Gromow-Witten Invariants and Elliptic Genera
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Chapter 7

Open String Instantons

Much as chapters 1 and 2 laid the foundations for closed string instantons, this chapter counts instantons for open strings. The starting point is the super-potential (7.1.1), with $n^2$ in the denominator, which should be compared to its closed-string counterpart (2.3.4), the prepotential $F_0 = \sum_{d>0} n_d^0 \text{Li}_3(q^d) = \sum_{n,d>0} \frac{1}{n^3} n_d^0 q^{dn}$ with $n^3$ in the denominator. We admit this result, the only one needed for our treatment, and it will spare us a cumbersome equivalent to the closed-string derivations of chapter 2.

The techniques used here have nothing in common with their closed-strings counterparts. They are more physical in substance, especially as they involve the intuition from the duality between the B-model of topological strings and Chern-Simons theory. In particular, instead of the closed-string notation $n_d^r$ for the BPS-invariants, we denote the open-string invariants here by the more customary $d_{k,m}$. This chapter draws on [AV-00] and on [G2-03], can be read independently and will not be used in the sequel.

7.1 Background

Open string instantons are holomorphic maps from Riemann surfaces with boundaries to the CY threefold target space. It is understood that the boundaries of the instanton end on special Lagrangian1 submanifolds of the threefold. In

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1Let us recall the lengthy definitions impelled by special Lagrangian submanifolds (see [J-00] for instance): A closed k-form $\varphi$ on an m-dimensional Riemannian manifold $M$ is a calibration iff for every (oriented) k-plane $V$ (subspace of $T_x M$) we have $\varphi|_V \leq \text{vol}_V$ (volume form on $V$). Then, an oriented submanifold $N \subset M$ is a calibrated iff $\varphi|_{T_x N} = \text{vol}_{T_x N}$ for all $x \in N$.

In $\mathbb{C}^n$ with complex coordinates $(z_1, \ldots, z_m)$ define the real 2-form $\omega := \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m)$ and the complex m-form $\Omega := dz_1 \wedge \cdots \wedge dz_m$ with $\text{Re}\Omega$ and $\text{Im}\Omega$ real m-forms. A real m-dimensional submanifold $L$ of $\mathbb{C}^n$ is Lagrangian iff $\omega|_L \equiv 0$, and it is special Lagrangian if it also satisfies $\text{Im}\Omega|_L \equiv 0$. An alternative definition is: $L$ is special Lagrangian iff it is calibrated w.r.t. $\text{Re}\Omega$.

Similarly, for a CY m-fold $M$ with Kähler form $\omega$ and holomorphic $(m,0)$-form $\Omega$, a real m-dimensional submanifold $N \subset M$ is called special Lagrangian iff $\omega|_N \equiv \text{Im}\Omega|_N \equiv 0$ (that
other words, we are interested in the problem of counting holomorphic disks with boundary on a Lagrangian submanifold.

In [AV-00], Aganagic and Vafa used Mirror Symmetry to determine these open string instantons: the B-model superpotential can be computed exactly and mapped to the A-model superpotential in the large volume limit of the CY threefold. The latter cannot be computed exactly, but contains instanton corrections, i.e. holomorphic disks ending on "A-model branes" (also called A-branes); comparison with the B-model superpotential allows us to determine the contribution of these instantons and their degeneracy.

The disk amplitude in the large volume limit (i.e. $e^v \to 0$), which in the type II context has the interpretation of superpotential corrections to $4d$ N=1 susy (see next section), is expected to be of the form [OV-99] (A-model superpotential):

$$W_A = \sum_{n \geq 1} \frac{d_{k,m}}{n^2} q^n y^{nm},$$

(7.1.1)

where $q = e^{-t}$ and $y = e^v$ are the (exponentiated) closed and open string complexified Kähler classes, measuring respectively the volume of compact curves and holomorphic disks embedded in the threefold. The coefficients $d_{k,m}$ are the numbers of primitive holomorphic disks labelled by the classes $k$ and $m$ – two vectors in the homologies $H_2$ of the threefold and $H_1$ of the brane respectively.

The tables given by [AV-00] exhibit the integrality of the coefficients $d_{k,m}$ for their two examples of threefolds: the resolved conifold and the degenerate $\mathbb{P}^1 \times \mathbb{P}^1$. These one-modulus cases are particularly simple due to the simple nature of the mirror map: for $\mathbb{P}^1$ (both examples), the relation between $t$ and $\hat{t}$ is rational: $q = \hat{q}/(1 + \hat{q})^2$. Such a closed form is not the rule, and all our other examples will exhibit the mirror map as a power series solution to the Picard-Fuchs differential equation.

The next four sections are a reminder of the method used by [AV-00]; it rests on the equivalence of the A- and B-model under mirror symmetry. On one hand, the A-model string amplitude is re-interpreted in topological string theory as counting holomorphic maps from Riemann surfaces (with boundary) to the target space; on the other hand, the B-model amplitude is obtained via Chern-Simons reduction to the world-volume of the B-brane.

The two last sections are original work; we prove the integrality of the open string instanton numbers for two examples from [AV-00]: the resolved conifold and the degenerate local $\mathbb{P}^1 \times \mathbb{P}^1$.  

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is, if it is calibrated w.r.t. $\text{Re}\Omega|_N$.
7.2 The A-model

For the A-model, we consider a $U(1)$ linear sigma model, i.e. a complex Kähler manifold $Y$ obtained by quotienting the hypersurface

$$L_Y := \{ \sum_{i=1}^{n} Q_i |\Phi_i|^2 = r^2 \} \quad (7.2.1)$$

of $\mathbb{C}^n$ by a $U(1)$ subgroup of the isometry group of $\mathbb{C}^n$. The charges $Q_i$ are integers, and if they sum up to 0, $Y$ is a complex $(n-1)$ dimensional CY manifold (non-compact, as the directions with negative charge are non-compact).

More generally, we view $\mathbb{C}^n$ as a torus fibration $T^n \to L$, where the base is just $\mathbb{R}^n$ parametrised by the $|\Phi_i|$. We can also consider a real $k$-dimensional subset $L_Y$ of $L$ given by $(n-k)$ equations (7.2.1) for $(n-k)$ sets of charges $Q_i^\alpha$, and then divide the fibration by $U(1)^{n-k}$ to obtain a fibration $Y = (T^k \to L_Y)$ which is a complex $k$-dimensional non-compact CY manifold.

Note that the base $L_Y$ is a Lagrangian submanifold of $Y$: since $L$ is given by fixing values of the arguments $\theta^i$ of the complex variables $\Phi^i$, the Kähler form $\omega = \sum_{i=1}^{n} d|\Phi_i|^2 \wedge d\theta^i$ vanishes on it. The base $L_Y$ is our first example of a Dk-brane.

Other examples of Dk-branes are obtained by considering rational linear subspaces of $L_Y$, i.e. submanifolds $D^r \subset L_Y$ of real dimension $r \leq k$, given by $(k-r)$ constraints

$$\sum_{i=1}^{n} q_i^\alpha |\Phi_i|^2 = c^\alpha \quad (7.2.2)$$

with integers $q_i^\alpha$, $\alpha = 1, \ldots, k-r$. Since the slope of $D^r$ is rational, the $(k-r)$-dimensional subspace of the fibre $T^k$ above any point of $D^r$ and orthogonal - w.r.t. $\omega$ - to $T_pD^r$ is itself a torus $T^{k-r}$. That is, we have a new Dk-brane given by the fibration $T^{k-r} \to D^r$. As a submanifold of $Y$, it is special Lagrangian iff $\sum q_i^\alpha = 0$. All these special Lagrangian submanifolds of $Y$ are called A-branes.

It is known [KKLM-99] that type IIA string theory on a CY threefold $Y$, with 6-branes filling space-time and wrapping special Lagrangian cycles of $Y$ leads to a 4d N=1 susy with a superpotential entirely generated by open-string worldsheet instantons. For the D-brane RR charges to satisfy Gauss' law, we need the $Y$ to be non-compact so as to offer directions for the D-brane flux to escape. This configuration corresponds to our A-model, so that the above disk amplitude (7.1.1) computes corrections to the 4d N=1 superpotential.

7.3 The B-model

The mirror equation to (7.2.1) is written in terms of $n$ dual twisted chiral fields $Y_i$, related to the original fields by the mirror map $\text{Re}(Y_i) = -|\Phi_i|^2$, as in [HV-00]:

$$\sum_{i=1}^{n} Q_i Y_i = -t \quad \text{or} \quad \prod_{i=1}^{n} y_i^{Q_i} = e^{-t} \quad \text{with} \quad y_i := e^{Y_i} \quad (7.3.1)$$
where \( t := r + i\theta \) is the complexified Kähler parameter, i.e. the Fayet-Iliopoulos term \( r \) of (7.2.1) combined with the \( U(1) \theta \) angle. The \( y_i \) are homogeneous coordinates for \( \mathbb{P}^{n-1} \). When \( L_Y \) is given by a set of \((n - k)\) equations, (7.3.1) also consists of \((n - k)\) equations for different Kähler parameters \( t_\alpha \).

Note that (7.3.1) is not yet the equation of the mirror CY space. The B-model is a Landau-Ginsburg theory with superpotential

\[
W(y_i) = \sum_{i=1}^{n} y_i,
\]

in which \((n - k)\) of the complex variables \( y_i \) can be substituted by (7.3.1), leaving just \( k \) of them. The mirror CY space is compact or not according to whether we add or not a gauge-invariant superpotential term \( PG(\phi_i) \) to the original theory. In the first case, the CY is given by an orbifold of the hypersurface \( \{ W(y_i) = 0 \} \), thus \((k - 2)\)-dimensional. In the second case, it is given by \( \{ W(y_i) = xz \} \), where \( x, z \) are affine (and not projective!) coordinates giving rise to non-compact directions, thus \( k\)-dimensional. (Note that sometimes the \( y_i \) variables occurring in \( W(y_i) \) are rescaled to new variables \( \tilde{y}_i \) such that these appear with powers different from 1.)

As for the B-brane, the mirror of equation (7.2.2) is

\[
\prod_{i=1}^{n} y_i^{\alpha_\alpha} = e^\alpha e^{-\alpha}, \quad \alpha = 1, \ldots, k - r, \quad (7.3.2)
\]

as a subspace of the mirror CY. We have allowed a phase \( e^\alpha \) to occur; in other words, we have complexified \( e^\alpha \). Thus the B-brane is a holomorphic submanifold of complex dimension \( k - (k - r) = r \), i.e. it is a D(2r)-brane, where \( r \) was the real dimension of the base of the A-brane.

### 7.4 Topological Strings and Chern-Simons action

In order to extract instanton numbers from our description of A-branes and their mirror B-branes, we need an alternative way of computing the B-model superpotential. We find salvation in topological string theory, where the A-model string amplitude counts holomorphic maps from Riemann surfaces with boundary to the target space with the boundary ending on A-branes, while the B-model amplitude computes the holomorphic Chern-Simons action reduced to the world-volume of the B-brane [OV-99]. Hence it only works for the CY threefolds, i.e. from now on we restrict to \( k = 3 \). Hence the A-brane is a D3-brane (special Lagrangian); the B-brane is a D2-brane if we have 2 charges \( q_1, q_2 \) or a D4-brane if we have 1 charge \( q_1 \).

Thus, the B-model counterpart of the superpotential (7.1.1) is the holomorphic CS action for \( N \) B-branes wrapping the mirror CY threefold \( Y^* \):

\[
W_B = \int_{Y^*} \Omega \wedge \text{Tr} \left[ A\bar{\partial}A + \frac{2}{3} A^3 \right] \quad (7.4.1)
\]
for a holomorphic $U(N)$ gauge field $A \in H^{0,1}(Y^*, \text{adj})$ with values in the adjoint of $U(N)$. This action was for a 6-brane wrapping all of $Y^*$; but since our B-branes are 2- or 4-dimensional, we shall have to dimensionally reduce the action.

We shall be interested in the cases where the B-brane is a D2-brane, i.e. a holomorphic curve $C$; that is the case $r = 1$ with $r$ being the real dimension of the base of the A-brane. Then the components of the gauge field $A$ are sections of the normal bundle $N(C)$, call them $s^i$, and it is not difficult to show that the reduced Chern-Simons action is

$$W_B(C) = \int_C \Omega_{ijz} s^i \bar{\partial}_z s^j dz \bar{dz},$$

Note that the sections $s^i$ parametrise deformations of $C$, and if we want to keep $C$ holomorphic we need $s^i$ to be holomorphic too (though of course the original CS action also makes sense for non-holomorphic $s^i$). Yet the reduced CS action will vanish in the light of $\bar{\partial}_z s^i(z) = 0$. This is clearly unattractive for our purposes. A way of obtaining a non-vanishing result is to take $C$ non-compact with boundary conditions for the B-brane at infinity: these are obstructions to holomorphic deformations of $C$ and will yield non-zero values for the variation of $W_B(C)$ under such deformations.

Non-compactness B-brane $C$ implies a non-compact mirror CY given, by $\{W(y_i) = xz\}$ for $k = 3$ homogeneous coordinates $y_i$. This equation reads $\{F(u, v) = xz\}$ for two affine complex variables $u, v$, say $y_1 = e^u, y_2 = e^v$. Since $r = 1$, the B-brane is given by $k - 1 = 2$ equations in the variables $y_i$, hence fixing $W(y_i)$ or $F(u, v)$ to a constant value. If this value is 0, the B-brane will split into two submanifolds $\{x = 0\}$ and $\{z = 0\}$ and hence deformations will be obstructed (as otherwise the brane would pick up a boundary) and the B-model superpotential $W_B(C)$ will not vanish, as desired.

Note that we can similarly obtain configurations where the A-brane will split in two: for instance, a charge $q = (1, -1, 0, \ldots, 0)$ restricts the Lagrangian submanifold $L_Y$ to $\{\Phi_1^2 - \Phi_2^2 = c\}$ and for vanishing $c$ the A-brane will enter a phase where it splits into $\{\Phi_1 = \Phi_2\}$ and $\{\Phi_1 = -\Phi_2\}$.

To finish off the computation of the B-model Chern-Simons action, we fix the values of one of the affine parameters $u, v$ of the B-brane at infinity to some constant value (say $v \to v_*$ for large $|z|$). This parameter $v$ measures, on the A-model side, the size of the holomorphic disk ending on the brane. We then choose $u$ and $v$ as the two sections of the normal bundle $N(C)$. $C$ itself is parametrised by $z$, and the last variable $x$ parametrising the B-brane is set to 0. We write the holomorphic 3-form as $\Omega = udv \frac{dz}{z}$ and obtain for the above integral:

$$W_B(C) = \int_C \frac{dz}{z} u \bar{\partial}_z v dz = \int_{v_*}^v u dv.$$  

This has the form of an Abel-Jacobi map for the 1-form $udv$ on the Riemann surface $\{F(u, v) = 0\}$, each point of which parametrises a different B-brane $C$. 

Thus, comparing the A- and B-models:

\[ \partial_v W_B = u = \ldots \{ F(u, v) = 0 \} \ldots \]  

\[ = \partial_v \left( \sum_{n \geq 1} \sum_{k,m} \frac{d_{k,m}}{n^2} (e^{-t})^{nk}(e^{v})^{nm} \right) = \partial_v W_A \]

and the dots mean that we solve \( F(u, v) = 0 \) for \( u \) to obtain an expression dependent on \( v \) and – through (7.3.1) – on \( e^{-t} \).

### 7.5 Example: The Resolved Conifold

We now turn to a non-compact example of a CY threefold, namely the resolved conifold: \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P} \), a rank two concave bundle over the complex line. The four fields \( \Phi_1, \ldots, \Phi_4 \) have charges \((1,1,-1,-1)\). The \( H_2 \) of the CY threefold is thus \( H_2(\mathbb{P}) = \mathbb{Z} \), while the A-brane is a Lagrangian submanifold cutting the base \( \mathbb{P} \) in a circle \( S^1 \), and \( H_1(S^1) = \mathbb{Z} \). Thus both \( k \) and \( m \) are merely integers and \( t, v \) merely complex numbers.

#### 7.5.1 The Instanton Numbers \( d_{k,m} \)

The input from the B-model is an explicit expression for the derivative of the superpotential [AV-00] (see [G2-03] for details of the double-series expansion):

\[
\partial_v W = \log \left( \frac{1 - e^v}{2} + \frac{1}{2} \sqrt{(1 - e^v)^2 + 4e^{-t+v}} \right),
\]

\[
= \log \left( \frac{1 - y}{2} + \frac{1}{2} \sqrt{(1 - y)^2 + 4qy} \right)
= \sum_{k \geq 0, m \geq k} \frac{(-1)^{k+1}}{m+k} \binom{m+k}{m} \binom{m}{k} q^k y^m
= \sum_{k \geq 0, m \geq k} C_{k,m} q^k y^m \quad \text{with } C_{0,0} = 0,
\]

where \( v \) is the (rescaled) natural variable in the phase where the mirror B-brane degenerates to two submanifolds passing through the South pole of the resolved conifold. This \( v \) also measures the size of the minimal holomorphic disk passing through the South pole and ending on the Lagrangian submanifold. Precisely when the submanifold splits into several components can we wrap the A-brane around any of those, and guarantee that it will not deform (as it would otherwise acquire a boundary). This phase is characterised by \( e^v \rightarrow 0 \), agreeing with the large volume limit on the A-model side.

Comparing this to the A-model expression (7.1.1)

\[
\partial_v W = -\sum_{k,m} m d_{k,m} \log(1 - q^k y^m) = \sum_{k,m} \left( \sum_{l \uparrow (k,m)} \frac{d_{k,m}}{l^2} \right) q^k y^m,
\]
7.5. **EXAMPLE: THE RESOLVED CONIFOLD**

we can recursively extract the values of all $d_{k,m}$ from the relation

$$C_{k,m} = \sum_{l \mid (k,m)} d_{k,l} \frac{m}{l^2}. \quad (7.5.2)$$

This gives table 7.1. And we discover with awe that all the topological invariants

$$d_{k,m} \text{ are integers!}$$

### 7.5.2 Proof of Integrality

In [G2-03] we proved analytically their integrality by deriving first the congruences for a prime $p > 3$ and integers $n, k, l$:

$$\binom{np^l}{kp^l} \equiv \binom{np^{l-1}}{kp^{l-1}} \mod p^{2l}, \quad (p^k - 1)! \equiv -1 \mod p^l \quad (k \geq l)$$

The dash notation indicates that we skip all multiples of $p$; thus

$$p^{k!'} := p^{k!}/(p^{k-1}! \cdot (p)_{p^{k-1}}).$$

These are generalisations of Wolstenholme’s theorem

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3 \quad \text{for} \quad p > 3, \quad (\text{and} \mod p^2 \quad \forall p).$$

and of Wilson’s theorem

$$(p-1)! \equiv -1 \mod p \quad \text{for} \quad p > 2, \quad (\text{and} \quad 1 \mod p \quad \text{for} \quad p = 2).$$

The proofs of these congruences are found in appendix E. Using them, let us prove the integrality of the topological invariants $d_{k,m}$:
Proposition 7.5.3. With

\[ C_{k,m} = \frac{(-1)^{k+1}}{k} \binom{m + k - 1}{k-1} \binom{m}{k} = \frac{(-1)^{k+1}}{m+k} \binom{m+k}{k} \binom{m}{k} \]

the instanton numbers \( d_{k,m} \) in (7.5.2) are all integers.

**Proof.** We proceed step by step, according to the greatest common divisor (gcd) of \( k \) and \( m \):

1. \((k,m) = 1\): From (7.5.2) we have \( C_{k,m} = d_{k,m} \). So for \( d_{k,m} \) to be integer, we need \( C_{k,m} \) to be 0 mod \( m \). Note that in general, \((n,k) = 1 \) implies \( n|\binom{n}{k} \), since \( \binom{n}{k} \equiv \binom{n}{k} \). Thus \( C_{k,m} \in \mathbb{Z} \) and even \( m\mathbb{Z} \).

2. \((k,m) = p^l\): for \( p \) prime. This time

\[
C_{k,m} = d_{k,m} m + d_{\frac{k}{p}, \frac{m}{p}, \frac{m}{p^2}} + \cdots + d_{\frac{k}{p}, \frac{m}{p}, \frac{m}{p^{2l}}} \\
= d_{k,m} m + \frac{1}{p} \frac{C_{k, \frac{m}{p}}}{\frac{m}{p}}
\]

Thus for \( d_{k,m} \) to be integer, we need \( C_{k,m} \equiv \frac{1}{p} C_{\frac{k}{p}, \frac{m}{p}, \frac{m}{p^2}} \mod m \), i.e. \( pC_{p^{l-1}k, p^{l-1}m} \equiv C_{p^l-1k, p^l-1m} \mod mp^{l+1} \) for \((k,m) = 1\), i.e.

\[
\left( \binom{p^l(m+k)}{p^l k} \binom{p^l m}{p^l k} - \binom{p^{l-1}(m+k)}{p^{l-1} k} \binom{p^{l-1} m}{p^{l-1} k} \right) \equiv 0 \mod mp^{2l}.
\]

Lemma E.0.1 tells us that the congruence is valid mod \( p^{3l} \) \((p > 3) \) or mod \( p^{3l-1} \) \((\forall p)\), hence also mod \( p^{2l} \) for any prime \( p \). Since \( m|\binom{p^l m}{p^l k} \), both terms also contain a factor of \( m \), and the congruence is valid mod \( mp^{2l} \).

3. \((k,m) = pq\): for primes \( p \) and \( q \). Again by (7.5.2) we have

\[
C_{k,m} = d_{k,m} m + d_{\frac{k}{p}, \frac{m}{p}, \frac{m}{p^2}} + d_{\frac{k}{q}, \frac{m}{q}, \frac{m}{q^2}} + d_{\frac{k}{pq}, \frac{m}{pq}, \frac{m}{pq^2}} \\
= d_{k,m} m + \frac{1}{p} \frac{C_{k, \frac{m}{p}}}{\frac{m}{p}} + \frac{1}{q} \frac{C_{k, \frac{m}{q}}}{\frac{m}{q}} - \frac{1}{pq} \frac{C_{k, \frac{m}{pq}}}{\frac{m}{pq}} \quad (7.5.4)
\]

Thus we need \( pqC_{pqk,pqm} - qC_{qk,qm} - pC_{pk,pkm} + C_{k,m} \equiv 0 \mod mp^2q^2 \) for \((k,m) = 1\), i.e.

\[
\left( pq(m+k) \right) \left( \binom{pqm}{pqk} \binom{qkm}{qk} - \binom{p(m+k)}{pk} \binom{pm}{pk} + \binom{m+k}{k} \binom{m}{k} \right) \equiv 0 \mod mq^2p^2.
\]

Again by Lemma E.0.1, the first difference is 0 mod \( p^3 \) \((p > 3) \) or mod \( p^2 \) \((\forall p)\), so is the last difference, and we can factor out \( p^2 \), hence also \( q^2 \). As before, we can also take out a factor of \( m \), and the whole line is thus 0 mod \( mp^2q^2 \).

4. \((k,m) = pqr\): for primes \( p, q, \) and \( r \). As before, the principle of inclusion and exclusion yields the requirement

\[
pqrC_{pqrk,pqrm} - qrC_{qrk,qrm} - prC_{prk,prm} - pqC_{pqk,pqkm} + rC_{rk,rm} + qC_{qk,qm} + pC_{pk,pkm} - C_{k,m} \equiv 0 \mod mp^2q^2r^2
\]
for \((m, k) = 1\). Reasoning as above and noting that the four pairs \((pqrC_{pqrk,pqrm} - qrC_{qrk,qrm})\), \((pC_{prk,prm} - rC_{rk,rm})\), \((pqC_{pqk,pqm} - qC_{qk,qm})\) and \((pC_{pk,pm} - C_{k,m})\) are all \(0 \mod p^2\), we find that the requirement is met.

\((k, m) = p'q'\) for \(p, q\) prime. Now we have

\[
C_{k,m} = d_{k,m}m + \frac{1}{p}C_{k,\frac{m}{p}} + \frac{1}{q}C_{\frac{k}{q},\frac{m}{q}} - \frac{1}{pq}C_{\frac{k}{pq},\frac{m}{pq}},
\]

so we are back at a combination of the cases \((k, m) = p^l\) and \((k, m) = pq\), and the same reasoning will show that \(d_{k,m}\) is again integer.

Having covered the cases of \((k, m)\) being product of primes and powers of primes, inductive reasoning will show that the same conclusion will be met in the most general case where \((k, m) = p_1^{l_1} \ldots p_j^{l_j}\).

### 7.6 Example: Degenerate \(\mathbb{P} \times \mathbb{P}\)

Our second example of non-compact CY3 is a concave line bundle over two complex lines: \(O(-4) \to \mathbb{P} \times \mathbb{P}\), with Kähler moduli \(t_1, t_2\) describing the sizes of the two complex lines (or real spheres). The two sets of charges \(Q_1, Q_2\) of the five fields \(\Phi_1, \ldots, \Phi_5\) are (1, 1, 0, 0, -2) and (0, 0, 1, 1, -2). An easy mirror map is only known for the degenerate case where the size of the second \(\mathbb{P}\) goes to infinity; that is we retain only one modulus, \(t_1\), with associated variable \(q = e^{-t_1}\). And so, as in the previous example, \(k\) and \(m\) are both integers.

#### 7.6.1 The Instanton Numbers \(d_{k,m}\)

This time the input from the B-model is [AV-00]:

\[
\partial_v W = \log \left( \frac{1 + q - y}{2} + \frac{1}{2}(1 + q - y)^2 - 4q \right)
= \sum_{k \geq 0, m \geq 1} -\frac{1}{m} \binom{m + k - 1}{k} q^k y^m
\quad \text{(7.6.1)}
= \sum_{k, m \geq 0} C_{k,m} q^k y^m \quad \text{with } C_{k,0} = 0,
\]

where \(v\) is the (rescaled) natural variable in the phase where the projection of the A-brane on the base is a circle on the \(\mathbb{P}\) of infinite volume. See [G2-03] for details leading to the double series expansion.

As before, we extract the topological invariants \(d_{k,m}\) recursively as in (7.5.2) and find table 7.2. Again, the integrality of the invariants is non-trivial and encouraging. Proof of integrality goes along the lines of the previous example [G2-03]:
### Table 7.2: Holomorphic disc numbers for the A-brane on the degenerate $O(K) \to \mathbb{P}^1 \times \mathbb{P}^1$ in phase II.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$d_{0,m}$</th>
<th>$d_{1,m}$</th>
<th>$d_{2,m}$</th>
<th>$d_{3,m}$</th>
<th>$d_{4,m}$</th>
<th>$d_{5,m}$</th>
<th>$d_{6,m}$</th>
</tr>
</thead>
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<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td></td>
</tr>
<tr>
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<td>$0$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$-4$</td>
<td>$-6$</td>
<td>$-9$</td>
<td>$-12$</td>
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<tr>
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<td>$-4$</td>
<td>$-11$</td>
<td>$-25$</td>
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<tr>
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<td>$-1$</td>
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<td>$-87$</td>
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<td>$-1764$</td>
<td>$-5926$</td>
</tr>
</tbody>
</table>

7.6.2 Proof of Integrality

**Proposition 7.6.2.** With

$$C_{k,m} = -\frac{1}{m} \binom{m+k-1}{k}^2 = -\frac{1}{k} \binom{m+k-1}{k} \binom{m+k-1}{k-1} = -\frac{m}{(m+k)^2} \binom{m+k}{m}^2,$$

the instanton numbers $d_{k,m}$ in (7.5.2) are all integers.

**Proof.** Note that only the last expression for the $C_{k,m}$ is suitable for the $m = 0$ case: $C_{k,0} = 0$. We proceed again inductively on the nature of the gcd of $k$ and $m$.

$(k, m) = 1$: As before, we need $C_{k,m} \equiv 0 \bmod m$, which is readily seen from the third expression for the $C_{k,m}$.

$(k, m) = p^l$: As before, the requirement boils down to $pC_{p^l k, p^l m} \equiv C_{p^l-1 k, p^l-1 m}$ mod $mp^l$ for $(k, m) = 1$, ie

$$\frac{m}{(m+k)^2} \left[ \left( \frac{p^l(m+k)}{p^l m} \right)^2 - \left( \frac{p^l-1(m+k)}{p^l-1 m} \right)^2 \right] \equiv 0 \bmod mp^{2l},$$

ie

$$\left( \frac{p^l(m+k)}{p^l m} \right) \equiv \pm \left( \frac{p^l-1(m+k)}{p^l-1 m} \right) \equiv 0 \bmod p^{2l},$$

which is again fine by Lemma E.0.1.

$(k, m) = pq$: The same requirement as in (7.5.4) stipulates

$$\left( \frac{pq(m+k)}{pqk} \right)^2 - \left( \frac{q(m+k)}{qk} \right)^2 - \left( \frac{p(m+k)}{pk} \right)^2 + \left( \frac{m+k}{k} \right)^2 \equiv 0 \bmod q^2 p^2.$$

for $(k, m) = 1$. Again, by Lemma E.0.1, the first difference is $0 \bmod p^3$, so is the second, and similarly for mod $q^3$.

The cases $(k, m) = pqr$ and $(k, m) = p^l q$ can be imported without change from the previous example, and thus the integrality of the $d_{k,m}$ is proved for the most general case of $(k, m) = p_1^{l_1} \cdots p_j^{l_j}$.  

$\square$