Gromow-Witten Invariants and Elliptic Genera

Grüenberg, D.B.

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Chapter 6

N=4 Characters in Gepner Models and Elliptic Genera

This chapter leaves the subject of topological strings, Gromov-Witten potentials and heterotic string partition functions behind, and focuses on the elliptic genus and its decomposition into ‘orbits’ NS_i. The latter are themselves linear combinations of characters \( ch_0^{NS}(\tau, z) \) weighted by modular functions \( F_i(\tau) \) (not to be confused with the GW-potentials \( F_g \)). We review the properties of characters of the N=4 SCA in the context of a non-linear sigma model on K3. We shall show how they are used to span the ‘orbits’, and how the orbits produce topological invariants like the elliptic genus. We shall derive the same expression for the K3 elliptic genus using three different Gepner models (\( 1^6, 2^4 \) and \( 4^3 \) theories), detailing the orbits and verifying that their coefficients \( F_i \) are given by elementary modular functions.

After recalling the background of the material, we recall in section 6.2 the N=4 characters for the \( c = 6 \) SCA of the non-linear sigma model with K3 target space. We also show how massless and massive characters are used to span the orbits, without yet detailing the construction of these orbits. The subsequent section is an expanded version of the results of [EOTY-89] on topological invariants for K3 based on computations with the orbits.

The original results and the crux of the chapter lie in sections 6.4, 6.5 and 6.6, where we reveal in detail the orbits for the \( 1^6, 2^4 \) and \( 4^3 \) Gepner models respectively, compute the functions \( F_i \) and develop several lemmas on theta functions. The last section, no less original, studies Gepner models of mixed levels, like \( 1^2 2^2 \), \( 1^4 4 \) and \( 1^2 4^2 \) – the first of which is a toroidal model and the other two are K3 models.

In section 6.4, we also explore a certain function \( a(\tau) \) which is essential in [BBG-94] for deriving Ramanujan identities. In particular, we study the function

\[
\frac{1}{\eta} \sum_{n \in \mathbb{Z}} (-1)^n (6n + 1)^k q^{(6n+1)^2/24}
\]
for \( k = 1, 2, 3, 4 \) (Prop. 6.4.15 and thereafter), and relate sums of cubes of theta functions to a single theta function (lemma 6.4.11).

### 6.1 Background

In the mid eighties, many efforts were deployed to find the characters of the N=2 superconformal algebra (SCA), culminating in the work of [RY-87,G-88]. The N=2 characters are defined by \( \text{tr} \, q^{L_0-c/24} y^{J_0} \) with \( L_0 \) the Virasoro operator and \( J_0 \) the \( U(1) \) charge.\(^1\) They fall into two classes: those for continuous central charge \( c > 3 \) and those for discrete \( c < 3 \), namely \( c = 3k/(k + 2) \) with \( k \) being the level.

In the first class, the characters for massive representations are proportional to \( \vartheta_3(z)/\eta^3 \) (in the NS sector), while those for massless representations have an extra denominator of \( 1 + \text{sign}(m)q^{|m|-1/2} \), where \( m \) is a quantum number labelling the conformal dimension \( h \) and the \( U(1) \) charge \( Q \). Unitarity constrains \((h,Q)\) to lie inside a polygonal domain of the plane. Massless representations are those hitting the unitary bound, i.e. with \((h,Q)\) on the boundary of the polygon. Massive representations are those with \((h,Q)\) in the interior of the polygon; they have vanishing Witten index.

In the second class, with discrete \( c < 3 \), the characters are spanned by theta functions and the coefficients are the mysterious string functions of [KP-84].

In the late eighties, characters for the N=4 SCA were also unravelled [ET1-88]. The N=4 algebra contains an affine \( su(2) \) Kac-Moody subalgebra of level \( k \), and the central charge of the SCA is \( c = 6k \). For \( k = 1 \), the massive characters are proportional to \( \vartheta_3(z)^2/\eta^3 \) (in the NS sector), while the massless characters have again an extra denominator. For higher level \( k \), the characters have an additional factor of \( \chi^l_{k-1} \), which denotes the \( su(2) \) affine characters (or a slight deformation of them in the massless case) with \( l \) being the isospin quantum number, \( 0 \leq l \leq k/2 \). Unitarity requires \( h \geq l \) (NS sector) and massless representations hit this bound. Massless N=4 characters were found to be expressible as \( \sum_{\nu} \mathcal{A}_{l,\nu} \chi^l_{k-1} \) for some branching functions \( \mathcal{A}_{l,\nu} \), and even expressible as an infinite sum of N=2 characters taken at double or triple points (these are special points in the \((h, Q)\) plane).

This correspondence between N=2 and N=4 characters was furthermore enhanced in [EOTY-89], where Gepner models were used to write N=4 characters - or rather orbits - as tensor products of several characters of the N=2 minimal theories. Then finite sums over these orbits \( NS_i \) or \( R_i \) yield the traces for the N=4 characters, modular invariant partition functions, elliptic genera, or other topological invariants like

\[
\Phi = \text{tr}_{NS,R}^{N=4} (-1)^F q^{L_0-c/24} y^{J_0} = \sum_i D_i \, NS'_i \, R'_i
\]

\(^1\)We use the common variables \( q = e^{2\pi iz} \) and \( y = e^{2\pi iz} \) where \( z \) keeps track of the \( U(1) \) theta angle.
for some combinatorial factors $D_i$. Since these objects are topological, they should not depend on the particular Gepner model at hand. Gepner models are special points in the moduli space of $K3$ surfaces [NW-99] where the above trace factorizes into a product of NS and R orbits. That is, we only go to points where the formula holds. At different such points, we have different sets of orbits NS$_i$ and of coefficients $D_i$. Moreover, each orbit should be expressible as a sum of a massless and a massive N=4 character: $\text{NS}_i(\tau, z) = \text{ch}^{\text{NS}}(\tau, z) + F_i(\tau) \text{ch}^{\text{NS}}(\tau, z)$.

In the context of $K3$ compactifications, the non-linear sigma model has central charge $c = 6$, thus the $su(2)$ subalgebra of the N=4 SCA has level $k = 1$. In the following, we shall give explicit expression of the functions $F_i(\tau)$ in the case of the $1^6$ and $2^4$ theories (and lay the cornerstone for the $4^3$ theory) and find that they are essentially given by quotients of Dedekind $\eta$ functions, thus reflecting the modular nature of the characters and topological invariants. We also derive the expression for $\Phi$ in both theories and gather on the way useful results on theta functions and other tools of analytic number theory.

### 6.2 N=4 Characters

We first write down the characters of the N=4 SCA with central charge $c = 6$ and level $k = 1$, i.e. corresponding to a sigma model with $K3$ target space. We give here explicitly the characters of the NS sector, and refer to spectral flow for their counterparts in the R sector. They depend on two variables, $q = e^{2\pi i r}$ and $y = e^{2\pi i z}$ for the modular parameter and the $U(1)$ theta angle respectively. Representations are parametrised by highest weight $h$ and isospin $l$ and unitarity implies $h \geq l$ (NS sector). Our N=4 SCA is the enhancement of a N=2 Gepner model by adding $SU(2)$ currents $J^\pm$, and the latter’s characters are defined by

$$\text{ch}^{\text{NS}}(\tau, z) := \text{tr}^{\text{NS}} q^{L_0 - c/24} y^J$$

#### 6.2.1 The Characters

We rewrite the familiar expressions of [ET1-88] for N=4 characters in a more useful parametrisation. There are two kinds of characters: we denote massless characters (with isospin $l$) by $\hat{\text{ch}}^{\text{NS}}_l(\tau, z)$ and massive ones (with highest weight $h$) by $\text{ch}^{\text{NS}}_h(\tau, z)$.

**Massless** representations saturate the unitarity bound: $h = l$ and $l = 0, \frac{1}{2}$:

$$\hat{\text{ch}}^{\text{NS}}_0 = -\frac{\theta_3(z)}{\eta^3} \sum_n q^{n^2/2-1/8} y^n \frac{1 - yq^{n-1/2}}{1 + yq^{n-1/2}} = 2 \left( \frac{\theta_1(z)}{\theta_3} \right)^2 + \left( \frac{q^{-1/8}}{\eta} - 2h_3 \right) \left( \frac{\theta_3(z)}{\eta} \right)^2$$

$$\hat{\text{ch}}^{\text{NS}}_{1/2} = \frac{\theta_3(z)}{\eta^3} \sum_n q^{n^2/2-1/8} y^n \frac{1}{1 + yq^{n-1/2}} = - \left( \frac{\theta_1(z)}{\theta_3} \right)^2 + h_3 \left( \frac{\theta_3(z)}{\eta} \right)^2,$$

with

$$h_3(\tau) := \frac{1}{\eta \theta_3} \sum_n q^{n^2/2-1/8} \frac{1 + q^{n-1/2}}{1 + q^{-n-1/2}}$$

(6.2.1)
The above equalities follow from the fact that the left hand sides are so-called theta functions of characteristic $(0, 0; -4\pi i, -2\pi i\tau)$ of degree 2 (see appendix D below), hence can be spanned by $\vartheta_1(z)^2$ and $\vartheta_3(z)^2$. The coefficients are obtained by evaluating the l.h.s. at $z = \frac{1+\tau}{2}$ and $z = 0$ resp., bearing in mind that $\vartheta_3\left(\frac{1+\tau}{2}\right) = 0$ and $\vartheta_1(0) = 0$. For $z = \frac{1+\tau}{2}$, note that the term $(1 - q^0)$ in the product expression of $\vartheta_3\left(\frac{1+\tau}{2}\right)$ cancels the denominator of the $n = 0$ term of the sum, yielding $2q^{-1/4}$ and $-q^{-1/4}$ for the left hand sides.

**Massive** representations are simpler and exist for $h > 0$ and $l = 0$:

$$ch^\text{NS}_h = q^{h-1/8} \frac{\vartheta_3(z)^2}{\eta^3}. \quad (6.2.2)$$

**Spectral flow** yields the R character (idem for massive characters):

$$\hat{\text{ch}}^R_t(\tau, z) = yq^{1/4} \hat{\text{ch}}^{\text{NS}}_{t-1}(\tau, z + \frac{\tau}{2}). \quad (6.2.3)$$

Thus for instance, the **Witten index** is given by

$$I = \text{tr}_n q^{L_0-c/24}(-1)^F = \hat{\text{ch}}^R_{1/2}(\tau, 1/2) = -q^{1/4} \hat{\text{ch}}^{\text{NS}}_0(\tau, 1+\tau) = \begin{cases} -2, & l = 0 \\ 1, & l = \frac{1}{2} \end{cases}$$

since $\vartheta_3\left(\frac{1+\tau}{2}\right) = 0$ and $\vartheta_1\left(\frac{1+\tau}{2}\right) = q^{-1/8}\vartheta_3$. For the massive characters, the Witten index vanishes: $\vartheta_3\left(\frac{1+\tau}{2}\right) = 0$ in (6.2.2).

### 6.2.2 The Orbits

The non-linear sigma models on $K3$ have three kinds of NS “orbits”: graviton, massless and massive orbits. Their construction will be detailed in the explicit computations below, sections 6.4, 6.5 and 6.6. For now, we only need to know that they can be spanned by massless and massive N=4 characters. The graviton orbit, for example, contains the massless character $\hat{\text{ch}}^{\text{NS}}_0$ and a sum of massive characters

$$\sum_{n \geq 1} c_n \text{ch}^\text{NS}_n = \left(\sum_{n \geq 1} c_n q^n\right) \text{ch}^\text{NS}_0 =: F_1(\tau) \text{ch}^\text{NS}_0.$$ 

Thus the graviton orbit has coordinates $(1, F_1)$ in the basis $\{\hat{\text{ch}}^{\text{NS}}_0, \text{ch}^\text{NS}_0\}$. Similarly for the other orbits. From the examples of the next sections, it will appear that the massless orbits have always coordinates $(1, F_i)$, and the massive orbits have coordinates $(0, F_j)$ (hence the name!). We use the subscripts $1, i, j$ for the different orbits: $1$ for the graviton orbit, $i = 2, \ldots, d$ for the massless orbits and $j = d+1, \ldots, d+d'$ for the massive orbits. Writing the N=4 NS$_i$ characters (for the three kinds of orbits) in the basis $\{\text{ch}^\text{NS}_{0,1/2}, \text{ch}^\text{NS}_0\}$ defines the functions $F_i(\tau)$:

$$\begin{align*}
\text{NS}_1(\tau, z) &= \text{ch}^\text{NS}_0(\tau, z) + F_1(\tau) \text{ch}^\text{NS}_0(\tau, z) \\
\text{NS}_i(\tau, z) &= \text{ch}_{1/2}^\text{NS}(\tau, z) + F_i(\tau) \text{ch}_0^\text{NS}(\tau, z) \\
\text{NS}_j(\tau, z) &= F_j(\tau) \text{ch}_0^\text{NS}(\tau, z)
\end{align*} \quad (6.2.4)$$
The set of functions \( F_i \) is determined by the particular Gepner model under study. Spectral flow generates again the Ramond counterparts, \( R_i \), and subsequent \((-1)^F\) insertion – denoted by a prime – yields the Witten index of the orbit (non-vanishing for massless characters only): \( R'_i = I_i = -2, R'_i = I_i = 1, R'_j = 0.\)

The action of the modular group transforms all these orbits into each other. For instance, the \( S \)-transformation defines a real matrix \( S_{ij} \):

\[
\text{NS}_i(\tau, z) = -\sum_j S_{ij} \text{NS}_j \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) e^{-2\pi i z^2/\tau}.
\]

Define \( D_i := S_{1,i}/S_{i,1} \), which are combinatorial factors of tensoring representations when using a Gepner model. For instance, in the \( 1^8 \) theory: \( D_i = (1, 20, 270, 30) \). Using \( D_i \), we form the modular invariant partition function for the \( K3 \) \( \sigma \)-model:

\[
Z(\tau, \bar{\tau}; z, \bar{z}) = \text{tr} q^{L_0 - \frac{1}{24}} q^{L_0 - \frac{1}{24}} y^{J_0} \bar{y}^{J_0} = \frac{1}{2} \sum_{i=1}^{d+d'} D_i (|\text{NS}_i|^2 + |\text{NS}_i'|^2 + |R_i|^2 + |R'_i|^2),
\]

where the prime represents \((-1)^F\) insertion. The last term evaluated at \( y = 1 \) is but the Witten index \( I \) and summing over it gives the Euler character:

\[
\chi = \sum_{i=1}^{d+d'} D_i I_i^2 = D_1 2^2 + D_2 + \ldots D_d = 4 + h^{1,1} = 24,
\]

as the sum of \( D_i \) over the massless orbit always adds up to the Hodge number \( h^{1,1} \) of the orbifold: 20 in our case of \( K3 \).

### 6.3 Topological invariants

In [EOTY-89,KKY-93], the authors studied the \( c=6 \) SCA of a sigma-model with \( K3 \) target space. The holonomy of the \( K3 \) manifold allows for two more \( SU(2) \) currents \( J^\pm \), i.e. conformal fields of weight 1 and \( U(1) \) charge \( J_0 = 2 J^3_0 = \pm 2 \), that generate the transformation of double spectral flow (i.e. NS→R→NS) and extend the \( \mathbb{N}=2 \) algebra to \( \mathbb{N}=4 \).

#### 6.3.1 \( K3 \) Elliptic Genera Revisited

Recall from section 3.4 that the elliptic genus of a \( (c, \bar{c}) = (6, 6) \) heterotic sigma model is, geometrically, a double sum whose coefficients are the indices of Dirac operators for certain vector bundles over \( K3 \):

\[
\Phi(\tau, z) = \sum_{n,r} c_{n,r} q^n y^r
\]

with \( c_{n,r} := \text{ind} \mathcal{D}_{E_{n,r}} = \int_{K3} \text{ch}(E_{n,r}) \text{td}(K3) \)
where the bundle $E_{n,r}$ is defined by

$$
\sum_{n,r} E_{n,r} q^n y^r := y^{-1} \bigotimes_{n \geq 1} \left( \bigwedge_{-q^{-1}} y T_{K3} \otimes \bigwedge_{-q^{-1}} \bar{T}_{K3} \otimes S_q^n T_{K3} \otimes S_q^n \bar{T}_{K3} \right),
$$

and $\bigwedge q E = \bigoplus_{k \geq 0} q^k \bigwedge E$, $S_q E = \bigoplus_{k \geq 0} q^k S^k E$. ($\bigwedge^k$ and $S^k$ denote the $k$'th exterior and symmetric products.)

The elliptic genus also has a topological expression, given by a trace over the left and right Ramond sectors with $(-1)^F$ insertion:

$$
\Phi(\tau, z) := \text{tr}_{R,R} (-1)^{F_L + F_R} q^{L_0 - 1/4} \bar{q}^{\bar{L}_0 - 1/4} y^0
$$

$$
= 24 \left( \frac{\vartheta_3(z)}{\vartheta_3} \right)^2 + 2 \frac{\vartheta_2^4 - \vartheta_4^4}{\eta^4} \left( \frac{\vartheta_4(z)}{\eta} \right)^2,
$$

where the second expression will be proved in the next subsection. Note that because we have no $\bar{y}^0$ for the right movers, the $(-1)^F_R$ insertion in the R-sector yields only a contribution from the zero modes ($q^0$-terms). Indeed, for any higher state, Susy ensures the existence of another state with opposite $(-1)^F$ eigenvalue. Thus the above expressions are independent of $\bar{q}$ and we could have dropped that variable from the definition. Note also that the fermion parity operator $(-1)^F = (-1)^{F_L + F_R} = (-1)^{F_L - F_R}$ is sometimes written $e^{i\pi(J_0 - \bar{J}_0)}$.

The $U(1)$ charge $J_0$ helps to distinguish between bosons and fermions, and its values are in $\mathbb{Z}$ for the NS sector and in $\mathbb{Z} + \frac{c}{6}$ for the R sector. Thus the difference between left- and right-moving $U(1)$ charge is always an integer for the NS-NS or R-R sectors.

At the special values of $z = \frac{1+i}{2}, \frac{r}{2}, \frac{1}{2}$ and 0, we obtain specific topological invariants [EOTY-89], using (C.0.11),(C.0.12) and dropping the extra $q^{-1/4}$ and $-q^{-1/4}$ in the first two cases:

**Dirac index**:

$$
\Phi^+_A := \text{tr}_{NS,R} (-1)^F_R q^{L_0 - 1/4} = 2\vartheta_3^2(\vartheta_2^4 - \vartheta_4^4)/\eta^6
$$

$$
\Phi^-_A := \text{tr}_{NS,R} (-1)^{F_L + F_R} q^{L_0 - 1/4} = -2\vartheta_2^2(\vartheta_2^4 + \vartheta_4^4)/\eta^6
$$

**Hirzebruch genus**: 

$$
\Phi_\sigma := \text{tr}_{NS,R} (-1)^F_R q^{L_0 - 1/4} = 2\vartheta_2^2(\vartheta_4^4 + \vartheta_2^4)/\eta^6
$$

**Euler character**: 

$$
\Phi_\chi := \text{tr}_{NS,R} (-1)^{F_L + F_R} q^{L_0 - 1/4} = 24
$$

Whence a shift $z \to z + \frac{r}{2}$ generates spectral flow R→NS, while $z \to z + \frac{1}{2}$ is responsible for an additional factor of $(-1)^{F_L}$. The elliptic genus evaluated at specific points thus yields the partition function for different spin structures; at $z = 0$, we obtain the Witten index – or the bosonic partition function if we have no spin structures.

We note that the above indices or genera are universal and do not depend on the $K3$ moduli. Since they hold for any complex structure, they are rightly called topological invariants.
6.3. TOPOLOGICAL INVARIANTS

6.3.2 Derivation by Orbits

We shall prove (6.3.1) by actually computing $\Phi^+_A$ with its $z$ dependence restored, \textit{i.e.} we consider the NS, R sector. This will allow us to work with the functions $F_i$ which we defined by the left-moving NS$_i$ orbits. Note that in the following, the $\tau$-dependence shall be understood and not always explicitly written. The prescription is to replace the trace by a sum over all orbits:

$$\Phi^+_A(\tau, z) := \text{tr}_{NS,R} q^{L_0-1/4} J_0(-1)^{F_R} q^{L_0-1/4} = \sum_{i=1}^{d+d'} D_i \text{NS}_i(\tau, z) \bar{R}'_i(\bar{r}, \bar{z} = 0)$$

This factorization of NS and R sectors will be confirmed by the concrete examples of the next sections.\textsuperscript{2} Note that in the right-moving sector, $\bar{R}'_i(\bar{r}, 0)$ is but the Witten index $I_1 = -2, I_1 = 1$ and $I_2 = 0$. Hence the trace consists of two parts only, one for the graviton orbit and one from the massless orbit. In (6.2.1), we can interpret the coefficient of $\vartheta_1(z)^2 / \vartheta_3$ as $-I_1$ for $i = 1, \ldots, d$ and similarly for the coefficient of $h_3$. Bearing this in mind, (6.2.4) gives us:

$$\sum_{i=1}^{d} D_i \text{NS}_i(z) \bar{R}'_i(0) = \sum_i D_i \left[ \text{ch}^{\text{NS}}_{0,1}(z) + F_i \text{ch}^{\text{NS}}_0(z) \right] I_i$$

$$= \left( -\sum_{i=1}^{d} D_i I_1^2 \right) \left( \vartheta_1(z) \right)^2 + \left( \frac{q^{-\frac{1}{2}}}{\eta} \left( D_1 I_1 + \sum_{i=1}^{d} D_i I_i F_i \right) + h_3 \sum_{i=1}^{d} D_i I_i^2 \right) \left( \frac{\vartheta_3(z)}{\eta} \right)^2$$

Since this is a topological invariant, it should be independent of the Gepner model at hand, \textit{i.e.} of the particular set of functions $F_i(\tau)$. That is, for different Gepner models we have different sets (of variable length) of orbits NS$_i$ and functions $F_i$, but the above sum yields always the same result. In sections 6.4 and 6.5 below, we show (for the $1^6$ and $2^4$ theories) how the large bracket yields $2(\vartheta_1^2 - \vartheta_3^2)/\eta^4$. Hence our Dirac index becomes:

$$\Phi^+_A(z) = -24 \left( \frac{\vartheta_1(z)}{\vartheta_3} \right)^2 + 2 \frac{\vartheta_2^4 - \vartheta_4^4}{\eta^4} \left( \frac{\vartheta_3(z)}{\eta} \right)^2$$

and the $z = 0$ value gives back the invariant of (6.3.2).

To arrive at the elliptic genus (6.3.1), we need to insert $(-1)^{F_L}$ and perform spectral flow for the left-movers. This corresponds to shifts $z \rightarrow z + \frac{1}{2}$ and $z \rightarrow z + \frac{i\tau}{2}$ respectively. The first of these operations yields $\Phi^-_A(z) = \sum_{i=1}^{d} D_i \text{NS}_i(z) \bar{R}'_i(0)$ and combination with the second yields $\Phi(z) = \sum_{i=1}^{d} D_i \text{R}_i(z) \bar{R}'_i(0)$ as in (6.3.1).

\textsuperscript{2}A thorough treatment of this factorization into tensor products of Hilbert spaces can be found in Katrin Wendland's PhD thesis [W-00]
6.3.3 Alternative derivation by orbifolds

The expression for the elliptic genus (6.3.1) can also be derived from orbifold models of the $K3$ surface, as was shown in [EOTY-89]. These models are formed by dividing the product of two complex tori $T \times T$ by the action of the symmetry group $\mathbb{Z}_n$:

$$z_1 \rightarrow z_1 e^{2\pi i/n} \quad \text{and} \quad z_2 \rightarrow z_2 e^{-2\pi i/n}. \quad (6.3.4)$$

Essentially four types occur, corresponding to $n = 2, 3, 4, 6$.

The partition function for these models consists of an untwisted piece and a twisted one. The untwisted piece is the fermionic contribution (in the NS sector, say) $|\vartheta_3(z)/\eta|$ times the bosonic lattice function $\Gamma_{2,2}(G, B)/|\eta|^4$.

The twisted piece consists of two complex fermions and two complex bosons, twisted by some power of the $\mathbb{Z}_n$ symmetry generator $e^{2\pi i/n}$, that is the $U(1)$ theta angle $z = 2\pi \theta$ will be shifted by $(s + r\tau)/n$. For the fermions (in the NS sector, say), we have again $\vartheta_3(z)/\eta$ (yet with twists in opposite direction, see (6.3.4)), while for the bosons we have $\eta/\vartheta_1$. Thus the twisted partition function is the sum

$$\sum_{r,s}' n_{r,s}|Z_{r,s}|^2, \quad Z_{r,s} := \frac{\vartheta_3(z + (s + r\tau)/n) \vartheta_3(z - (s + r\tau)/n)}{\vartheta_1((s + r\tau)/n)^2},$$

where the prime on the sum signifies omission of $r = s = 0$. The weights $n_{r,s}$ are defined by $n_{0,s} := (s \sin(\pi s/n))^4/n$ and $n_{r,s} := n_{s,n-r}$. Concretely, these weights all equal 8 for $n = 2$ and 3 for $n = 3$; while for $n = 4$ the three weights forming at the half-periods $(r, s = 0, 2)$ equal 4 and the remaining twelve weights equal 1. For $n = 6$, the three half-period weights equal 16/6, the eight third-period weights equal 9/6 while the remaining twenty-four weights equal 1/6. In all cases, the important observation is that the sum of the weights equals 24: $\sum_{r,s}' n_{r,s} = 24$.

By the Riemann addition formula (C.0.14), the $(r, s)$-block can be rewritten as

$$Z_{r,s}(z) = \left( \frac{\vartheta_1(z)}{\vartheta_3} \right)^2 + \left( \frac{\vartheta_3((s + r\tau)/n)}{\vartheta_1((s + r\tau)/n)} \right)^2 \left( \frac{\vartheta_3(z)}{\vartheta_3} \right)^2,$$

while its equivalent for the R-sector with $(-1)^F$ insertion is

$$q^{1/4} y Z_{r,s}(z + (1 + \tau)/2) = \left( \frac{\vartheta_3(z)}{\vartheta_3} \right)^2 - \left( \frac{\vartheta_3((s + r\tau)/n)}{\vartheta_1((s + r\tau)/n)} \right)^2 \left( \frac{\vartheta_1(z)}{\vartheta_3} \right)^2.$$
With these building blocks, we can now compute the elliptic genus:

\[ \Phi(\tau, z) = \text{tr}_{n,n} (-1)^F q^{L_0-1/4} y^{j_0} \bar{q}^{\bar{L}_0-1/4} \]

\[ = \sum_{r,s} n_{r,s} q^{1/4} y Z_{r,s}(z + (1 + \tau)/2) \bar{q}^{1/4} \bar{Z}_{r,s}((1 + \tau)/2) \]

\[ = \left( \sum_{r,s} n_{r,s} \left( \frac{\vartheta_3(z)}{\vartheta_3} \right)^2 - \left( \sum_{r,s} n_{r,s} \left( \frac{\vartheta_3((s + r\tau)/n)}{\vartheta_1((s + r\tau)/n)} \right)^2 \right) \left( \frac{\vartheta_1(z)}{\vartheta_3} \right)^2 \right) \]

\[ = 24 \left( \frac{\vartheta_3(z)}{\vartheta_3} \right)^2 + 2 \frac{\vartheta_2^4 - \vartheta_4^4}{\eta^4} \left( \frac{\vartheta_1(z)}{\eta} \right)^2, \]

where we have used (C.0.36) to transform

\[ \sum_{r,s} n_{r,s} \left( \frac{\vartheta_3((s + r\tau)/n)}{\vartheta_1((s + r\tau)/n)} \right)^2 = \left( \frac{\vartheta_3}{2\pi \eta^4} \right)^2 \sum_{r,s} n_{r,s} \left( \varphi((s + r\tau)/n) - \text{const} \right) \]

and this last sum equals \(-24\text{const}\), by repeated use of (C.0.37) for equal values of \(n_{r,s}\). The constant itself equals \(\frac{\pi^2}{3}(\vartheta_2^4 - \vartheta_4^4)\). So we do indeed recover (6.3.1).

### 6.4 Computations in \(1^6\) Theory

For clarity, we shall now detail the ideas developed at the beginning of this section, and show what we mean under "orbits" and functions \(F_i\) in the concrete example of the \(1^6\) theory. This Gepner model is based on the tensoring of six times the same \(k = 1\), N=2 SCFT. That is, the N=4 characters will be tensor products of six N=2 characters. So we first present those N=2 characters.

#### 6.4.1 General Considerations

In general, for values of the central charge between 0 and 3, unitary representations of N=2 superconformal algebras exist at discrete values of the central charge, namely at \(c = 3k/(k + 2)\). The highest weight states have conformal dimension and \(U(1)\) charge parametrised by two quantum numbers \(l, m\) (isospin and its third component) [RY-87]:

\[ h_{l,m} = \frac{l(l + 2) - m^2}{4(k + 2)} \quad Q_{l,m} = \frac{m}{k + 2} \]

where \(0 \leq l \leq k\), \(-l \leq m \leq l\), \(l \equiv m \text{ mod } 2\). The NS characters of these N=2 theories are linear combinations of \(su(2)\) theta functions:

\[ \text{ch}^{\text{NS}}_{l,m}(y,q) = \sum_{m' = -k+1}^k c_{l,m',m} \theta_{k+2}(m' - mk, k(k+2)) \left( \frac{\tau}{2}, \frac{z}{k + 2} \right) \]

\[ \theta_{m,k}(\tau, z) := \sum_{n \in \mathbb{Z} + m/2k} q^{kn^2} y^{kn}, \quad \theta_{m,k} = \theta_{m+2k, k} \]
For later purposes, note the behaviour under 'full' \((z \to z + \tau)\) spectral flow:

\[
\theta_m, k(k+2) \left(\frac{\tau}{2}, \frac{z}{k+2}\right) = q^{-\frac{k}{2(k+2)}y} q^{-\frac{k}{2(k+2)}} \theta_{m+2k, k(k+2)} \left(\frac{\tau}{2}, \frac{z}{k+2}\right).
\]

The coefficients \(c_{l,m}\) are the string functions of Kac and Peterson [KP-84] for \(l \equiv m \mod 2\); for the \(su(2)\) affine Lie algebra they have an alternative definition via the Weyl-Kac formula:

\[
\frac{\theta_{l+1,k+2} - \theta_{l-1,k+2}}{\theta_{1,2} - \theta_{-1,2}} = \sum_{m=-k+1}^{k} c_{l,m} \theta_{m,k}.
\]

Since the l.h.s. and r.h.s. have expansions with powers of \(y\) in \(\mathbb{Z} + l/2\) and \(\mathbb{Z} + m/2\) resp., we see that \(c_{l,m} = 0\) if \(l \neq m \mod 2\). Of course, for each level \(k\) we have different set of string functions. Note also the symmetries:

\(c_{l,m} = c_{l,-m} = c_{l,m+2k} = c_{k-l,k-m}\). For the case of the affine \(su(2)\) algebra \(A_1^{(1)}\), the string functions are merely proportional to Hecke 'indefinite' modular forms:

\[
c_{l,m} = \eta(\tau)^{-3} \sum_{-|x| < y \leq |x|} \text{sign}(x) q^{(k+2)x^2 - ky^2}
\]

where \(x, y\) are such that \((x, y)\) or \((1/2 - x, 1/2 + y)\) are \(\in \mathbb{Z}^2 + (\frac{l+1}{2(k+2)}, \frac{m}{2k})\).

For our present case of \(k = 1, c = 1\), the latter sum can be remarkably rewritten as

\[
\sum_{-|x| < y \leq |x|} \text{sign}(x) q^{3x^2 - y^2} = \sum_{j \geq 0} (-1)^{j+1} q^{(3(2j+1)^2 - (6j+1)^2)/24} \sum_{|l| \leq j/2} \eta(\tau)^2
\]

The last equality is another remarkable result of [KP-84]. Our string functions at level one thus become:

\[
c_{0,0} = c_{1,1} = c_{1,-1} = \frac{1}{\eta(\tau)}, \quad c_{0,1} = c_{1,0} = 0.
\]

### 6.4.2 Characters and Orbits

So we are in a position to write down the three minimal \(N=2\) characters, obtained for \(l = m = 0\), \(l = m = 1\) and \(l = -m = 1\):

\[
A := \text{ch}^{NS}_{0,0}(y, q) = \frac{1}{\eta} \theta_{0,3}(\frac{\tau}{2}, \frac{3}{3}) = \frac{1}{\eta} \sum \frac{q^{k}}{n^2} y^n = \frac{1}{\eta} \vartheta_3(z\lvert 3\tau)
\]

\[
B := \text{ch}^{NS}_{1,0}(y, q) = \frac{1}{\eta} \theta_{2,3}(\frac{\tau}{2}, \frac{3}{3}) = \frac{1}{\eta} \sum \frac{q^{(n+1)^2}}{n^2} y^{n+\frac{1}{2}} = \frac{1}{\eta} q^{1/6} y^{1/3} \vartheta_3(z + \tau\lvert 3\tau)
\]

\[
C := \text{ch}^{NS}_{1,-1}(y, q) = \frac{1}{\eta} \theta_{4,3}(\frac{\tau}{2}, \frac{3}{3}) = \frac{1}{\eta} \sum \frac{q^{(n+2)^2}}{n^2} y^{n+\frac{2}{3}} = \frac{1}{\eta} q^{2/3} y^{2/3} \vartheta_3(z + 2\tau\lvert 3\tau)
\]

(6.4.2)
Under spectral flow, the three $su(2)$ theta functions are shifted into one each other:

$$
\theta_{m,3}(\tau, z) \rightarrow z^{1/2} \theta_{m+1,3}(\tau, z) \rightarrow \tau q^{1/6} y^{-1/3} \theta_{m+2,3}(\tau, z),
$$

so that under 'full' spectral flow, the three characters are cyclically permuted:

$$
A \xrightarrow{z \rightarrow z + \tau} B \rightarrow C \rightarrow A,
$$

where we have omitted the incrementing factors of $q^{-1/6} y^{-1/3}$.

To build the various orbits of the $1^6$ theory, we consider all possible homogeneous polynomials of degree 6 in $A, B, C$, respecting the following two rules:

1. the orbit must be holomorphic, i.e. its Fourier expansion must have integer powers of $y$;

2. the orbit must be covariant under full spectral flow: $NS_i(z \rightarrow z + \tau) = q^{-1/2} y^{-2} NS_i(z)$.

Condition (1) excludes combinations like $A^5 B$, $A^4 B^2$ or $A^3 B^2 C$,... Condition (2) requires invariance of the orbit under cyclic permutation of $A, B, C$, and also guarantees it to be a theta function of characteristic $(0, 0; -4\pi i, 2\pi i)$ and degree 2. Thus each orbit can be spanned by $ch_{0,1}^{NS}$ and $ch_{0}^{NS}$, or alternatively by $\vartheta_1(z|\tau)^2$ and $\vartheta_3(z|\tau)^2$.

The four possible orbits respecting the above rules are:

$$
\begin{align*}
NS_1 &= A^6 + B^6 + C^6 \\
NS_2 &= A^3 B^3 + B^3 C^3 + C^3 A^3 \\
NS_3 &= A^2 B^2 C^2 \\
NS_4 &= A^4 BC + B^4 CA + C^4 AB.
\end{align*}
(6.4.3)
$$

We recall the definition of the combinatorial factors $D_i$ associated to a particular model: The above orbits can be checked to have the following modular behaviour [EOTY-89]:

$$
NS_i = -\sum_j S_{ij} NS_j \left(-\frac{1}{\tau}, \frac{z}{\tau}\right) e^{-2\pi i z^2/\tau}, \quad S_{ij} = \frac{1}{27} \begin{pmatrix}
3 & 60 & 270 & 90 \\
3 & -21 & 27 & 9 \\
1 & 2 & 9 & -6 \\
3 & 6 & -54 & 9
\end{pmatrix}
(6.4.4)
$$

Then the $D_i$ are defined by $D_i := S_{1,i}/S_{i,1}$, that is $(1, 20, 270, 30)$ in the present case of $1^6$ theory. Note that the first column $(S_{1,i})$ is just the number of summands in the orbit $NS_i$, while the first row $(S_{i,1})$ is $S_{i,1}$ times the number of permutations of the factors in any summand of $NS_i$. The same trick will allow a quick determination of the $D_i$ in the $2^4$ or $4^3$ theories.
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We note that at $y = -q^{-1/2}$, that is $z = \frac{1 + \tau}{2}$, the massive character vanishes and so does $B$, $\text{ch}(h = 0) = 0 = B$, while $A = e^{i\pi/3}C = q^{-1/24}$. Thus at $y = -q^{-1/2}$:

$$\begin{align*}
\text{NS}_1 &= 2q^{-1/4} = \text{ch}_0^{\text{NS}}(z = \frac{1 + \tau}{2}) \\
\text{NS}_2 &= -q^{-1/4} = \text{ch}_{\frac{1}{2}}^{\text{NS}}(z = \frac{1 + \tau}{2}) \\
\text{NS}_3 &= 0 \\
\text{NS}_4 &= 0,
\end{align*}$$

(6.4.5)

and we recognize that the first orbit is the graviton orbit, the second is the massless orbit (only one), while the third and fourth orbits are massive.

6.4.3 The functions $F_j$

We will now compute the functions $F_j$ for the massive orbits. For $F_3$ this is pretty easy, while $F_4$ is more involved. $F_1$ and $F_2$ do not seem to have appealing expressions. Let us start with $F_3$:

$$\begin{align*}
\text{NS}_3 &= A^2B^2C^2 = \frac{q^{2/3}}{\eta^6}(\vartheta_3(z|3\tau) \vartheta_3(z + \tau|3\tau) \vartheta_3(z - \tau|3\tau))^2 \\
&= \frac{q^{2/3}}{\eta^6} \left(\vartheta_3(z|\tau) \prod (1 - q^{3n})^3 \right)^2 \\
&= \frac{\eta(3\tau)^6}{\eta^5} \vartheta_3(z|\tau)^2 \\
&= F_3 \text{ch}_0^{\text{NS}} = F_3 q^{-1/8} \frac{\vartheta_3(z|\tau)^2}{\eta^3}
\end{align*}$$

from which we find that

$$F_3 = q^{1/8} \frac{\eta(3\tau)^6}{\eta^5}. \quad (6.4.6)$$

Similarly, for $F_4$ we have:

$$\begin{align*}
\text{NS}_4 &= ABC(A^3 + B^3 + C^3) \\
&= \frac{\eta(3\tau)^3}{\eta^4} \vartheta_3(z|\tau) \left[\vartheta_3(z|3\tau)^3 + q^{1/2} y \vartheta_3(z + \tau|3\tau)^3 + q^{2/3} y^{2/3} \vartheta_3(z + 2\tau|3\tau)^3\right] \\
&= F_4 \text{ch}(h = 0) = F_4 q^{-1/8} \frac{\vartheta_3(z|\tau)^2}{\eta^3}.
\end{align*}$$

Thus we see that the large bracket with the sum of cubes of theta functions must be proportional to $\vartheta_3(z|\tau)$. This is indeed the content of lemma 6.4.11 below, and we then obtain for $F_4$:

$$F_4 = q^{1/8} \frac{\eta(3\tau)^3}{\eta^4} a(\tau), \quad (6.4.7)$$
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where $a(\tau)$ is a function already studied in [BBG-94]:

$$a(\tau) := \frac{1}{\eta} \sum_z (-1)^n (6n + 1) \frac{q^{(6n+1)^2}/24}{q^{k^2+kl+l^2}}.$$

6.4.4 Dirac Index

Although we could not find interesting expressions for $F_1$ and $F_2$, we shall nonetheless derive (6.3.3), that is we shall show:

**Proposition 6.4.8.**

$$\sum_{i=1}^{d} D_i N S_i I_i = -2 NS_1 + 20 NS_2 = -24 \frac{\vartheta_1(z|\tau)^2}{\vartheta_3^2} + 2(\vartheta_2^4 - \vartheta_4^4) \frac{\vartheta_3(z|\tau)^2}{\eta^6}. \quad (6.4.9)$$

**Proof.** Because the Dirac index (l.h.s.) is spanned by the massless and massive characters $c_0$ and $c$, or equivalently by $\vartheta_1(z|\tau)^2$ and $\vartheta_3(z|\tau)^2$, we only need to recover the constants multiplying these two basis vectors. Due to (6.4.5) and the vanishing of $\vartheta_3(z|\tau)$ at $z = \frac{1+\tau}{2}$, we see that $\vartheta_1(z|\tau)^2$ is correctly multiplied by $-24/\vartheta_3^2$. To check the constant in front of $\vartheta_3(z|\tau)^2$ would only require setting $z = 0$, where $\vartheta_1(z|\tau)$ vanishes. The l.h.s. then would give $-2(A^6 + 2B^6) + 20(B^3(2A^3 + B^3))$ because $B = C$ at $z = 0$. However, we have not succeeded in showing directly that this equals $2(\vartheta_2^4 - \vartheta_4^4) \vartheta_3^2/\eta^6$. Presumably, this is an interesting corollary of the theorem.

Rather, to find the constants multiplying the two basis vectors, we shall differentiate both sides twice and set $z = \frac{1+\tau}{2}$. This last evaluation has the merit of making the character $C$ vanish, and giving also $A = -B = \frac{1}{\eta} \vartheta_3(\frac{1+\tau}{2}|3\tau)$. For $NS_1$, we have:

\[
\partial_2^2 |_{z=\frac{1+\tau}{2}} NS_1 = \partial_2^2 |_{z=\frac{1+\tau}{2}} (A^6 + B^6 + C^6) = \partial_2^2 |_{z=\frac{1+\tau}{2}} (A^6 + B^6) = 6A^4[A'' + 5A'A] + 6B^4[B'' + 5B'B].
\]

Recalling that $\vartheta_3$ and $\vartheta_3''$ are even functions of $z$, while $\vartheta_3'$ is odd, and that they are all periodic under $z \rightarrow z+1$, we note the following: $\vartheta_3(\frac{1+\tau}{2}|3\tau) = \vartheta_3(\frac{1+\tau}{2}|3\tau)$ and similarly for $\vartheta_3'$, but with an additional minus sign for $\vartheta_3''$. Thus for instance, we have at $z = \frac{1+\tau}{2}$:

\[
A = \vartheta_3(z|3\tau),
\]

\[
A |_{z=\frac{1+\tau}{2}} = \vartheta_3(\frac{1+\tau}{2}|3\tau),
\]

\[
A' |_{z=\frac{1+\tau}{2}} = -\vartheta_3'(\frac{1+\tau}{2}|3\tau),
\]

\[
A'' |_{z=\frac{1+\tau}{2}} = \vartheta_3''(\frac{1+\tau}{2}|3\tau).
\]

\[
B = q^{1/6} y^{1/3} \vartheta_3(z + \tau|3\tau),
\]

\[
B |_{z=\frac{1+\tau}{2}} = -\vartheta_3(\frac{1+\tau}{2}|3\tau),
\]

\[
B' |_{z=\frac{1+\tau}{2}} = [-\frac{2^3}{3} \vartheta_3 - \vartheta_3'(\frac{1+\tau}{2}|3\tau),
\]

\[
B'' |_{z=\frac{1+\tau}{2}} = [-\frac{4^3}{9} \vartheta_3 + \frac{4^3}{3} \vartheta_3' + \vartheta_3''(\frac{1+\tau}{2}|3\tau)]
\]
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In particular: $A'B + AB' = -\frac{2\pi i}{3} \vartheta_3$. Bearing this in mind, we obtain:

$$\vartheta_3''(\frac{1+i\tau}{2} | 3\tau)^4 \left[ 12 \vartheta_3'' \vartheta_3 + 60 \vartheta_3^{'2} + 48\pi i \vartheta_3'' \vartheta_3 - 16\pi^2 \vartheta_3^2 \right] (\frac{1+i\tau}{2} | 3\tau).$$

Similarly, for $\vartheta_3$, we have:

$$\vartheta_3''(\frac{1+i\tau}{2} | 3\tau)^4 \left[ -6 \vartheta_3 \vartheta_3'' + 6 \vartheta_3^{'2} + 4\pi^2 \vartheta_3^2 \right] (\frac{1+i\tau}{2} | 3\tau).$$

Thus the l.h.s. altogether yields:

$$\vartheta_3''(\frac{1+i\tau}{2} | 3\tau)(-2 \vartheta_3 + 20 \vartheta_3) = -\frac{4}{\vartheta^3} \left[ \vartheta_3^3 (36 \vartheta_3'' + 24\pi i \vartheta_3^{'2} - 4\pi^2 \vartheta_3) - 24\pi^2 \vartheta_3 \right] (\frac{1+i\tau}{2} | 3\tau)$$

$$= 16\pi^2 q^{-1/4} (6 + E_2),$$

where we have noted that $\vartheta_3(\frac{1+i\tau}{2} | 3\tau) = q^{-1/24} \eta$ and that the curved bracket is proportional to the second Eisenstein series $E_2 = \frac{1}{i\tau} \partial_{\tau} \log \eta$:

$$q^{1/24} (36 \vartheta_3'' + 24\pi i \vartheta_3^{'2} - 4\pi^2 \vartheta_3) = -4\pi^2 \sum_Z (-1)^n (6n + 1)^2 q^{(6n+1)^2/24}$$

$$= -4\pi^2 \frac{24}{2\pi i} \partial_{\tau} \sum_Z (-1)^n q^{(6n+1)^2/24}$$

$$= -4\pi^2 \frac{24}{2\pi i} \partial_{\tau} \eta$$

$$= -4\pi^2 \eta \ E_2,$$

(6.4.10)

by virtue of (C.0.5).

We now turn to the r.h.s. of (6.4.9) and shall differentiate twice. To this effect, we remind a few useful facts:

$$\vartheta_3'(z + \frac{1+i\tau}{2} | \tau) = \partial_{z} q^{-1/8} y^{1/2} \vartheta_3(z)$$

$$i\pi \vartheta_3(z + \frac{1-i\tau}{2} | \tau) + \vartheta_3^{-1/8} y^{1/2} \vartheta_3'(z | \tau) \left[ z \equiv 0 \right] - i\pi q^{-1/8} \vartheta_3$$

$$\vartheta_3'(z + \frac{1-i\tau}{2} | \tau) = \partial_{z} - i\pi q^{-1/8} y^{1/2} \vartheta_3(z)$$

$$i\pi \vartheta_3(z + \frac{1-i\tau}{2} | \tau) - i\pi q^{-1/8} y^{1/2} \vartheta_3'(z | \tau) \left[ z \equiv 0 \right] - 2\pi i q^{-1/8} \eta^3,$$

where we used (C.0.6). Similarly, we find:

$$\vartheta_3''(\frac{1-i\tau}{2} | \tau) = -\pi^2 q^{-1/8} \vartheta_3 + q^{-1/8} \vartheta_3'(0 | \tau)$$

$$\vartheta_3''(z + \frac{1-i\tau}{2} | \tau) = 4\pi^2 q^{-1/8} \eta^3.$$
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Taking (C.0.28) into account, we have overall for the r.h.s.:

$$\vartheta_2^2|_{z=\frac{1}{2}\tau} \left[ -24 \vartheta_1(z|\tau)^2 \vartheta_3^2 + 2(\vartheta_2^4 - \vartheta_4^4) \frac{\vartheta_3(z|\tau)^2}{\eta^6} \right] = 16\pi^2 q^{-1/4} (6 + E_2),$$

which was just the l.h.s..

6.4.5 Lemmas and Arithmetic Results

Lemma 6.4.11.

$$\vartheta_3(z|3\tau)^3 + q^{1/2} y \vartheta_3(z + \tau|3\tau)^3 + q^2 y^2 \vartheta_3(z + 2\tau|3\tau)^3 = a(\tau) \vartheta_3(z|\tau),$$

where $a(\tau) := \frac{1}{\eta} \sum_{n}(\tau n + 1) q^{6n+1}/24 = \sum_{k,l\in\mathbb{Z}} q^{k^2+kl+l^2}$.

Proof. That the r.h.s. is proportional to $\vartheta_3(z|3\tau)$ follows from the second proof that we shall give. To find the constant $a(\tau)$, we differentiate both sides wrt $z$ and set $z = \frac{1}{2}\tau$:

$$3 \vartheta_3(\frac{1}{2}+\tau|3\tau)^2 \left[ -2 \vartheta_3^2 - \frac{2\pi i}{3} \vartheta_3(\frac{1}{2}+\tau|3\tau) \right] = a(\tau) (-2\pi i q^{-1/8} \eta^3)$$

Note that $\vartheta_3(\frac{1}{2}+\tau|3\tau) = q^{-1/24} \eta$ and so we find

$$a(\tau) = \frac{1}{\eta} \sum_{n} (-1)^n (6n + 1) q^{(6n+1)^2/24}.$$  

For the second expression for $a$, we offer an alternative proof (with warm thanks to Sander Zwegers for this marvelous trick):

$$\text{l.h.s.} = \left( \sum_{Z} q^{3n^2} y^n \right)^3 + \left( \sum_{Z+\frac{1}{3}} q^{3n^2} y^n \right)^3 + \left( \sum_{Z+\frac{2}{3}} q^{3n^2} y^n \right)^3 = \sum_{n} q^{3(n_1^2+n_2^2+n_3^2)} y(n_1+n_2+n_3)$$

where $n = (n_1, n_2, n_3)$ on the r.h.s. sweeps through the set $S := \mathbb{Z}^3 \cup (\mathbb{Z} + \frac{1}{3})^3 \cup (\mathbb{Z} + \frac{2}{3})^3$. This set lies in 1 to 1 correspondence with all $k \in \mathbb{Z}^3$ via the following smart substitution:

$$n_1 = (k_1 + k_2 - k_3)/3 \quad k_1 = n_1 + n_2 + n_3$$
$$n_2 = (k_1 - 2k_2 - k_3)/3 \quad k_2 = n_1 - n_2$$
$$n_3 = (k_1 + k_2 + 2k_3)/3 \quad k_3 = -n_1 + n_3$$

Note that in the first definitions, all right hand sides are equal mod 1, which guarantees that all $n_i$ are in the same component of the set $S$; whence the 1-1 correspondence. Moreover:

$$n_1^2 + n_2^2 + n_3^2 = \frac{1}{3} k_1^2 + \frac{2}{3} (k_2^2 + k_3^2 + k_2k_3), \quad n_1 + n_2 + n_3 = k_1$$

Hence

$$\text{l.h.s.} = \left( \sum_{k_2,k_3\in\mathbb{Z}} q^{k_2^2+k_2k_3+k_3^2} \right) \sum_{Z} q^{k_1^2} y^{k_1} = \text{r.h.s.}$$
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**Corollary 6.4.12.**

For $z = 0$:
$$\vartheta_3(0|3\tau)^3 + 2q^{1/2} \vartheta_3(\tau|3\tau)^3 = a(\tau) \vartheta_3$$

For $z = 1/2$:
$$\vartheta_3(0|3\tau)^3 - 2q^{1/2} \vartheta_3(\tau|3\tau)^3 = a(\tau) \vartheta_3$$

For $z = 3\tau/2$:
$$\vartheta_3(0|3\tau)^3 + 2q^{1/4} \vartheta_3(\tau|3\tau)^3 = a(\tau) \vartheta_2$$

For the sake of instruction, we give a third proof of the above lemma, after reformulating it with a different constant of proportionality:

**Lemma 6.4.13.**

$$\vartheta_3(z|\tau)^3 + q^{1/6} y \vartheta_3(z + \frac{7}{3}|\tau)^3 + q^{2/3} y^2 \vartheta_3(z + \frac{2\tau}{3}|\tau)^3 = \left(6\frac{\eta(3\tau)^3}{\eta} + a(\tau)\right) \vartheta_3(z|\frac{\tau}{3})$$

where $a(\tau) = \frac{1}{\eta} \sum_{\omega}(\omega \tau + 1)(6n + 1) q^{(6n+1)^2/24}$

**Proof.** The advantage of having divided $\tau$ by 3 is that now all three terms on l.h.s. and the r.h.s. are theta functions of degree 3 and characteristic $(0,0; -6\tau i, 3\tau i\tau)$. The space of such functions is 3 dimensional and can be spanned by

$$\left\{\vartheta_3(z|\frac{\tau}{3}), \vartheta_3(z + \frac{7}{9}|\frac{\tau}{3}), \vartheta_3(z + \frac{2\tau}{3}|\frac{\tau}{3})\right\},$$

or by

$$\left\{\vartheta_3(3z|3\tau), y \vartheta_3(3z + \tau|3\tau), y^2 \vartheta_3(3z + 2\tau|3\tau)\right\},$$

etc.

Replacing $\tau \rightarrow 3\tau$, the l.h.s. as a whole is still a theta function of degree 1 and characteristic $(0,0; -2\pi i, \pi i \tau)$, hence must be proportional to $\vartheta_3(z|\tau)$. That is, all we have to do is to compute the constant of proportionality. To this end, we set $z = 0$ in the lemma and prove:

$$\vartheta_3(0|\tau)^3 + 2q^{1/6} \vartheta_3(\tau/3|\tau)^3 = \left(6\frac{\eta(3\tau)^3}{\eta} + a(\tau)\right) \vartheta_3(0|\frac{\tau}{3})$$

We quote from [FK-01], p. 273, a property describing how the cubes of theta functions can be spanned by basis vectors:

$$\vartheta' := e^{\frac{i\pi}{6}} \eta \vartheta_3(z + \frac{1+i\tau}{3}|\tau)^3 = -\pi e^{\frac{i\pi}{12}} q^{\frac{5}{6}} y^{-\frac{1}{2}} \eta(3\tau)^3 \left[z^{1/2} \vartheta_3(3z + \frac{1+i\tau}{3}|3\tau) - z^{-1/2} \vartheta_3(-3z + \frac{1+i\tau}{3}|3\tau)\right] + \vartheta' \vartheta_3(3z + 3\frac{1+i\tau}{3}|3\tau),$$

where

$$\vartheta' := \vartheta'([1/3])(0|3\tau) := \vartheta_3(0\tau) := e^{2\pi i/3} q^{1/6} y^{-1/3} \vartheta_3(z - \tau|3\tau),$$

such that

$$-\frac{3i}{\pi} e^{-i\pi/6} \vartheta' \frac{\eta}{\eta} = \frac{1}{\eta} \sum_{\omega}(\omega \tau + 1)(6n + 1) q^{(6n+1)^2/24} = a(\tau).$$

Special cases of this property are:

at $z = -\frac{1+i\tau}{2}$:
$$\vartheta_3(0|\tau)^3 = 6q^{\frac{3}{2}} \eta(3\tau)^3 \vartheta_3(\tau|3\tau) + a(\tau) \vartheta_3(0|3\tau)$$

at $z = \frac{-1}{2} + \frac{\tau}{6}$:
$$\vartheta_3(\frac{\tau}{3}|\tau)^3 = 3\frac{\eta(3\tau)^3}{\eta} \left[q^{-\frac{1}{6}} \vartheta_3(0|3\tau) + \vartheta_3(\tau|3\tau)\right] + a(\tau) \vartheta_3(\tau|3\tau),$$
6.4. **COMPUTATIONS IN $1^6$ THEORY**

so that

$$\vartheta_3(0|\tau)^3 + 2q^{1/6} \vartheta_3(\tau/3|\tau)^3 = \left(6\frac{\eta(3\tau)^3}{\eta} + a(\tau)\right) \left[\vartheta_3(0|3\tau) + 2q^{1/6} \vartheta_3(\tau|3\tau)\right]$$

Use lemma 6.6.4 to rewrite the square bracket as $\vartheta_3(0|\tau/3)$. □

Combining lemma 6.4.11 and 6.4.13, we arrive at an interesting observation, already noticed in [BBG-94]:

**Corollary 6.4.14.**

$$a(\tau/3) = 6\frac{\eta(3\tau)^3}{\eta} + a(\tau).$$

For completeness, we also observe that $a(\tau)$ can be written as the difference of two Lambert series (i.e. $\sum a_2q^n$):

$$a(\tau) = \frac{q^{2\delta}}{\eta} \vartheta_y y |z| \sum_{y} (-1)^n y^{6n+1} q^{n(3n+1)/2}$$

$$= \frac{q^{2\delta}}{\eta} \vartheta_y y \vartheta_3(6z + \frac{1+\tau}{2}|3\tau)$$

$$= \frac{q^{2\delta}}{\eta} \vartheta_y y \prod (1 - q^{3n})(1 - y^{6n} q^{3n-1})(1 - y^{-6n} q^{3n-2})$$

$$= 1 + 6 \sum_{n \geq 1} \left(\frac{-q^{3n-1}}{1 - q^{3n-1}} + \frac{q^{3n-2}}{1 - q^{3n-2}}\right)$$

$$= 1 + 6 \sum q^n \left(\sum_{d|n \not\equiv 1(2)} 1 - \sum_{d|n \equiv 2(3)} 1\right)$$

$$= 1 + 6 \sum \delta_{3,1}(n) q^n,$$

where $\delta_{k,l}(n)$ is the number of divisors of $n$ which are $\frac{k-l}{2}$ mod $k$ minus those which are $\frac{k+l}{2}$ mod $k$. For example, a well-known result of Jacobi states that the number of integer solutions to $x^2 + y^2 = n$ is $4\delta_{4,2}(n)$.

Many more beautiful properties about $a(\tau)$ are found in [BBG-94], such as

$$a(\tau) = \vartheta_3(0|2\tau) \vartheta_3(0|6\tau) + \vartheta_2(0|2\tau) \vartheta_2(0|6\tau).$$

We give a last property, of our own, relating to $a(\tau)^2$:

**Proposition 6.4.15.**

$$2a(\tau)^2 = 3E_2(3\tau) - E_2.$$
Proof. We apply the previous trick – of differentiating a Jacobi product – to the sum already encountered in (6.4.10):

\[ E_2 = \frac{1}{\eta} \sum_z (-1)^n (6n + 1)^2 q^{(6n+1)^2/24} \]

\[ = \frac{q^{1/24}}{\eta} \frac{1}{4\pi^2} \partial_z^2 \rho_0 \sum (-1)^n y^{6n+1} q^{(3n+1)/2} \]

\[ = \frac{q^{1/24}}{\eta} \frac{1}{4\pi^2} \partial_z^2 \rho_0 y \vartheta_3(6z + \frac{1 + \tau}{2} | 3\tau) \]

\[ = \frac{q^{1/24}}{\eta} \frac{1}{4\pi^2} \partial_z^2 \rho_0 y \prod (1 - q^{3n})(1 - y^{6n}q^{3n-1})(1 - y^{-6n}q^{3n-2}) \]

Abbreviating the last product by \( \Pi \), we have that \( q^{3\frac{1}{2}} \Pi_{|0} = \eta, \ \Pi' = \Pi \Sigma \) and \( \Pi'' = \Pi(\Sigma^2 + \Sigma') \), where \( \Sigma := 12\pi i \sum_{n \geq 1} \left( \frac{-y^{2n}q^{3n-1}}{1 - y^{3n-1}} + \frac{y^{-6}q^{3n-2}}{1 - y^{-6}q^{3n-2}} \right) \). In this notation, we also have \( \Sigma_{|0} = 2\pi i (a(\tau) - 1) \). Thus:

\[ E_2 = \left[ 1 + \frac{1}{i\pi} \Sigma - \frac{1}{4\pi^2}(\Sigma^2 + \Sigma') \right]_{|0} \]

\[ = 1 + 2(a - 1) + (a - 1)^2 - 36 \sum_{n \geq 1} \left( \frac{q^{3n-1}}{(1 - q^{3n-1})^2} + \frac{q^{3n-2}}{(1 - q^{3n-2})^2} \right) \]

\[ = a^2 + \frac{3}{2} [E_2 - E_2(3\tau)] . \]

\[ \square \]

Since the sums \( \frac{1}{\eta} \sum_z (-1)^n (6n + 1)^k q^{(6n+1)^2/24} \) yield enticing expressions for powers \( k = 1, 2 \) \((a(\tau), E_2 \text{ resp.})\), it is natural to wonder whether this extends to higher power. We have not found any alternative expression for the case \( k = 3 \), but can nonetheless relate it to \( a(\tau) \) and \( E_2 \). Again, we mimic the trick of the previous proposition:

\[ \frac{1}{\eta} \sum_z (-1)^n (6n + 1)^3 q^{(6n+1)^2/24} = a^3 + \frac{3a}{(2\pi i)^2} \Sigma'_{|0} + \frac{1}{(2\pi i)^3} \Sigma''_{|0} \]

\[ = a^3 + 3a \frac{3}{2} (E_2 - E_2(3\tau)) + \frac{1}{(2\pi i)^3} \Sigma''_{|0} , \]

with

\[ \Sigma''_{|0} = (2\pi i)^3 6^3 \sum_{n \geq 1} \left( \frac{-q^{3n-1}(1 + q^{3n-1})}{(1 - q^{3n-1})^3} + \frac{q^{3n-2}(1 + q^{3n-2})}{(1 - q^{3n-2})^3} \right) \]

\[ = (2\pi i)^3 6^3 \sum_{t \geq 1} t^2 q^t (1 - q^t) \]

\[ \frac{1}{1 - q^t} \]
6.5. COMPUTATIONS IN $2^4$ THEORY

The same recursion for the case $k = 4$ is even more involved and is not worth writing in full, due to the complicated nature of $\Sigma^{''}$:

$$\frac{1}{\eta} \sum_z (-1)^n (6n + 1)^4 q^{(6n+1)^2/24} = a^4 + \frac{6a^2}{(2\pi i)^2} \Sigma' |_{0} + \frac{4a}{(2\pi i)^3} \Sigma'' |_{0} + \frac{3}{(2\pi i)^4} \Sigma''' |_{0} + \frac{1}{(2\pi i)^4} \Sigma^{'''} |_{0}$$

$$= \frac{1}{\eta} \left( \frac{24}{2\pi i} \partial_{\tau} \right)^{2} \eta = 3E_{2}^{2} - 2E_{4}.$$

The last line is obtained using the covariant derivative for modular forms; it shows that $\partial_{\tau}^{k/2} \eta$ (for $k$ even) can be expressed as a polynomial in $E_{2}, E_{4}, E_{6}$. For instance, for $k = 6$ this is

$$\frac{1}{\eta} \sum_z (-1)^n (6n + 1)^6 q^{(6n+1)^2/24} = 16E_{6}^{2} - 30E_{2}E_{4} + 15E_{6}^{3}.$$

6.5 Computations in $2^4$ Theory

We mimic here the approach of the $1^6$ theory, as detailed in the previous section. This Gepner model is obtained by tensoring 4 times the $k = 2, N=2$ theory. Although we have more orbits and more cases to study, the mathematics are easier, due to the simpler properties enjoyed by theta functions with $\tau$ divided by 4 (instead of divided by 3 as in the $1^6$ theory).

6.5.1 Characters and Orbits:

This time we have six minimal $N=2$ characters, obtained for $l = 0$ ($m = 0$), $l = 1$ ($m = \pm 1$) and $l = 2$ ($m = 0, \pm 2$). The string functions at level 2, due to their symmetries, number only three:

$$c_{00} = c_{22}, \quad c_{20} = c_{02} \quad c_{1, -1} = c_{11}.$$ 

In the following characters, we use the shorthand $\theta_{m}$ for the $su(2)$ theta functions $\theta_{m,s}(\frac{\tau}{2}, \frac{\tau}{4})$:

$$A := \text{ch}_{0,0}^{N}(y, q) = c_{00} \theta_{0} + c_{02} \theta_{8} = \frac{1}{2\eta} \sqrt{\frac{\vartheta_{2}}{2\eta}} \vartheta_{3}(z|2\tau) + \frac{1}{2\eta} \sqrt{\frac{\vartheta_{4}}{2\eta}} \vartheta_{4}(z|2\tau)$$

$$B := \text{ch}_{2,0}^{N}(y, q) = c_{02} \theta_{0} + c_{00} \theta_{4} = \frac{1}{2\eta} \sqrt{\frac{\vartheta_{2}}{2\eta}} \vartheta_{2}(z|2\tau) + \frac{1}{2\eta} \sqrt{\frac{\vartheta_{4}}{2\eta}} i \vartheta_{1}(z|2\tau)$$

$$C := \text{ch}_{2,0}^{N}(y, q) = c_{02} \theta_{0} + c_{00} \theta_{8} = \frac{1}{2\eta} \sqrt{\frac{\vartheta_{2}}{2\eta}} \vartheta_{3}(z|2\tau) - \frac{1}{2\eta} \sqrt{\frac{\vartheta_{4}}{2\eta}} i \vartheta_{4}(z|2\tau)$$

$$D := \text{ch}_{-2,-2}^{N}(y, q) = c_{02} \theta_{4} + c_{00} \theta_{0} = \frac{1}{2\eta} \sqrt{\frac{\vartheta_{2}}{2\eta}} \vartheta_{2}(z|2\tau) - \frac{1}{2\eta} \sqrt{\frac{\vartheta_{4}}{2\eta}} i \vartheta_{1}(z|2\tau)$$

$$E := \text{ch}_{1,-1}^{N}(y, q) = c_{11} (\theta_{-2} + \theta_{6}) = \frac{\eta(2\tau)}{\eta^2} q^{16} y^{-\frac{1}{4}} \vartheta_{3}(z - \frac{\tau}{2}|2\tau)$$

$$F := \text{ch}_{1,1}^{N}(y, q) = c_{11} (\theta_{-6} + \theta_{2}) = \frac{\eta(2\tau)}{\eta^2} q^{16} y^{\frac{1}{4}} \vartheta_{3}(z + \frac{\tau}{2}|2\tau)$$

(6.5.1)
CHAPTER 6. N=4 CHARACTERS AND ELLIPTIC GENERA

The \( su(2) \) theta functions are related to the standard (or "Ur-") Jacobi theta function via

\[
\theta_{m,8}(\frac{\tau}{2}, \frac{z}{4}) = \sum_{n \in \mathbb{Z}} q^{(2n+\frac{m}{8})^2} y^{2n+\frac{m}{8}} = q^{\frac{m}{8}} y^\frac{m}{8} \theta_3(2z + \frac{m\tau}{2}|8\tau),
\]
\[
\theta_m + \theta_{m+8} = q^{\frac{m}{8}} y^\frac{m}{8} \theta_3(z + \frac{m\tau}{4}|2\tau).
\]

The rightmost column of (6.5.1) is obtained by rewriting \( ac + bd \) as \( \frac{1}{2} (a+b)(c+d) + \frac{1}{2} (a-b)(c-d) \) and using the explicit expression for the string functions from the next subsection.

Under spectral flow, these eight \( su(2) \) theta functions are shifted into one each other:

\[
\theta_{m,8}(\frac{\tau}{2}, \frac{z}{4}) \frac{z \to z + \frac{z}{2}}{q^{-1/6} y^{-1/4}} \theta_{m+2,8}(\frac{\tau}{2}, \frac{z}{4}) \frac{z \to z + \frac{z}{2}}{q^{-1/4} y^{-1/2}} \theta_{m+4,8}(\frac{\tau}{2}, \frac{z}{4}),
\]

so that under 'full' spectral flow, the six characters split into two groups which are cyclicly permuted:

\[
A^z \to B \to C \to D \to A, \quad E \to F \to E,
\]

where we have omitted the incrementing factors of \( q^{-1/4} y^{-1/2} \), etc.

To build the various orbits of the \( 2^4 \) theory, we consider all possible homogeneous polynomials of degree 4 in \( A, B, C, D, E, F \), respecting the following two rules:

1. the orbit must be holomorphic, i.e. its Fourier expansion must have integer powers of \( y \);
2. the orbit must be covariant under full spectral flow: \( \text{NS}_i(z \to z + \tau) = q^{-1} y^{-2} \text{NS}_i(z) \).

Note that \( A, C \) have integer \( y \)-expansion, \( B, D \) half-integer, and \( E, F \) have powers of \( y \) in \( \mathbb{Z} \pm \frac{1}{4} \) resp. Thus, condition (1) excludes combinations like \( A^3 B, A^3 E, A^2 B C, A^2 E \), etc. Condition (2) requires invariance of the orbit under cyclic permutation of \( A, B, C, D \) and \( E, F \) separately, and also guarantees it to be a theta function of characteristic \( (0,0;-4\pi i,2\pi i) \) and degree 2. Thus each orbit can be spanned by \( \text{ch}_0(l = 0, \frac{1}{2}) \) and \( \text{ch}(h = 0) \), or alternatively by \( \vartheta_1(z|\tau)^2 \) and \( \vartheta_3(z|\tau)^2 \).

The twelve possible orbits respecting the above rules are:

| \( \text{NS}_1 \) | \( = A^4 + B^4 + C^4 + D^4 \) | \( \text{NS}_7 \) | \( = AB^2 C + BC^2 D + CD^2 A + DA^2 B \) |
| \( \text{NS}_2 \) | \( = E^4 + F^4 \) | \( \text{NS}_8 \) | \( = ABCD \) |
| \( \text{NS}_3 \) | \( = A^2 B^2 + B^2 C^2 + C^2 D^2 + D^2 A^2 \) | \( \text{NS}_9 \) | \( = ABE^2 + BCF^2 + CDE^2 + DAF^2 \) |
| \( \text{NS}_4 \) | \( = ABF^2 + BCE^2 + CDF^2 + ADE^2 \) | \( \text{NS}_{10} \) | \( = (A^2 + B^2 + C^2 + D^2)EF \) |
| \( \text{NS}_5 \) | \( = B^2 D^2 + A^2 C^2 \) | \( \text{NS}_{11} \) | \( = E^2 F^2 \) |
| \( \text{NS}_6 \) | \( = AC^3 + BD^3 + CA^3 + BD^3 \) | \( \text{NS}_{12} \) | \( = (AC + BD)EF \) |

The combinatorial factors \( D_i \), defined after (6.4.4), associated to these orbits are \( (1, 2, 6, 12, 12, 4, 12, 96, 12, 12, 24, 48) \).
Due to the following relations among the characters,

\begin{align*}
AC &= BD \\
AB + CD &= F^2 \\
AD + BC &= E^2,
\end{align*}

(proved in lemma 6.5.7 with \( \tau \) replaced by \( \frac{1}{2} \)), we find that several orbits coincide:

\begin{align*}
NS_2 &= NS_4 \\
NS_5 &= 2 NS_8 \\
NS_6 &= NS_7 = \frac{1}{2} NS_9 = NS_{11}.
\end{align*}

### 6.5.2 String Functions:

Some explicit expressions for the string functions at level 2 are found in [KP-84], p. 220:

\[
c_{11} = \frac{\eta(2\tau)}{\eta^2} = \frac{1}{\eta^2} \sqrt{\frac{\vartheta_2}{2\eta}}, \quad c_{00} - c_{02} = \frac{\eta(\tau/2)}{\eta^2} = \frac{1}{\eta^2} \sqrt{\frac{\vartheta_4}{\eta}}
\]

The authors also give the complicated modular properties of the string functions. The latter expression, \( c_{00} - c_{02} \), upon shifting \( \tau \to \tau + 1 \), yields \( e^{-i\pi/8} (c_{00} + c_{02}) \).

Similarly, shifting \( \frac{1}{\eta} \sqrt{\frac{\vartheta_4}{\eta}} \) by \( \tau \to \tau + 1 \) yields \( e^{-i\pi/8} \frac{1}{\eta} \sqrt{\frac{\vartheta_2}{\eta}} \). Together with lemma 6.5.6 below, this gives:

\[
c_{00} + c_{02} = \frac{1}{\eta^2} \sqrt{\frac{\vartheta_4}{\eta}} = \frac{\eta}{\eta(2\tau)} \frac{\vartheta_2}{\eta} = \frac{1}{\eta} \sqrt{\frac{\vartheta_4}{\eta}} = \frac{1}{\eta} \sqrt{q^{-1/16} \vartheta(\tau) - q^{1/2} \vartheta(3\tau)}
\]

\[
\eta(\tau/2) \eta^2 = \frac{1}{\eta^2} \sqrt{\frac{\vartheta_4}{\eta}} = \frac{\eta}{\eta^2} \sqrt{q^{-1/16} \vartheta(\tau) + q^{1/2} \vartheta(3\tau)}
\]

Here, and for the remainder of this section, we use the shorthand \( \vartheta(z) := \vartheta_3(z|8\tau) \).

Thus:

\[
c_{00} = q^{-\frac{1}{16}} \frac{\vartheta(\tau)}{\vartheta(\tau)^2 - q \vartheta(3\tau)^2} = q^{\frac{1}{16}} \frac{\vartheta(\tau)}{\eta \eta(2\tau)}
\]

\[
c_{02} = q^{-\frac{1}{16}} \frac{q^{\frac{1}{2}} \vartheta(3\tau)}{\vartheta(\tau)^2 - q \vartheta(3\tau)^2} = q^{\frac{1}{16}} q^{\frac{1}{2}} \frac{\vartheta(3\tau)}{\eta \eta(2\tau)}
\]

\[\text{The form } \frac{1}{\eta} \sqrt{\frac{\vartheta_4}{\eta}} \text{ for the string functions } c_{11} \text{ and } c_{00} \pm c_{02} \text{ is expected from the fact that } \eta \phi_{ml} \text{ gives the character of the field } \phi_{lm} \text{ in the } \mathbb{Z}_k \text{ parafermion model [FZ-85, GQ-87]. For our case of } k = 2, \text{ the } \mathbb{Z}_2 \text{ parafermion model is just the Ising model, and its characters are the well known square roots of theta functions (with different spin structures). Thanks to Katrin Wendland for pointing at this and at her PhD thesis (p.50) [W-00] which already contains the explicit expressions for (6.5.1).}
and

\[
c_{00}^2 - c_{02}^2 = \frac{1}{\eta \eta(2\tau)} = \frac{q^{-\frac{1}{8}}}{\vartheta(\tau)^2 - q \vartheta(3\tau)^2}
\]
\[
c_{00} c_{02} = q^{\frac{3}{8}} \frac{\vartheta(\tau) \vartheta(3\tau)}{\eta^2 \eta(2\tau)^2} = \frac{\eta(8\tau)^2}{\eta^3 \eta(4\tau)}
\]

We now study the orbits at the special value of \( z = \frac{1+\tau}{2} \). Note first that at this value, \( \theta_{m,8}(\frac{\tau}{2}, \frac{1+\tau}{8}) = (-1)^{\frac{m^2}{2}} q^{(\frac{m}{2})^2+m/16} \vartheta_3((\frac{m}{2} + 1)\tau|8\tau) \). With lemma 6.5.6, our characters reduce to

\[
A = c_{00} \vartheta(\tau) + c_{02} (-q^{3/2}) \vartheta(5\tau) = q^{-1/16}
\]
\[
C = c_{02} \vartheta(\tau) + c_{00} (-q^{3/2}) \vartheta(5\tau) = 0
\]
\[
B = -i \left[ c_{02} \vartheta(\tau) - c_{00} q^{1/2} \vartheta(3\tau) \right] = 0
\]
\[
D = -i \left[ c_{02} \vartheta(\tau) - c_{00} q^{1/2} \vartheta(3\tau) \right] = -i q^{-1/16}
\]
\[
E = \frac{\eta(2\tau)}{\eta^2} q^{3/8} e^{-i\pi/4} \vartheta_3(\frac{1}{2}|2\tau) = e^{-i\pi/4} q^{-1/16}
\]
\[
F = \frac{\eta(2\tau)}{\eta^2} q^{3/8} e^{-i\pi/4} \vartheta_3(\frac{1}{2} + \tau|2\tau) = 0
\]

Plugging these values into the orbits \( NS_i \) yields \( NS_1 = 2q^{-1/4} \), \( NS_i = -q^{-1/4} \) \((i = 2, \ldots, 4)\), while the remaining \( NS_j \) vanish \((j = 5, \ldots, 12)\). We thus recognize from (6.2.4) the graviton, massless and massive orbits respectively.

### 6.5.3 The Functions \( F_i, F_j \) and the Dirac Index

In order to compute the functions \( F_i \) (6.2.1), we set \( z = 0 \), in which case \( \theta_{m,8}(\frac{\tau}{2}, 0) = q^{(\frac{m}{2})^2} \vartheta((\frac{m}{2})\tau) = \theta_{-m,8} \). Hence the characters simplify to:

\[
A + C = \frac{1}{\eta} \sqrt{\frac{d_3}{\eta}} \vartheta_3(0|2\tau) = \frac{\eta(2\tau)^4}{\eta(4\tau)^2 \eta(\tau/2)} = E^2/B
\]
\[
A - C = \frac{1}{\eta} \sqrt{\frac{d_3}{\eta}} \vartheta_4(0|2\tau) = \frac{\eta(\tau/2)}{\eta(2\tau)} = 1/E
\]
\[
B = D = \frac{1}{\eta} \sqrt{\frac{d_2}{\eta}} \vartheta_2(0|2\tau) = \frac{\eta \eta(4\tau)^2}{\eta(\tau/2)^2 \eta(2\tau)^2}
\]
\[
E = F = \frac{\eta(2\tau)}{\eta(\tau/2)} = \sqrt{\frac{1}{2} \frac{d_2}{d_4}}
\]

With these observation and the fact that \( AC = B^2 \) at \( z = 0 \) (6.5.2), the massive orbits at \( z = 0 \) have a rather simple form:

\[
NS_5 = 2NS_8 = 2B^4
\]
\[
NS_6 = NS_7 = \frac{1}{2} NS_9 = NS_{11} = E^4
\]
\[
NS_{10} = E^6/B^2 = NS_{12} NS_6/NS_5
\]
\[
NS_{12} = 2B^2 E^2
\]
Given that $\text{NS}_j = F_j \, q^{-\frac{1}{8}} \, \eta^2 \, \eta^7 = F_j \, \frac{q^{-\frac{1}{8}} \, \eta^7}{\eta(2\tau)^4 \, \eta(\tau/2)^4}$, we find the following values for $F_j$:

\[
F_6 = 2F_8 = 2q^{1/8} \, \frac{\eta(4\tau)^8}{\eta^9 \, \eta(2\tau)^4} \\
F_6 = F_7 = \frac{1}{2} F_9 = F_{11} = q^{1/8} \, \frac{\eta(2\tau)^8}{\eta^9} \\
F_{10} = q^{1/8} \, \frac{\eta(2\tau)^{14}}{\eta^9 \, \eta(4\tau)^4} \\
F_{12} = 2q^{1/8} \, \frac{\eta(2\tau)^2 \, \eta(4\tau)^4}{\eta^5} \\
F_5 F_{10} = F_6 F_{12}
\]

The massless orbits are a little less elegant, especially $\text{NS}_1$ and $\text{NS}_3$ which are not factorizable. For the latter, we shall need the following observation, again having set $z = 0$:

\[
(A^2 + C^2) = (A - C)^2 + 2B^2 = 1/E^2 + 2B^2 \\
\Rightarrow E^6 = B^2 + 4B^4 E^2
\]

and

\[
1/E^2 + 2B^2 = 2 \frac{\vartheta_4}{\vartheta_2} + \frac{\vartheta_2 (0|2\tau)^2}{\vartheta_2 \vartheta_4} = 2 \frac{\vartheta_4}{\vartheta_2} + \frac{\vartheta_2 (0|2\tau)^2}{\vartheta_2 \vartheta_4} = \frac{3 \vartheta_4^2 + \vartheta_2^2}{2 \vartheta_2 \vartheta_4}
\]

We note that the last expression cannot be factorized and so we give the graviton + massless orbits as they stand:

\[
\text{NS}_1 = (A^2 + C^2)^2 = (1/E^2 + 2B^2)^2 = \left(\frac{3 \vartheta_4^2 + \vartheta_3^2}{2 \vartheta_2 \vartheta_4}\right)^2 \\
\text{NS}_2 = \text{NS}_4 = 2 \, E^4 \\
\text{NS}_3 = (3 \vartheta_4^2 + \vartheta_3^2) \, \frac{\vartheta_2 (0|2\tau)^2}{8 \eta^6} = (3 \vartheta_4^2 + \vartheta_3^2) \, \frac{\vartheta_2 - \vartheta_3^2}{4 \vartheta_2 \vartheta_4} = (3 \vartheta_4^2 + \vartheta_3^2) \, \frac{\eta \, \eta(4\tau)^4}{2 \, \eta(\tau/2) \, \eta(2\tau)^4} \\
\text{NS}_5 = 2 \, \text{NS}_1 \, \text{NS}_5
\]

with extensive use of formulae in appendix C. Given that $\text{NS}_i = (h_3 + F_i \, \frac{q^{-1/8}}{\eta}) \, \frac{\vartheta_2^2}{\eta^6}$ for $i = 2, 3, 4$ (and $h_3$ replaced by $-2h_3$ for $\text{NS}_1$), we find the following values for $F_1$ and $F_4$:

\[
F_1 = q^{1/8} \, \eta \left[ \left( \frac{3 \vartheta_4^2 + \vartheta_3^2}{4 \, \eta^2} \right)^2 + 2 \, h_3 \right] - 1 \\
F_2 = q^{1/8} \, \eta \left[ 2 \left( \frac{\eta(2\tau)}{\eta} \right)^8 - h_3 \right] = 2 \, F_6 - q^{1/8} \, \eta \, h_3 \\
F_3 = q^{1/8} \, \eta \left[ 3 \vartheta_4^2 + \vartheta_3^2 \right] \frac{\eta(4\tau)^4}{2 \, \eta^4 \, \eta(2\tau)^4} - h_3 \right] \\
F_4 = F_2
\]

As in the $1^6$ theory, we shall again derive (6.3.3), that is we shall show:

**Proposition 6.5.4.**

\[
\sum_{i=1}^{d} D_i \text{NS}_i I_i = -2 \, \text{NS}_1 + 2 \, \text{NS}_2 + 6 \, \text{NS}_3 + 12 \, \text{NS}_4
\]

\[
= -24 \, \frac{\vartheta_3 (z|\tau)^2}{\vartheta_2^2} + 2(\vartheta_4 - \vartheta_4) \, \frac{\vartheta_3 (z|\tau)^2}{\eta^6}.
\]

\[
(6.5.5)
\]
Proof. Due to (6.5.3) and the vanishing of $\vartheta_3(z|\tau)$ at $z = \frac{1+\tau}{2}$, we see that $\vartheta_3(z|\tau)^2$ is correctly multiplied by $-24/\vartheta_3^2$. We only need to recover the factor multiplying $\vartheta_3(z|\tau)^2$. Unlike in $1^6$ theory, setting $z = 0$ on both sides will easily do the job:

$$\text{l.h.s.}|_{0} = \frac{1}{2} \frac{\vartheta_2^2}{\vartheta_4^2} \left[ (3 \vartheta_4^2 + \vartheta_2^2)^2 + (2 + 12) \vartheta_2^4 + 6(3 \vartheta_4^2 + \vartheta_2^2)^2 \right]$$

The square brackets yield a total of $16(\vartheta_2^4 - \vartheta_4^4)$, while the prefactor is $\frac{1}{2} \left( \frac{\vartheta_2}{\frac{3}{2} \eta^4} \right)^2$. Thus we obtain the r.h.s.

6.5.4 Lemmas:

Lemma 6.5.6. With the shorthand $\vartheta(z) := \vartheta_3(z|8\tau)$, we have:

1. $q \vartheta(5\tau) = \vartheta(3\tau)$
2. $\vartheta(0) - q \vartheta(4\tau) = \vartheta_4(0|2\tau) = \frac{\eta^2}{\eta(2\tau)}$
3. $\vartheta(0) + q \vartheta(4\tau) = \vartheta_3(0|2\tau) = \frac{\eta^2}{\eta(2\tau)}$
4. $\vartheta(\tau) + q^{1/2} \vartheta(3\tau) = q^{-1/8} \vartheta_2(0|\tau/2) = q^{-1/8} \frac{\eta^2}{\eta(\tau/2)}$
5. $\vartheta(\tau) - q^{1/2} \vartheta(3\tau) = \vartheta_2(\tau/2|2\tau) = q^{-1/8} \frac{\eta(2\tau)}{\eta(\tau/2)}$

Proof. These are all instances of the more general lemma 6.6.3. Alternatively:

1. directly from sum or product expression.
2. l.h.s. $= \sum q^{4n^2} - \sum q^{4(n+\frac{1}{2})^2} = \sum q^{4(\frac{n}{2})^2} = \vartheta_4(0|2\tau) = \prod (1 - q^{2n}) (1 - q^{2n})^{-1} = \text{r.h.s.}$
3. idem
4. l.h.s. $= q^{-1/8} \sum (q^{4(n-\frac{1}{2})^2} + q \sum q^{4(n+\frac{1}{2})^2}) = q^{-1/8} \sum q^{4(n-\frac{1}{2})^2} = \vartheta_3(\tau/2|2\tau)$
5. idem

Lemma 6.5.7.

\[ \vartheta_4(0|\frac{\tau}{2}) \left( \vartheta_1(z|\tau)^2 + \vartheta_4(z|\tau)^2 \right) = \vartheta_3(0|\frac{\tau}{2}) \left( \vartheta_3(z|\tau)^2 - \vartheta_2(z|\tau)^2 \right) \]
\[ = \vartheta_3(0|\frac{\tau}{2}) \vartheta_3(\frac{\tau}{2}|\frac{\tau}{4}) \vartheta_4(\frac{\tau}{4}|\frac{\tau}{4}) \]
\[ \vartheta_3(0|\frac{\tau}{2}) \vartheta_2(z|\tau)\vartheta_3(z|\tau) + \vartheta_4(0|\frac{\tau}{2}) i\vartheta_1(z|\tau)\vartheta_4(z|\tau) = \vartheta_2(0|\frac{\tau}{2}) q^{16} y^{-1} \vartheta_3(z - \frac{\tau}{4}|\tau)^2 \]
\[ \vartheta_3(0|\frac{\tau}{2}) \vartheta_2(z|\tau)\vartheta_3(z|\tau) - \vartheta_4(0|\frac{\tau}{2}) i\vartheta_1(z|\tau)\vartheta_4(z|\tau) = \vartheta_2(0|\frac{\tau}{2}) q^{16} y^{\frac{1}{2}} \vartheta_3(z + \frac{\tau}{4}|\tau)^2 \]
Proof. All three expressions of the first line are theta functions of degree two and characteristic \((0, 0; -4 \pi i, -2 \pi i r)\), i.e. elements of the two-dimensional vector space \(T_{2, -2 \pi i r}\) (see appendix D). In the other two lines, all terms upon extra multiplication with \(y\) are elements of \(T_{2, -3 \pi i r}\). So all we need to do is to verify the relations at two independent values of \(z\).

For all lines, verification at \(z = 0, \pi/2\) is immediate with (C.0.16), (C.0.17). The r.h.s. gives a handy factorised form of the l.h.s., which would not lend itself to straightforward factorisation as the sums \((i \vartheta_1 \pm \vartheta_4)\) cannot be made into a theta function.

## 6.6 Computations in \(4^3\) Theory

We mimic here the approach of the two previous sections. This Gepner model is obtained by tensoring 3 times the \(k = 4, N=2\) theory. We have considerably more orbits and more cases to study; the mathematics involve properties of theta functions with \(\tau\) divided by 3, 4 and 6.

### 6.6.1 Characters and Orbits:

This time we have 15 minimal \(N=2\) characters, obtained for \(l = 0\) \((m = 0)\), \(l = 1\) \((m = \pm 1)\), \(l = 2\) \((m = 0, \pm 2)\), \(l = 3\) \((m = \pm 1, 3)\), \(l = 4\) \((m = 0, \pm 2, \pm 4)\). The string functions at level 4, due to their symmetries, number only seven:

\[
\begin{align*}
\psi^{00} &= \psi^{44}, & \psi^{02} &= \psi^{42}, & \psi^{04} &= \psi^{40}, & \psi^{20} &= \psi^{24}, & \psi^{22}, & \\
\psi^{11} &= \psi^{33}, & \psi^{13} &= \psi^{31}.
\end{align*}
\]

In the following characters, \(\theta_m\) is a shortcut for the \(su(2)\) theta function evaluated as \(\theta_m, 24(\tau/2, \pi/6)\):

\[
\begin{align*}
A := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_{-12} + \psi^{00} \theta_0 + \psi^{02} \theta_{12} + \psi^{04} \theta_{24} \\
B := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_{20} + \psi^{04} \theta_{-16} + \psi^{02} \theta_{-4} + \psi^{00} \theta_8 \\
C := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_{-20} + \psi^{04} \theta_{-8} + \psi^{02} \theta_4 + \psi^{00} \theta_{16} \\
D := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_{-12} + \psi^{04} \theta_0 + \psi^{02} \theta_{12} + \psi^{00} \theta_{24} \\
E := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_{-4} + \psi^{04} \theta_8 + \psi^{02} \theta_{20} + \psi^{00} \theta_{-16} \\
F := \psi^{NS}_{14}(y, q) &= \psi^{02} \theta_4 + \psi^{04} \theta_{16} + \psi^{02} \theta_{-20} + \psi^{00} \theta_{-8} \\
G := \psi^{NS}_{14}(y, q) &= \psi^{22} \theta_{-20} + \psi^{20} \theta_{-8} + \psi^{22} \theta_4 + \psi^{20} \theta_{16} \\
H := \psi^{NS}_{14}(y, q) &= \psi^{22} \theta_{-12} + \psi^{20} \theta_0 + \psi^{22} \theta_{12} + \psi^{20} \theta_{24} \\
I := \psi^{NS}_{14}(y, q) &= \psi^{22} \theta_{-4} + \psi^{20} \theta_8 + \psi^{22} \theta_{20} + \psi^{20} \theta_{-16} \\
J := \psi^{NS}_{14}(y, q) &= \psi^{13} \theta_{-22} + \psi^{11} \theta_{-10} + \psi^{11} \theta_2 + \psi^{13} \theta_{14} \\
K := \psi^{NS}_{14}(y, q) &= \psi^{13} \theta_{-14} + \psi^{11} \theta_{-2} + \psi^{11} \theta_{10} + \psi^{13} \theta_{22} \\
L := \psi^{NS}_{14}(y, q) &= \psi^{11} \theta_{18} + \psi^{13} \theta_{-18} + \psi^{13} \theta_{-6} + \psi^{11} \theta_{6} \\
M := \psi^{NS}_{14}(y, q) &= \psi^{11} \theta_{22} + \psi^{13} \theta_{-10} + \psi^{13} \theta_2 + \psi^{11} \theta_{14} \\
N := \psi^{NS}_{14}(y, q) &= \psi^{11} \theta_{-14} + \psi^{13} \theta_{-2} + \psi^{13} \theta_{10} + \psi^{11} \theta_{22} \\
O := \psi^{NS}_{14}(y, q) &= \psi^{11} \theta_{-6} + \psi^{13} \theta_{6} + \psi^{13} \theta_{18} + \psi^{11} \theta_{-18}
\end{align*}
\]
The $su(2)$ theta functions are related to the standard Jacobi theta function via

$$\theta_{m,24}(\frac{z}{2}, \frac{\zeta}{6}) = \sum_{n \in \mathbb{Z}} q^{12(n+\frac{m}{24})^2} y^{4(n+\frac{m}{24})} = q^{(\frac{m}{24})^2/3} y^{\frac{m}{24}} \vartheta_3(4z + \frac{m\pi}{2}) |24\tau|$$

Under full spectral flow, these $su(2)$ theta functions are shifted into one each other:

$$\theta_{m,24}(\frac{z}{2}, \frac{\zeta}{6}) \rightarrow q^{-1/3} y^{-2/3} \theta_{m+8,24}(\frac{z}{2}, \frac{\zeta}{6}),$$

so that the fifteen characters split into three groups which are cyclicly permuted:

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow A,$$
$$G \rightarrow H \rightarrow I \rightarrow G,$$
$$J \rightarrow K \rightarrow M \rightarrow N \rightarrow O \rightarrow J,$$

where we have omitted the incrementing factors of $q^{-1/3} y^{-2/3}$, etc.

To build the various orbits of the $4^3$ theory, we consider all possible homogeneous polynomials of degree 3 in $A, B, ..., O$, respecting our usual rules. Note the following powers for the $y$-expansions:

- $A, D, H$ have powers of $y$ in $\mathbb{Z}$,
- $J, M$ in $\mathbb{Z} + \frac{1}{3}$,
- $B, E, I$ in $\mathbb{Z} - \frac{1}{3}$,
- $K, N$ in $\mathbb{Z} - \frac{1}{6}$,
- $C, F, G$ in $\mathbb{Z} + \frac{1}{3}$,
- $L, O$ in $\mathbb{Z} + \frac{1}{2}$.

The twenty-three possible orbits are:

- $NS_{1} = A^3 + B^3 + C^3 + D^3 + E^3 + F^3$
- $NS_{2} = G^3 + H^3 + I^3$
- $NS_{3} = (BC + EF)H + (CD + FA)I + (DE + AB)G$
- $NS_{4} = AO^2 + BI^2 + CK^2 + DL^2 + EM^2 + FN^2$
- $NS_{5} = (JL + MO)G + (KM + NJ)H + (LN + OK)I$
- $NS_{6} = AL^2 + BM^2 + CN^2 + DO^2 + EJ^2 + FK^2$
- $NS_{7} = A^2D + B^2E + C^2F + D^2A + E^2B + F^2C$
- $NS_{8} = ABC + BCD + CDE + DEF + EFA + FAB$
- $NS_{9} = AEC + BFD$
- $NS_{10} = GHI$
- $NS_{11} = ADH + BEI + CFG$
- $NS_{12} = (A^2 + D^2)H + (B^2 + E^2)I + (C^2 + F^2)G$
- $NS_{13} = (BF + EC)H + (AC + DF)I + (BD + AE)G$
- $NS_{14} = (A + D)H^2 + (B + E)I^2 + (C + F)G^2$
- $NS_{15} = (C + F)HI + (A + D)IG + (B + E)GH$
- $NS_{16} = (L^2 + O^2)H + (J^2 + M^2)I + (K^2 + N^2)G$
- $NS_{17} = (A + D)LO + (B + E)JM + (C + F)KN$
- $NS_{18} = HLO + IJM + GKN$
- $NS_{19} = AJK + BKL + CLM + DMN + ENO + FOJ$
- $NS_{20} = DJK + EKL + FLM + AMN + BNO + COJ$
- $NS_{21} = (JK + MN)H + (KL + NO)I + (LM + OJ)G$
- $NS_{22} = CIL + DKM + ELN + FMO + ANJ + BOK$
- $NS_{23} = FJL + AKM + BLN + CMO + DNJ + EOK.$

The coefficients $D_i$, see after (6.4.4), are thus $(1; 2,6,3,6,3; 3,6,18,36,12,3,6,3; 3,6,12,6,6,6,6,6).$
6.6.2 String Functions:

Some explicit expressions for the string functions at level 2 are found in [KP-84], pp. 219-220. We use the notation $\eta_n$ for $\eta(n\tau)$:

$$c_{02} = \frac{\eta_{12}^2}{\eta^2 \eta_6}, \quad c_{00} - c_{04} = \frac{1}{\eta_2}, \quad c_{11} + c_{13} = \frac{1}{\eta_{1/2}},$$

$$c_{00} + c_{04} - 2c_{02} + 2c_{20} - 2c_{22} = \frac{\eta_{1/12}^2}{\eta^2 \eta_{1/6}}$$

The behaviour under $S$ and $T$ transformation is also outlined by the authors. For example, $T$ transforms the third equation into

$$c_{11} - c_{13} = \frac{\eta_{1/2} \eta_2}{\eta^3},$$

where we discarded $e^{i\pi/24}$ on both sides. Similarly, $T^6$ (i.e. $\tau \rightarrow \tau + 6$) transforms the last equation into

$$-(c_{00} + c_{04} + 2c_{02} + 2c_{20} + 2c_{22}) = -\frac{\eta_{1/6}^5}{\eta^2 \eta_{1/12}^2 \eta_{1/3}^2}, \quad \text{since} \quad \eta_{1/2} \xrightarrow{T} \frac{\eta^3}{\eta_{1/2} \eta_2} e^{i\pi/24}.$$

Furthermore,

$$c_{00} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} (c_{00} + c_{04} + 2c_{02} + 2c_{20} + 2c_{22} + 2\sqrt{3}(c_{11} + c_{13}))$$

$$c_{04} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} (c_{00} + c_{04} + 2c_{02} + 2c_{20} + 2c_{22} - 2\sqrt{3}(c_{11} + c_{13}))$$

$$c_{02} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} (c_{00} + c_{04} - 2c_{02} + 2c_{20} - 2c_{22})$$

$$c_{20} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} (2c_{00} + 2c_{04} + 4c_{02} - 2c_{20} - 2c_{22})$$

$$c_{00} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} (2c_{00} + 2c_{04} - 4c_{02} - 2c_{20} + 2c_{22}),$$

and so

$$c_{00} + c_{04} \xrightarrow{S} \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} \left(2c_{00} + 2c_{04} + 4c_{02} + 4c_{20} + 4c_{22}\right)$$

$$= \frac{1}{2\sqrt{6}} \sqrt{\frac{i}{\tau}} \left(\frac{\eta_{1/6}^5}{\eta^2 \eta_{1/12}^2 \eta_{1/3}^2}\right) \xrightarrow{S} \frac{\eta_{1/6}^5}{\eta^2 \eta_{1/12}^2 \eta_{1/3}^2}$$

since $\eta_n \xrightarrow{S} \sqrt{-i\tau/n} \eta_{1/n}$ for rational $n$. We thus find

$$c_{00} + c_{04} = \frac{\eta_{1/6}^5}{\eta^2 \eta_{1/12}^2 \eta_{1/3}^2} = \frac{\vartheta_3(0|6\tau)}{\eta^2}$$

$$c_{00} - c_{04} = \frac{1}{\eta_2} = \frac{\vartheta_4(0|2\tau)}{\eta^2}$$

$$c_{02} = \frac{\eta_{1/12}^2}{\eta^2 \eta_6} = \frac{\vartheta_2(0|6\tau)}{2\eta^2}$$
\[
\begin{align*}
c_{20} + c_{22} &= \frac{1}{2\eta^2} \left( \frac{\eta_6^{12}}{\eta_4^{1/12}} - \vartheta_3(0|6\tau) - \vartheta_2(0|6\tau) \right) \\
&= \frac{1}{2\eta^2} \left( \vartheta_3(0|\tau/6) - \vartheta_3(0|3\tau/2) \right) = \frac{q^{1/12}}{\eta^2} \vartheta_3(\frac{\tau}{2} | \frac{3\tau}{2}) \\
c_{20} - c_{22} &= \frac{1}{2\eta^2} \left( \frac{\eta_6^{12}}{\eta_4^{1/12}} - \vartheta_3(0|6\tau) + \vartheta_2(0|6\tau) \right) \\
&= \frac{1}{2\eta^2} \left( \vartheta_4(0|\tau/6) - \vartheta_4(0|3\tau/2) \right) = -\frac{q^{1/12}}{\eta^2} \vartheta_4(\frac{\tau}{2} | \frac{3\tau}{2}),
\end{align*}
\]

where we have used lemma 6.6.3, 6.6.4. Thus

\[
\begin{align*}
c_{20} &= \frac{q^{1/3}}{\eta^2} \vartheta_3(2\tau|6\tau) = \frac{q^{1/12}}{\eta^2} \vartheta_2(\tau|6\tau) \\
c_{22} &= \frac{q^{1/3}}{\eta^2} \vartheta_2(2\tau|6\tau) = \frac{q^{1/12}}{\eta^2} \vartheta_3(\tau|6\tau).
\end{align*}
\]

6.6.3 Relation with 16 theory

With the above values of the string functions, we shall show some coincidences of the characters of 43 theory with those of 16 theory. We first note:

\[
\begin{align*}
\theta_m + \theta_{m+24} &= \sum q^{3(n + \frac{m}{24})^2} y^{2(n + \frac{m}{24})} = q^{\left( \frac{m}{24} \right)^2/3} y^{\frac{m}{12}} \vartheta_3(2z + \frac{\tau \eta_4}{4}|6\tau) \\
\theta_m - \theta_{m+24} &= \sum (-1)^n q^{3(n + \frac{m}{24})^2} y^{2(n + \frac{m}{24})} = q^{\left( \frac{m}{24} \right)^2/3} y^{\frac{m}{12}} \vartheta_4(2z + \frac{\tau \eta_4}{4}|6\tau)
\end{align*}
\]

\[
\begin{align*}
\theta_m + \theta_{m+12} + \theta_{m+24} + \theta_{m+36} &= \sum q^{\frac{3(n + \frac{m}{24})^2}{3}} y^{n + \frac{m}{12}} = q^{\left( \frac{m}{24} \right)^2/3} y^{\frac{m}{12}} \vartheta_3(z + \frac{\tau \eta_4}{8} | 3\tau) \\
\theta_m - \theta_{m+12} + \theta_{m+24} - \theta_{m+36} &= \sum (-1)^n q^{\frac{3(n + \frac{m}{24})^2}{3}} y^{n + \frac{m}{12}} = q^{\left( \frac{m}{24} \right)^2/3} y^{\frac{m}{12}} \vartheta_4(z + \frac{\tau \eta_4}{8} | 3\tau)
\end{align*}
\]

\[
\begin{align*}
\theta_m + \theta_{m+12} - \theta_{m+24} - \theta_{m+36} &= \sum \delta_n q^{\frac{3(n + \frac{m}{24})^2}{3}} y^{n + \frac{m}{12}}, \\
\text{for } m \equiv 2 \mod 8 : &= \sum (q e^{2\pi i \frac{3}{4}})^{\frac{3(n + \frac{m}{24})^2}{3}} y^{n + \frac{m}{12}} \\
&= q^{\left( \frac{m}{24} \right)^2/3 + 3/16} y^{\frac{m}{12}} \eta_{\frac{m}{24}} \eta_{\frac{m}{3}} \eta_{\frac{m}{8}} \vartheta_3(z + \frac{3}{4} + \frac{m}{8}) | 3\tau) \\
&\times \vartheta_3(z + \frac{3}{4} - \frac{m}{8}) | 3\tau) \\
\text{and for } m \equiv -2 \mod 8 : &= \sum (-1)^n (q e^{2\pi i \frac{3}{4}})^{\frac{3(n + \frac{m}{24})^2}{3}} y^{n + \frac{m}{12}} \\
&= q^{\left( \frac{m}{24} \right)^2/3 + 3/16} y^{\frac{m}{12}} \eta_{\frac{m}{24}} \eta_{\frac{m}{3}} \eta_{\frac{m}{8}} \vartheta_4(z + \frac{3}{4} + \frac{m}{8}) | 3\tau) \\
&\times \vartheta_4(z + \frac{3}{4} - \frac{m}{8}) | 3\tau)
\end{align*}
\]

where \( \delta_n := +, +, -, - \) for \( n \equiv 0, 1, 2, 3 \mod 4 \). The trick for the last formula is the same as in the special case leading to (6.6.2).
Thus, using lemma 6.6.6:

\[ A + D = 2c_{02} \vartheta_2(2z|6\tau) + (c_{00} + c_{04}) \vartheta_3(2z|6\tau) = \frac{1}{\eta^2} \vartheta_3(z|3\tau)^2 = A^2_{[1^6 \text{ th.}]} \]
\[ B + E = \frac{q^{1/3} \eta^{2/3}}{\eta} \vartheta_3(z + \tau|3\tau)^2 = B^2_{[1^6 \text{ th.}]} \]
\[ C + F = \frac{q^{4/3} \eta^{4/3}}{\eta} \vartheta_3(z + 2\tau|3\tau)^2 = C^2_{[1^6 \text{ th.}]} \]

\[ G = c_{22} q^{1/3} \eta^{1/3} \vartheta_3(2z + \tau|6\tau) + c_{20} q^{1/3} \vartheta_2(2z + \tau|6\tau) = \frac{q^{1/3} \eta^{1/3}}{\eta} \vartheta_3(z|3\tau) \vartheta_3(z + \tau|3\tau) = AB_{[1^6 \text{ th.}]} \]
\[ H = \frac{q^{1/3} \eta^{1/3}}{\eta} \vartheta_3(z + \tau|3\tau) \vartheta_3(z + 2\tau|3\tau) = BC_{[1^6 \text{ th.}]} \]
\[ I = \frac{q^{1/3} \eta^{1/3}}{\eta} \vartheta_3(z|3\tau) \vartheta_3(z + 2\tau|3\tau) = AC_{[1^6 \text{ th.}]} \]
(6.6.1)

where the r.h.s.'s are taken from (6.4.2). In particular, this implies \((A + D)H = GI\), etc, and for the orbits:

<table>
<thead>
<tr>
<th>4\textsuperscript{3} theory</th>
<th>1\textsuperscript{6} theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS\textsubscript{1} + 3 NS\textsubscript{7}</td>
<td>= NS\textsubscript{1}</td>
</tr>
<tr>
<td>NS\textsubscript{2}</td>
<td>= NS\textsubscript{2}</td>
</tr>
<tr>
<td>NS\textsubscript{3} + NS\textsubscript{13}</td>
<td>= NS\textsubscript{2}</td>
</tr>
<tr>
<td>$\frac{1}{3}$ NS\textsubscript{14} = NS\textsubscript{10} = NS\textsubscript{8} + NS\textsubscript{9}</td>
<td>= NS\textsubscript{3}</td>
</tr>
<tr>
<td>NS\textsubscript{15} = NS\textsubscript{12} + 2 NS\textsubscript{11}</td>
<td>= NS\textsubscript{4}</td>
</tr>
</tbody>
</table>

These relations will be useful for the study of Gepner models with mixed levels, in particular the 1\textsuperscript{4}4 theory.

6.6.4 Characters at $z = \frac{1+\tau}{2}$:

We now study the orbits at the special value of $z = \frac{1+\tau}{2}$. Note first that at this value, $\theta_m = \theta_{m,24}(\frac{1}{2}, \frac{1+\tau}{12}) = e^{2\pi i \frac{m}{24}} q^{(\frac{m}{24}+m)/24} \vartheta_3((\frac{m}{2} + 2)\tau|24\tau)$, as well as:

\[
\theta_8 = \theta_{-16}, \quad \theta_{-22} = -\theta_{14}, \quad \theta_{-10} = -\theta_{2}
\]
\[
\theta_m + \theta_{m+12} + \theta_{m+24} + \theta_{m+36} = e^{2\pi i \frac{m}{24}} q^{(\frac{m}{24}+m)/24} \vartheta_4((\frac{m}{2} + \frac{1}{2})\tau|\frac{3\tau}{2})
\]
\[
\theta_m + \theta_{m+12} - \theta_{m+24} - \theta_{m+36} = e^{2\pi i \frac{m}{24}} \sum \delta_n q^{\frac{3}{2}(n+\frac{m}{3})^2+\frac{1}{2}(n+\frac{m}{3})}
\]

where $\delta_n = +, +, -, -$ for $n \equiv 0, 1, 2, 3 \mod 4$. Without $\delta_n$, we would recover the sum of the four theta functions with only $+$ signs. Note that $\delta_n$ can be removed if we replace $q^{1/2}$ with $-q^{1/2}$ in the sum and additionally multiply the sum by some root of unity, since $(-1)^{\frac{3}{2}(n+\frac{m}{3})^2+n+\frac{m}{3}} = i^{\frac{m^2}{12} + \frac{m}{3} i^{3n^2+2+\frac{m}{3} n} n}$.

For $m = 6$ or $-2$, we recover $e^{-2\pi i/16} \delta_n$. Thus we obtain the last line from the sum with only $+$ signs by inserting $-q^{1/2}$ in the latter’s result. For $m = 6$ and $-2$:

\[
\theta_{-6} + \theta_{18} + \theta_{-18} + \theta_{-6} = -iq^{-\frac{1}{12}} \vartheta_4(-\frac{3\tau}{4}) = -iq^{-\frac{1}{12}} \eta(\frac{\tau}{2})
\]
\[
\theta_{-2} + \theta_{10} + \theta_{22} + \theta_{-14} = q^{-\frac{1}{12}} \eta(\frac{\tau}{2})
\]

On the r.h.s., replacing $q^{1/2}$ with $-q^{1/2}$ in $q^{-\frac{1}{12}} \eta(\tau/2)$ yields:

\[
q^{-\frac{1}{12}} \prod(1 - q^n)(1 - q^{n-1/2}) \rightarrow e^{-2\pi i/16} q^{-\frac{1}{12}} \prod(1 - q^n)(1 + q^{n-1/2}) = e^{-2\pi i/16} q^{-\frac{1}{12}} \eta(\tau/2) \eta(2\tau)
\]
Thus:

\[ \theta_6 + \theta_{18} - \theta_{-18} - \theta_{-6} = -i q^{-1/12} \frac{\eta^{3}}{\eta(\tau/2)\eta(2\tau)} \]

\[ \theta_{-2} + \theta_{10} - \theta_{22} - \theta_{-14} = q^{-1/12} \frac{\eta^{3}}{\eta(\tau/2)\eta(2\tau)}. \]  

(6.6.2)

This will be useful for the characters \( K, L, N, O \).

With lemmas 6.6.7, 6.6.8 and 6.6.9 our characters at \( z = \frac{1+i\tau}{2} \) reduce to

\[ A = \frac{1}{2\eta^2} \left[ \partial_2(0|6\tau)(\theta_{-12} + \theta_{12}) + \partial_3(0|6\tau)(\theta_0 + \theta_{24}) + \partial_4(0|2\tau)(\theta_0 - \theta_{24}) \right] \]

\[ = \frac{1}{2\eta^2} \left[ \partial_2(0|6\tau)(-q^{1/4})\partial_3(2\tau|6\tau) + \partial_3(0|6\tau) \partial_3(\tau|6\tau) + \partial_4(0|2\tau) \partial_4(\tau|6\tau) \right] \]

\[ = q^{-1/12} \eta^2 \]

\[ B = C = D = E = 0 \]

\[ F = e^{-2\pi i/3} q^{-1/12} \]

\[ G = \frac{q^{1/12}}{\eta^2} \partial_3(\tau|6\tau)(\theta_{-20} + \theta_4) + \frac{q^{1/3}}{\eta^2} \partial_3(2\tau|6\tau)(\theta_{-8} + \theta_{16}) = 0 \]

\[ H = 0 \]

\[ I = e^{-2\pi i/6} q^{-1/12} \]

\[ J = c_{13}(\theta_{-22} + \theta_{14}) + c_{11}(\theta_{-10} + \theta_{2}) = 0 \]

\[ K = \frac{1}{2}(c_{11} + c_{13})(\theta_{-14} + \theta_{-2} + \theta_{10} + \theta_{22}) + \frac{1}{2}(c_{11} - c_{13})(-\theta_{-14} + \theta_{-2} + \theta_{10} - \theta_{22}) \]

\[ = e^{-2\pi i/12} q^{-1/12} \]

\[ L = M = N = 0 \]

\[ O = -i q^{-1/12}. \]

Plugging these values into the orbits \( NS_i \) yields \( NS_1 = 2q^{-1/4} \), \( NS_i = -q^{-1/4} \) \((i = 2, \ldots, 6)\), while the remaining \( NS_j \) vanish \((j = 7, \ldots, 23)\). We thus recognize from (6.2.4) the graviton, massless and massive orbits respectively. Also, the value of the elliptic genus at \( z = 0 \) is \( \Phi(0) = \sum_{i=1}^{d} D_i |R_i(0)|^2 = \sum_{i=1}^{d} D_i | - q^{1/4}NS_i((\frac{1+i\tau}{2})|^2 = \sum_{i=1}^{d} D_i I_i^2 = 4D_1 + D_2 + \cdots + D_{6} = 24 \), which is the correct coefficient for a \( K3 \) model (6.3.1).

**6.6.5 Characters at \( z = 0 \):**

In order to compute the functions \( F_i \) or the Dirac genus, we set \( z = 0 \), in which case

\[ \theta_m = \theta_{m,24}(\frac{\tau}{2}, 0) = \sum q^{12(n + \frac{m}{48})^2} = \]

\[ q^{(\frac{m}{8})^2/3} \partial_3(\frac{m\tau}{2}|24\tau) = \theta_{-m} \]

\[ \theta_m + \theta_{m+12} - \theta_{m+24} - \theta_{m+36} = \sum \delta_n q^{\frac{3}{4}(n + \frac{m}{48})^2} = \sum \delta_n q^{\frac{3}{4}(n+1 - \frac{m}{48})^2} \]

(for \( m = -10, -2 \):)

\[ = \sum \delta_n q^{\frac{3}{4}(n-\frac{1}{8})^2} = \]

\[ = e^{-2\pi i(\frac{m}{8})^2/3} \sum (-1)^n (q^{1/2})^{\frac{3}{2}(n-\frac{1}{8})^2} = \frac{\eta^3}{\eta(\tau/2)\eta_2}, \]

where \( \delta_n := +, +, -, - \) and the last line is obtained by replacing \( q^{1/2} \) by \( q^{-1/2} \)

in \( \eta(\tau/2) \) (same trick as earlier).
With lemma 6.6.6, the characters at $z = 0$ take the following values:

\[ A = \frac{1}{2\pi i} \vartheta_3(0|3\tau)^2 + \frac{1}{2\pi i} \vartheta_4(0|2\tau) \vartheta_4(0|6\tau) =: A_1 + A_2 \]
\[ B = F = \frac{1}{2\pi i} \vartheta_3(\tau|3\tau)^2 + \frac{1}{2\pi i} \vartheta_4(0|2\tau) \vartheta_4(2\tau|6\tau) =: B_1 + B_2 \]
\[ C = E = B_1 - B_2 \]
\[ D = A_1 - A_2 \]
\[ G = I = \frac{2^{1/6}}{\eta} \vartheta_3(0|3\tau) \vartheta_3(\tau|3\tau) \]
\[ H = \frac{2^{1/3}}{\eta} \vartheta_3(\tau|3\tau)^2 = 2B_1 \]
\[ J = K = \frac{2^{1/48}}{\eta^{1/2}} \vartheta_3(\frac{\tau}{4}\frac{3\tau}{4}) + \frac{1}{2} =: J_1 + \frac{1}{2} \]
\[ L = O = \frac{1}{2\eta^{1/2}} \vartheta_2(0|3\tau) \]
\[ M = N = J_1 - \frac{1}{2}. \]

We did not succeed in factorizing them, as we did for the $1^6$ and $2^4$ theories. The corresponding values for the orbits are not particularly enlightening. We only note the following coincidence: $NS_{22} = NS_{23} = 2A_1(J_1^2 - \frac{1}{4}) + 4LJ_1B_1$. Due to 6.2.4, this equality holds in general (not only at $z = 0$) since $F_{22} = F_{23}$.

Thus we shall also refrain from giving here horrendous expressions (unfactorized) for the functions $F_i$ and the Dirac index. But they can be easily written down on the basis of the above information. For instance, proving the value of the Dirac index (6.3.3) boils down to verifying its coefficient at $z = 0$:

\[
\sum_{i=1}^{d} D_i \; \text{NS}_i(0) \; I_i = -2 \; \text{NS}_1 + 2 \; \text{NS}_2 + 6 \; \text{NS}_3 + 3 \; \text{NS}_4 + 6 \; \text{NS}_5 + 3 \; \text{NS}_6
\]
\[
= 32B_1^3 - 4A_1^3 - 12A_1A_2^2 - 48B_1B_2^2 + 4G^3 + \\
+ 24G^2(A_1B_1 + A_2B_2 + J_1L) + 24B_1J_1^2 + 12A_1L^2
\]
\[
\equiv 2 \frac{3}{\eta^6} \vartheta_3^4 - \vartheta_3^2,
\]

which is an arduous manipulation with theta functions identities (left to the reader).

### 6.6.6 Lemmas and arithmetic results:

**Lemma 6.6.3.**

\[ \vartheta_3(z|\tau) = \vartheta_3(2z|4\tau) + \vartheta_2(2z|4\tau) \]
\[ \vartheta_4(z|\tau) = \vartheta_3(2z|4\tau) - \vartheta_2(2z|4\tau) \]

*Proof.* Directly from Fourier expansion. 

**Lemma 6.6.4.**

\[ \vartheta_2(0|\tau) = \vartheta_2(0|9\tau) + 2q^{1/2} \vartheta_2(3\tau|9\tau) \]
\[ \vartheta_3(0|\tau) = \vartheta_3(0|9\tau) + 2q^{1/2} \vartheta_3(3\tau|9\tau) \]
\[ \vartheta_4(0|\tau) = \vartheta_4(0|9\tau) - 2q^{1/2} \vartheta_4(3\tau|9\tau) \]
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\textbf{Proof.} Idem. For instance, the middle line goes like

\[ \text{l.h.s.} = \sum q^{(3n)^2/2} + \sum q^{(3n+1)^2/2} + \sum q^{(3n+2)^2/2} = \sum q^{(3n)^2/2 + 2} \sum q^{(3n+1)^2/2} = \text{r.h.s.} \]

\( \square \)

\textbf{Lemma 6.6.5.}

\[ \vartheta_3\left( \frac{1}{3} \right) = \vartheta_3(0|6\tau) - q^{1/3} \vartheta_3(2\tau|6\tau) = \vartheta_3(0|6\tau) - q^{1/12} \vartheta_3(\tau|6\tau) \]

\[ \vartheta_2\left( \frac{1}{3} \right) = -\vartheta_2(0|6\tau) + q^{1/3} \vartheta_2(2\tau|6\tau) = -\vartheta_2(0|6\tau) + q^{1/12} \vartheta_3(\tau|6\tau) \]

\textbf{Proof.} Idem. For instance, the first line:

\[ \text{l.h.s.} = \sum q^{n^2/3} e^{2\pi in/3} = \sum q^{3n^2/3} + (e^{2\pi i/3} + e^{-2\pi i/3}) \sum q^{3(n+\frac{1}{3})^2/3} = \text{r.h.s.} \]

\( \square \)

\textbf{Lemma 6.6.6.}

\[ \vartheta_3(z|\tau) \vartheta_3(z'|\tau) + \vartheta_2(z|\tau) \vartheta_2(z'|\tau) = \vartheta_3(z + z'|\tau) \vartheta_3(z - z'|\tau) \]

\[ \vartheta_3(z|\tau) \vartheta_3(z'|\tau) - \vartheta_2(z|\tau) \vartheta_2(z'|\tau) = \vartheta_4(z + z'|\tau) \vartheta_4(z - z'|\tau) \]

\textbf{Proof.} Idem. For instance, the first line:

\[ \text{l.h.s.} = \sum q^{(m^2+n^2)/2} y^m y^n = \sum q^{(k^2+l^2)/2} y^{(k+l)/2} y^{(k+l)/2} = \text{r.h.s.} \]

where we made the substitution \( k = m + n, \ l = m - n. \)

For the second line, one just needs to introduce \((-1)^{2n}\) and \((-1)^{k-l}\) in the two sums respectively.

\( \square \)

\textbf{Lemma 6.6.7.}

\[ \vartheta_3(0|6\tau) \vartheta_3(\tau|6\tau) - \vartheta_2(0|6\tau) \vartheta_2(\tau|6\tau) = q^{-1/12} \eta^2 \]

\[ \vartheta_3(0|\frac{1}{3}) \vartheta_3(\frac{1}{3}|\frac{1}{3}) - \vartheta_2(0|\frac{1}{3}) \vartheta_2(\frac{1}{3}|\frac{1}{3}) = 6\eta^2 \]

\[ \vartheta_3(0|\frac{2}{3}) \vartheta_3(\frac{1}{3}|\frac{2}{3}) + \vartheta_2(0|\frac{2}{3}) \vartheta_2(\frac{1}{3}|\frac{2}{3}) = 3\eta^2 \]

\[ \vartheta_3(0|\frac{2}{3}) \vartheta_4(\tau|\frac{2}{3}) - \vartheta_4(0|\frac{2}{3}) \vartheta_3(\tau|\frac{2}{3}) = -2q^{-1/3} \eta^2 \]

\textbf{Proof.} The first line is just lemma 6.6.6 with \( z = 0 \) and \((z'|\tau)\) replaced by \((1 + \tau|6\tau)\). The second line is obtained from the first by \( S \) transformation, \( i.e. \tau \rightarrow -1/\tau \). The third line is obtained by rewriting the second line as \( ab + cd = (a + c)(b + d)/2 + (a - c)(b - d)/2 \) and using lemma 6.6.3. The fourth line is again an \( S \) transformation of the previous line. Applying the rewriting trick on this last line, we would of course fall back on the first line.

We note that the first line in Fourier series gives us an interesting formula:

\[ \eta^2 = \sum (-1)^n q^{3(m^2+n^2)} \text{ where } (m,n) \in \left( \frac{1}{2}, \frac{1}{3} \right) \cup (0, \frac{1}{3}) + \mathbb{Z}^2 \]. This is to be compared with the previous formula 6.4.1. The crucial difference is that we presently have a positive definite quadratic form in the exponent of \( q \), whereas previously the form was indefinite (this accounts for the extra constraint on \( x, y \) there).

In the first line, use lemma 6.6.4 to replace \( 2q^{1/12} \vartheta_3(\tau|6\tau) \) by \( \vartheta_2(0|\frac{2}{3}) \vartheta_3(\tau|\frac{2}{3}) \) and similarly for \( 2q^{1/12} \vartheta_2(\tau|6\tau) \), and obtain:
6.7. COMPUTATIONS IN MIXED THEORIES

Lemma 6.6.8.

\[ \vartheta_2(0|\frac{2\tau}{3}) \vartheta_3(0|6\tau) - \vartheta_3(0|\frac{2\tau}{3}) \vartheta_2(0|6\tau) = 2\eta^2 \]

\[ \vartheta_4(0|\frac{2\tau}{3}) \vartheta_3(0|\frac{\tau}{6}) - \vartheta_3(0|\frac{2\tau}{3}) \vartheta_4(0|\frac{\tau}{6}) = 4\eta^2, \]

where an \( S \) transformation connects the two lines. Unlike in the previous lemma, performing the rewriting trick will not give two more variants; here this trick is just equivalent to the \( S \) transformation itself.

Now, in the third line of lemma 6.6.7, use lemma 6.6.5 to replace \( \vartheta_2(\frac{1}{3}|\frac{2\tau}{3}) \) by \(-\vartheta_2(0|6\tau) + q^{1/3} \vartheta_2(2\tau|6\tau)\) and similarly for \( \vartheta_3(\frac{1}{3}|\frac{2\tau}{3})\), and obtain the first line of

Lemma 6.6.9.

\[ \vartheta_3(0|\frac{2\tau}{3}) \vartheta_3(\tau|6\tau) - \vartheta_2(0|\frac{2\tau}{3}) \vartheta_2(\tau|6\tau) = q^{-1/12} \eta^2 \]

\[ \vartheta_3(0|\frac{2\tau}{3}) \vartheta_3(\frac{1}{3}|\frac{\tau}{6}) - \vartheta_4(0|\frac{3\tau}{2}) \vartheta_4(\frac{1}{3}|\frac{\tau}{6}) = 2\eta^2 \]

\[ \vartheta_3(0|6\tau) \vartheta_2(\frac{2}{3}|\frac{2\tau}{3}) + \vartheta_3(0|6\tau) \vartheta_3(\frac{2}{3}|\frac{2\tau}{3}) = \eta^2 \]

\[ \vartheta_4(0|\frac{\tau}{6}) \vartheta_3(\tau|\frac{3\tau}{2}) - \vartheta_3(0|\frac{\tau}{6}) \vartheta_4(\tau|\frac{3\tau}{2}) = q^{-1/3} \eta^2. \]

Proof. Again, the successive lines are obtained by \( S \) transformation, the rewriting trick, and \( S \) transformation.

\[ \square \]

6.7. Computations in Mixed Theories

We have met with success the construction of \( N=4 \) characters in pure Gepner models like \( 1^6, 2^4, 4^3 \) theories. Other Gepner models for the \( N=4 \) SCFT on \( K3 \) are mixed tensor products of \( N=2 \) theories, like \( 1^3 2^2, 1^4 4^2, 1^2 4^2, 2^6, 1^2 2^4, \ldots \). All these products \( k_1^{n_1} \ldots k_j^{n_j} \) are formed with the requirement that the central charge equal six: \( c = \sum n_i \frac{3k_i}{k_i + 2} = 6 \). We shall investigate the first three cases of such theories with mixed levels \( k_i \) and see that they do not necessarily share the previous structure characteristic of pure theories. Specifically, the notion of gravitational, massless and massive orbits, with values at \( z = \frac{1+\tau}{2} \) equal to \( 2q^{-1/4}, -q^{-1/4} \) and 0 respectively – expected from (6.2.4), only applies to \( K3 \) models like the \( 1^4 4^2 \) or \( 1^2 2^4 \) theories below. The other CY twofold, the complex torus, gives a model whose orbits all vanish at \( z = \frac{1+\tau}{2} \), yielding a zero Euler characteristic as in the \( 1^2 2^2 \) theory below.

6.7.1 Computations in \( 1^3 2^2 \) Theory

The NS\(_1\) orbits are tensor products of three orbits from the \( k = 1 \) theory (section 6.4) and two orbits from the \( k = 2 \) theory (section 6.5). The former theory has characters \( A, B, C \) (with powers of \( y \) in \( Z + 0, \frac{1}{3}, \frac{2}{3} \) resp.), while the latter's characters we denote by \( \bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F} \) (with powers of \( y \) in \( Z + 0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4} \) resp.). In order to have only integer powers of \( y \) and cyclic permutation in the orbits, the only possible combinations are products of \( ABC, A^2 + B^3 + C^3 \) with \( \bar{A} \bar{C} + \bar{B} \bar{D}, \bar{E} \bar{F}, \bar{A}^2 + \bar{C}^2 + \bar{B}^2 + \bar{D}^2 \). Each of these products vanishes at \( z = \frac{1+\tau}{2} \), so all orbits are massive and the Dirac index vanishes. This is to be expected as
both the $I^3$ and the $I^2$ theories are toroidal models (with complex and Kähler moduli $\tau = \rho = e^{2\pi i/3}$ and $\tau = \rho = i$ resp.), hence so is their tensor product. That is, the target space of the sigma model is not $K3$ but a complex two-torus.

6.7.2 Computation in $I^4$ Theory

We denote the characters of the $k = 4$ theory by a bar over the letters. Our usual two rules (integer powers of $y$ and invariance under cyclic permutation) restrict the orbits to be of the following form:

\[
\begin{align*}
\text{NS}_1 &= A^4(\bar{A} + \bar{D}) + B^4(\bar{B} + \bar{E}) + C^4(\bar{C} + \bar{F}) \\
\text{NS}_2 &= A^3B(\bar{B} + \bar{E}) + B^3C(\bar{C} + \bar{F}) + C^3A(\bar{A} + \bar{D}) \\
\text{NS}_3 &= A^3C(\bar{C} + \bar{F}) + B^3A(\bar{D} + \bar{A}) + C^3B(\bar{E} + \bar{B}) \\
\text{NS}_4 &= A^2B^2\bar{G} + B^2C^2\bar{H} + C^2A^2I \\
\text{NS}_5 &= ABC(\bar{A}\bar{H} + \bar{B}\bar{I} + \bar{C}\bar{G}) \\
\text{NS}_6 &= A^4\bar{H} + B^4\bar{I} + C^4\bar{G} \\
\text{NS}_7 &= A^3B\bar{I} + B^3C\bar{G} + C^3A\bar{H} \\
\text{NS}_8 &= A^3CG + B^3AH + C^3BI \\
\end{align*}
\]

Due to the relations between the $I^3$ characters and the $I^6$ characters established in (6.6.1), the orbits 2,3,4 are equal, and so are the orbits 6,7,8. Thus we obtain consecutively the orbits NS$_1$, NS$_2$, 3 NS$_3$ and NS$_4$ of $I^6$ theory (6.4.3), which proves the equivalence of both models!

The coefficients $D_i = S_{1,i}/S_{i,1}$ defined after (6.4.4) are (1; 4,4,12; 24; 2,8,8), where $S_{1,1} = (6;6,6,3;3,3,3)$ are the numbers of terms in each orbit and $S_{1,i} = (1;4,4,6;12;1,4,4)$ is $S_{1,1}$ times the number of permutations of the factors in any term of orbit NS$_i$ (look only at the $I^4$ factors). Thus for the Dirac index, we have correctly $-2$ NS$_1 + (4+4+12)$ NS$_2$, as in (6.4.9).

6.7.3 Computation in $I^24^2$ Theory

This time, we are even allowed to include the characters $J, K, \ldots, O$ from the $k = 4$ theory. The orbits take the form

\[
\begin{align*}
\text{NS}_1 &= A^2(\bar{A}^2 + \bar{D}^2) + B^2(\bar{B}^2 + \bar{E}^2) + C^2(\bar{C}^2 + \bar{F}^2) \\
\text{NS}_2 &= A^2(\bar{I}^2 + \bar{D}^2) + B^2(\bar{J}^2 + \bar{M}^2) + C^2(\bar{K}^2 + \bar{N}^2) \\
\text{NS}_3 &= A^2(\bar{B} + \bar{E})\bar{G} + B^2(\bar{C} + \bar{F})\bar{H} + C^2\bar{D} + \bar{A})I \\
\text{NS}_4 &= A^2(\bar{C} + \bar{F})I + B^2(\bar{D} + \bar{A})\bar{G} + C^2(\bar{E} + \bar{B})\bar{H} \\
\text{NS}_5 &= A\bar{B}\bar{G}^2 + B\bar{C}\bar{H}^2 + C\bar{A}\bar{I}^2 \\
\text{NS}_6 &= AB(\bar{A}B + \bar{D}\bar{E}) + BC(\bar{B}\bar{C} + \bar{E}\bar{F}) + CA(\bar{C}\bar{D} + \bar{F}\bar{A}) \\
\text{NS}_7 &= AB(\bar{I}\bar{J} + \bar{O}\bar{M}) + BC(\bar{M}\bar{K} + \bar{J}\bar{N}) + CA(\bar{N}\bar{L} + \bar{K}\bar{O}) \\
\text{NS}_8 &= A^2\bar{R}^2 + B^2\bar{T}^2 + C^2\bar{G}^2 \\
\text{NS}_9 &= AB(\bar{C} + \bar{F})\bar{G} + BC(\bar{A} + \bar{D})\bar{H} + CA(\bar{B} + \bar{E})I \\
\text{NS}_{10} &= AB\bar{H}I + BC\bar{I}\bar{G} + C\bar{A}\bar{H} \\
\text{NS}_{11} &= A^2(\bar{B}\bar{C} + \bar{E}\bar{F}) + B^2(\bar{C}\bar{D} + \bar{F}\bar{A}) + C^2(\bar{D}\bar{E} + \bar{A}\bar{B}) \\
\text{NS}_{12} &= A^2(\bar{B}\bar{F} + \bar{E}\bar{C}) + B^2(\bar{C}\bar{A} + \bar{F}\bar{D}) + C^2(\bar{D}\bar{B} + \bar{E}\bar{A}) \\
\text{NS}_{13} &= A^2\bar{I}\bar{G} + B^2\bar{G}\bar{H} + C^2\bar{H}\bar{I} \\
\text{NS}_{14} &= A^2(\bar{A} + \bar{D})\bar{H} + B^2(\bar{B} + \bar{E})I + C^2(\bar{C} + \bar{F})\bar{G} \\
\text{NS}_{15} &= AB(\bar{B} + \bar{E})\bar{H} + BC(\bar{C} + \bar{F})I + CA(\bar{A} + \bar{D})\bar{G} \\
\text{NS}_{16} &= AB(\bar{A} + \bar{D})I + BC(\bar{B} + \bar{E})\bar{G} + CA(\bar{C} + \bar{F})\bar{H} \\
\end{align*}
\]
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\( NS_{17} = A^2 D + B^2 D \bar{E} + C^2 \bar{F} \)
\( NS_{18} = A^2 D + B^2 J M + C^2 K N \)
\( NS_{19} = AB \bar{C} \bar{D} + BC \bar{E} \bar{A} + CA \bar{E} \bar{B} \)
\( NS_{20} = AB(\bar{A} \bar{E} + \bar{D} B) + BC(B \bar{F} + \bar{E} C) + CA(\bar{C} \bar{A} + \bar{F} D) \)
\( NS_{21} = AB(\bar{L} M + \bar{O} J) + BC(M \bar{N} + J K) + CA(\bar{N} O + K \bar{L}) \)
\( NS_{22} = A^2 (J K + M \bar{N}) + B^2 (\bar{K} L + \bar{N} O) + C^2 (L \bar{M} + \bar{O} J) \)
\( NS_{23} = A^2 (J N + M \bar{K}) + B^2 (\bar{K} \bar{O} + N \bar{L}) + C^2 (L \bar{J} + \bar{O} M) \)
\( NS_{24} = AB \bar{K} \bar{N} + BC \bar{L} \bar{O} + CA \bar{J} \bar{M} \)
\( NS_{25} = AB(K^2 + \bar{N}^2) + BC(L^2 + \bar{O}^2) + CA(J^2 + \bar{N}^2) \)
\( NS_{26} = AB(C^2 + F^2) + BC(\bar{D}^2 + \bar{A}^2) + CA(E^2 + B^2) \)

At \( z = \frac{1+\tau}{2} \), the first two orbits give \( 2q^{-1/4} \) and \(-2q^{-1/4} \) resp., while the next five orbits give \( q^{-1/4} \); the other orbits all give 0. This embarrassing second orbit prevents us to classify it as either a graviton, massless or massive orbit. The coefficients \( D_i \) are \((1,1,2,2,4,4,4; 2,4,8,2,2,4,2,4,2,8,2,2)\). So the value of the elliptic genus at \( z = 0 \) is \( \Phi(0) = \sum_{i=1}^{d} |R_i(0)|^2 = \sum_{i=1}^{d} D_i T_i^2 = 4D_1 + 4D_2 + D_3 + \cdots + D_7 = 24 \), so this is a K3 model with the appropriate Euler character.

Due to the relations between the \( 4^3 \) characters and the \( 1^6 \) characters established in (6.6.1), we find the following relations between the orbits:

<table>
<thead>
<tr>
<th>( 1^2 4^2 ) theory</th>
<th>( 4^3 ) theory</th>
<th>( 1^2 4^2 ) theory</th>
<th>( 4^3 ) theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( NS_1 )</td>
<td>( NS_1 + NS_7 )</td>
<td>( NS_8 = NS_9 = NS_{10} )</td>
<td>( NS_{14} = 3 NS_{10} )</td>
</tr>
<tr>
<td>( NS_2 )</td>
<td>( NS_4 + NS_6 )</td>
<td>( NS_{11} )</td>
<td>( 2 NS_8 )</td>
</tr>
<tr>
<td>( NS_3 = NS_4 = NS_5 )</td>
<td>( NS_3 + NS_{13} )</td>
<td>( NS_{12} )</td>
<td>( NS_8 + 3 NS_9 )</td>
</tr>
<tr>
<td>( NS_6 )</td>
<td>( NS_3 )</td>
<td>( NS_{13} = NS_{14} = NS_{15} )</td>
<td>( NS_{15} = NS_{12} + 2 NS_{11} )</td>
</tr>
<tr>
<td>( NS_7 )</td>
<td>( NS_5 )</td>
<td></td>
<td>( NS_{16} )</td>
</tr>
</tbody>
</table>

In addition, some of these orbits match even those of \( 1^6 \) theory:

<table>
<thead>
<tr>
<th>( 1^2 4^2 ) theory</th>
<th>( 4^3 ) theory</th>
<th>( 1^6 ) theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( NS_1 + 2 NS_{16} )</td>
<td>( NS_1 + 3 NS_7 )</td>
<td>( NS_1 )</td>
</tr>
<tr>
<td>( NS_3 = NS_4 = NS_5 = NS_8 + NS_{20} )</td>
<td></td>
<td>( NS_{13} )</td>
</tr>
<tr>
<td>( NS_8 = NS_9 = NS_{10} )</td>
<td>( NS_{14} = 3 NS_{10} )</td>
<td>( 3 NS_3 )</td>
</tr>
<tr>
<td>( NS_{11} + NS_{12} )</td>
<td>( 3 (NS_8 + NS_9) )</td>
<td>( 3 NS_3 )</td>
</tr>
<tr>
<td>( NS_{13} = NS_{14} = NS_{15} )</td>
<td>( NS_{15} = NS_{12} + 2 NS_{11} )</td>
<td>( NS_4 )</td>
</tr>
</tbody>
</table>
CHAPTER 6. N=4 CHARACTERS AND ELLIPTIC GENERA