fermions and sum over their spin structures:

$$-\frac{i}{\pi} \frac{1}{2} \sum_{\text{even}} (-1)^{a+b} \bar{\vartheta} \left( \frac{\bar{\vartheta}^{[a]}_{b}}{\eta} \right) \frac{\bar{\vartheta}^{[a+h]}_{b+g} \bar{\vartheta}^{[a-h]}_{b-g}}{\eta^3} =$$

$$\frac{1}{24 \eta^4} \left[ \frac{1}{2} \left( \bar{\vartheta}^{[a+b]}_{2} \bar{\vartheta}^{[a-h]}_{1} \bar{\vartheta}^{[a-h]}_{g} - (\bar{\vartheta}^{[a-b]}_{3} + \bar{\vartheta}^{[a]}_{4}) \bar{\vartheta}^{[a+h]}_{g} \bar{\vartheta}^{[a-h]}_{1} \right) + (\bar{\vartheta}^{[a-h]}_{2} + \bar{\vartheta}^{[a]}_{3}) \bar{\vartheta}^{[a-h]}_{1+g} \bar{\vartheta}^{[a-h]}_{1-g} \right]$$

$$= -\frac{1}{2} \eta^2 \bar{\vartheta}^{[1-h]}_{1+g} \bar{\vartheta}^{[1-h]}_{1-g} (-1)^{1+h}$$

where we have used (C.0.30) and (C.0.11); and the use of (C.0.12) will convince the reader that whatever the combination of $g$ and $h$, the square brackets yield $-12 \eta^5 \bar{\vartheta}^{[1-h]}_{1+g} \bar{\vartheta}^{[1-h]}_{1-g} (-1)^{1+h}$. In particular, this vanishes for $(g, h) = (0, 0)$.

If we multiply it with the $K3$ bosons and the uncompactified bosons ($1/\eta^2 \bar{\vartheta}^2$ for two transverse bosons in the LCG), we obtain, noting that $\vartheta^{[1-h]}_{1-g} (-1)^{1+h} = \vartheta^{[1+h]}_{1+g}$:

$$-8 \frac{\eta^2}{\vartheta^{[1+h]}_{1+g} \vartheta^{[1-h]}_{1-g}}, \quad (h, g) \neq (0, 0). \quad (5.3.6)$$
Taking further into account the remaining \( E_8 \) fermions with orbifold projection, and summing explicitly over the \( g, h \) blocks of the orbifold:

\[
-8 \frac{\bar{\eta}^2}{\eta^8} \sum_{(h, g) \neq (0, 0)} \sum_{a, b = 0}^1 \frac{\partial \phi[a, b] \partial \phi[\alpha, h]}{\partial \phi[\alpha + h] \partial \phi[-\alpha - g]}
\]

\[
= -8 \frac{\bar{\eta}^2}{\eta^8} \frac{1}{4} \left( \frac{\partial_2 \phi_2^2 - \partial_4 \phi_4^2}{\eta^3} + \frac{\partial_3 \phi_3^2 + \partial_5 \phi_5^2}{\eta_4} - \frac{\partial_3 \phi_3^2 + \partial_5 \phi_5^2}{\eta_2} \right)
\]

\[
= -8 \frac{\bar{\eta}^2}{\eta^8} \frac{1}{4} \frac{1}{\eta^6} \left( \partial_3 \phi_3^4 (\partial_2 \phi_2^1 - \partial_4 \phi_4^4) + \partial_2 \phi_2^1 (\partial_3 \phi_3^4 + \partial_4 \phi_4^1) - \partial_3 \phi_3^4 (\partial_3 \phi_3^4 + \partial_4 \phi_4^1) \right)
\]

\[
= \frac{\bar{\eta}^2}{\eta^14} \quad E_6
\]

(5.3.7)

where the last bracket yields \(-2 \ E_6\) according to (C.0.51).

Finally, combining the non-projected \( E_8 \) bosons with the \( T^2 \) bosons gives us the lattice function \( \Gamma_{10.2}/\eta^{10}\bar{\eta}^2 \), and we obtain for the overall expression of the integrand of (5.1.2) without the gauge theory factor:

\[
\frac{-1}{4\pi^2} \frac{1}{\bar{\eta}^2 \eta^2} \sum_{\text{even}} (-1)^{a+b} 4\pi i \partial T \left( \frac{\partial \phi[a]}{\eta} \right) C_{\text{int}}[\phi] = \frac{\Gamma_{10.2} \ E_6}{\eta^{24}} = \Gamma_{2.2} \frac{E_4 \ E_6}{\eta^{24}}. \quad (5.3.8)
\]

This is just the “new susy index” (5.2.2), obtained in a different approach (SCFT for a sigma-model) but with the gauge bundle still split into one factor with bosonic realisation and the other with fermionic one. Thus our explicit example of \( T^2 \times K3 \) with its two independent calculations is a verification that the integrands of (5.1.2) and (5.2.1) are indeed the same!

To give a feeling of the information content of these partition functions about the orbifold theory at hand, we show how the massless spectrum can be arrived at. We use the partition functions in (5.3.5) and claim that the twisted sector contains the massless states \( 8 (56,1,1) \) and \( 32 (1,2,1) \) in the \( E_7 \times SU(2) \times E_8 \). The remaining \( 56, 2, 1 \) \( + 4 (1,1,1) \) are in the untwisted sector. Now the twisted sector in (5.3.5) corresponds to \( h = 1 \) and the bosonic part of it is given by \( a = 0 \) (for which we have \( \phi_3, \phi_4 \), i.e. half-integer powers of \( q \), i.e. the NS sector). The right-moving fermions (including two non-compact transverse fermions \( \phi/[\beta]/\eta \) give us

\[
\frac{1}{2\eta^4} (\phi_3^2 \phi_2^2 + \phi_4^2 \phi_4^2) \quad (g = 0 \text{ and } g = 1 \text{ resp.}). \quad (5.3.9)
\]

Using this expression, we look at the lowest powers (massless states) in the full
5.4. **Examples of Threshold Corrections**

In section 5.3, we devised a convenient way for calculating the integrand of the one-loop threshold corrections, by combining the ingredients of the partition function of the heterotic orbifold compactification. We will now proceed to incorporate the gauge theory factor, that is the trace of the Casimir operator or the square brackets in (5.1.2).

Let us go back to our $N=2$ $\mathbb{Z}_2$ orbifold $T^2 \times K3$ of section 5.3 with gauge group $E_8 \times E_7 \times SU(2)$. We shall let $[Q_i^2 - \frac{k_i}{4\pi\tau_2}]$ act on the three factors of the gauge group separately, by replacing $Q_i^2$ with $(\frac{\partial_{\theta_1}}{2\pi i})^2|_{v_j=0}$ acting on the character $\chi_R(v_j)$ of the corresponding factor. That is, we first write the character with a $v$-dependence and then differentiate twice w.r.t. only one of the $v_j$, say $v_1$. For instance, the $E_8$ character is $\chi_{E_8} = \Gamma_8/\eta^8 = E_4/\eta^8 = (\theta_2^8 + \theta_3^8 + \theta_4^8)/2\eta^8$, and with $v$-dependence it becomes

$$\chi_{E_8}(v_j) = \frac{1}{2} \sum_{a,b=0} \prod_{j=1}^{8} \frac{\theta_b^8(v_j)}{\eta^8}.$$  \hfill (5.4.1)

When acting with $[(\frac{\partial_{\theta_1}}{2\pi i})^2|_{v_j=0} - \frac{k_i}{4\pi\tau_2}]$, the operator on the left can be replaced by $\frac{1}{i\pi} \partial_r$, see (C.0.26), with an extra factor of 1/8 since the other theta functions are now also affected by the $\partial_r$. Taking out this factor of 1/8, we are left with the covariant derivative $D_4$ of (C.0.58), and the whole expression is, by (C.0.60):

$$\left[ (\frac{\partial_{\theta_1}}{2\pi i})^2|_{v_j=0} - \frac{k_i}{4\pi\tau_2} \right] \chi_{E_8}(v_j) = \frac{E_2 E_4 - E_6}{12 \eta^8}.$$  \hfill (5.4.2)

Since all other factors of the gauge group or of the internal partition function remain the same, we simply ought to replace the $\Gamma_8/\eta^8$ factor of (5.3.8) by the
above result, and the complete threshold \( (5.1.2) \) is:

\[
\Delta_{E_8} = \int_{\mathcal{F}} d^2 \tau \tau_2 \left[ -\frac{1}{12} \Gamma_{2,2} \frac{\hat{E}_2 E_4 E_6 - E_6^2}{\eta^{24}} + 60 \right],
\]

where we bear in mind that, as before, the lattice function \( \Gamma_{2,2} \) depends on the torus moduli: \( T, U \in \mathcal{H} \). We note that the (nearly) modular form in the fraction has a Fourier expansion starting with \( 720 + \ldots \), that \( \Gamma_{2,2} \) starts with \( 1 + \ldots \), so that dividing by \(-12\), we do indeed obtain \(-60\) as the constant term. This agrees with the beta function coefficient \( b_{E_8} \) of the corresponding orbifold (see appendix E.2 of [G2-04]).

Similarly, for the threshold corresponding to the \( E_7 \) factor of the gauge group, we exchange the roles of the bosonic and fermionic \( E_8 \) partition functions. That is we keep \( \Gamma_8/\eta^8 \) unchanged and let \( \left( \frac{\partial}{2 \pi} \right)^2 \mid_{\nu_j=0} - \frac{k_i}{4 \pi \tau_2} \) act on the \( E_7 \) character \((5.3.4)\) with \( v_1\)-dependence:

\[
\chi_{E_7}(v_1) = \frac{1}{2} \sum_{a,b} \theta^{[a]}(v_1) \theta^{[b]} \left[ \frac{\partial^{a+h_1} \theta^{[a-h]}_{b-g}}{\eta^8} \right]
\]

We remember from \((5.3.7)\) that these were combined with \( \tilde{\eta}^2/\partial d \) of \((5.3.6)\) and summed over \( g, h \) to yield \( E_6 \). This is basically a product of 12 theta functions, of which only the first will have the \( v_1 \) dependence, so that we can replace the \( \partial_{v_1}^2 \), by \( \frac{1}{12} \partial_{\tau} \) to have \( D_6 \ E_6 \), again according to \((C.0.26)\). The covariant derivative yields \( E_4^2 - \hat{E}_2 \ E_6 \), and combining this with the remaining toroidal bosons \( \Gamma_{2,2} \) and \( \Gamma_8/\eta^8 \), we obtain the desired threshold corrections:

\[
\Delta_{E_7} = \int_{\mathcal{F}} d^2 \tau \tau_2 \left[ -\frac{1}{12} \Gamma_{2,2} \frac{\hat{E}_2 E_4 E_6 - E_4^3}{\eta^{24}} - 84 \right],
\]

where the fraction has a Fourier expansion starting with \(-1008 + \ldots \), which yields 84 when divided by 12. This again agrees with the beta function coefficient \( b_{E_7} \) of the corresponding orbifold.

Note that \((5.4.3)\) contains also the character for the \( SU(2) \) subgroup; so that if we were to compute the trace of the \( SU(2) \) Casimir, we would obtain the same result for the threshold and consequently for \( b_{SU(2)} \); 84, also agreeing with the table of beta-function coefficients in appendix E.2 of [G2-04].

It is interesting to note that the difference of our two thresholds can be easily computed: recalling that \( \eta^{24} = \Delta = (E_4^3 - E_6^2)/1728 \), we evaluate the integral using the trick displayed in [DKL-91] ("lattice unfolding technique"):

\[
\Delta_{E_8} - \Delta_{E_7} = -144 \int_{\mathcal{F}} d^2 \tau (\Gamma_{2,2} - 1) = 144 \log \left( 4\pi^2 T_2 U_2 |\eta(T)\eta(U)|^4 \right),
\]

which scales as \( \sim T_2 \), i.e. as the volume of the torus, in the decompactification limit.
5.4. EXAMPLES OF THRESHOLD CORRECTIONS

The above case of $N=2$ orbifold is closely related to the $N=1$ orbifold where we introduce a second $Z_2$ twist to obtain a $Z_2 \times Z_2$ orbifold with gauge group $E_8 \times E_6 \times U(1)^2$. However, the construction is independent of the untwisted moduli $(T_i, U_i)$ of the three twisted 2-planes, and the threshold corrections are not affected by this $N=1$ sector. So the threshold corrections carry over from the $N=2$ sectors (only one 2-plane twisted): $\Delta_{E_8}$ is as in (5.4.2) and $\Delta_{E_6}$ as in (5.4.4). Consequently, the constant term of the whole integrand is $3/2$ of what it used to be ($3$ for the three 2-planes and $1/2$ due to the extra $Z_2$ twist).

We easily understand this for the $E_8$ factor: since all matter multiplets are neutral, only vector multiplets contribute to the beta function coefficient, and the same proportion of $3/2$ is found by comparing the beta function coefficient for $N=2$ and $N=1$ vector multiplets: $-6/12$ and $-3/4$ (see [G2-04]).

The reader will wonder what the above explicit expressions for one-loop threshold corrections to gauge couplings have to do with the Gromov-Witten invariants we started with in the first chapter. The answer was in fact already given in section 4.3: these threshold corrections $\Delta_1$ occur in an ODE for the prepotential $F_0$, i.e. the genus-0 GW potential. The resolution of the integrals over the fundamental domain yield polylog expressions that can be reorganised as in (4.3.1). This convenient sum allowed [HM-95] to extract the coefficients $c(-T^2)$ and attribute an enumerative meaning to them, as GW invariants often have.