Gromow-Witten Invariants and Elliptic Genera
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Appendix B

Momentum on a Lattice

B.1 Lattice $\Gamma_{1,1}$

Motivation

The quantised momentum on a lattice of signature $(1,1)$ is easiest to understand in the CFT context of free bosons on a torus. The torus is specified by the periods $1, \tau$ on the complex plane. The fermionic coordinates satisfy $\psi(w + 1) = \pm \psi(w + \tau) = \pm \psi(w)$, where the sign ambiguity in both periods allows for four different spin structures. The bosonic coordinates are assumed periodic, $X = X + 2\pi r$, i.e. compactified on the one-dimensional target space $S^1$, a circle of radius $r$. That is, they are maps $X : T^2 \to S^1$. Their compact coordinate allows for an infinite number of configurations (not just 4 as for the fermions), labelled by $n, n' \in \mathbb{Z}$: $X(w + 1) = X(w) + 2\pi rn$ and $X(w + 1) = X(w) + 2\pi rn'$.

There are two approaches to calculating the partition function $F$:

- Hamiltonian formulation: the partition function is given by a trace over the Hilbert space:
  \[ \text{tr} \ q^{L_0 - \frac{1}{2}L_0} q^{L_0 - \frac{1}{2}} \]

- Lagrangian formulation: the partition function is given by the functional integral
  \[ \int e^{-S} = \frac{1}{\eta\eta'} \sum_{n,m \in \mathbb{Z}} q^{\frac{1}{2}n^2} q^{\frac{1}{2}m^2} \]

In the latter, we have introduced the left and right components of the momentum vector $p = (p_L, p_R)$:

\[ p_L := \frac{m}{2r} + nr = \frac{p}{2} + \omega \]
\[ p_R := \frac{m}{2r} - nr = \frac{p}{2} - \omega \]
where the scalar momentum \( p = m/r \) and winding \( \omega = nr \) are quantised in terms of integers \( m \) and \( n \).

Comparison of the two approaches shows that the Hilbert space consists of sectors labelled by \( m, n \), each sector generated by the state \( |m, n\rangle \) playing the role of the ground state, \( i.e. \alpha_l|m, n\rangle = 0, \forall l > 0 \).

In the \( |m, n\rangle \) sector, the bosonic Fock subspace \( \mathcal{F}_l \) generated by the creation operator \( \alpha_{-l} \) contains the states \( |m, n\rangle, \alpha_{-l}|m, n\rangle, \alpha_{-l}|m, n\rangle, \ldots \). When the Virasoro operator \( L_0 = \sum_{l \geq 1} \alpha_{-l} \alpha_l + \frac{1}{2} \alpha_0^2 \) acts on them, we obtain \( q^{l0} \alpha_l^j |m, n\rangle = q^j \alpha_{-l} |m, n\rangle \) for \( j > 0 \), and \( q^{l0} |m, n\rangle = q^{\alpha_0^2/2} |m, n\rangle \) for \( j = 0 \). Thus the \( j > 0 \) part yields the expected \( \eta^{-1} \) of the bosonic partition function:

\[
\text{tr}_{\mathcal{F}_l} q^{L_0} = \prod_{l \geq 1} \text{tr}_{\mathcal{F}_l} q^{L_0} = \prod_{l \geq 1} (1 + q^l + q^{2l} + \ldots) = \prod_{l \geq 1} (1 - q^l)^{-1} = \frac{q^{1/24}}{\eta},
\]

independently of the sector \( |m, n\rangle \). So the remaining \( j = 0 \) part, \( i.e. \ q^{\alpha_0^2/2} \), must account for the remaining \( q^{\alpha_0^2/2} \) of the Lagrangian approach. Hence we identify \( \alpha_0 = p_l \) and \( \bar{\alpha}_0 = p_R \), that is

\[
L_0|m, n\rangle = \frac{1}{2} \alpha_0^2 |m, n\rangle = \frac{1}{2} \left( \frac{m}{2r} + nr \right)^2 |m, n\rangle,
\]

and similarly for \( \bar{L}_0 \). Then we see that the state \( |m, n\rangle \) has energy eigenvalue \( \frac{m^2}{2r} + n^2 r^2 \) under \( H = L_0 + \bar{L}_0 \), and (integer) momentum eigenvalue \( mn \) under \( \tilde{P} = L_0 - \bar{L}_0 \). Thus \( |0, 0\rangle \) is the actual ground state.

**Lattice and Inner Product**

We can rewrite the momentum vector in the basis \( \{k, \bar{k}\} \), where \( k := (\frac{1}{2r}, \frac{1}{2r}), \bar{k} := (r, -r) \) generate the even self-dual Lorentzian lattice of signature \((1, 1)\), denoted by \( \Gamma_{1,1} \):

\[
p = (p_L, p_R) = m \left( \frac{1}{2r}, \frac{1}{2r} \right) + n(r, -r) = \left( \begin{array}{c} m \\ n \end{array} \right)
\]

The inner product associated to \( \Gamma_{1,1} \) reads \( (x, y) \cdot (x', y') = xx' - yy' \). This implies for the vertical vectors \( \left( \begin{array}{c} m \\ n \end{array} \right) \cdot \left( \begin{array}{c} m' \\ n' \end{array} \right) = mn' + m'n \). The basis vectors satisfy \( k^2 = \bar{k}^2 = 0, k \cdot \bar{k} = 1 \), while the square momentum is \( p^2 = p_L^2 - p_R^2 = 2mn \).

**B.2 Lattice \( \Gamma_{2,2} \)**

**Motivation**

When the bosonic coordinate \( X \) is compactified on a torus instead of circle (as in the previous case of \( \Gamma_{1,1} \)), we have two radii \( r_1, r_2 \) to care about. So we have to sum over four integers \( m_1, m_2, n_1, n_2 \) to take all configurations of \( X \) into account, \( i.e. \) we end up with a Siegel theta function for the \( \Gamma_{2,2} \) lattice. Each of \( p_L, p_R \) is now a vector in \( \mathbb{R}^2 \).
In the previous section, the Narain sum over $\Gamma_{1,1}$ only depended on one real modulus $r$, reflecting the fact that the moduli space of $\Gamma_{1,1}$ lattices is one-dimensional – equal to $O(1,1)/O(1) \times O(1)$, i.e. $O(1,1)/\mathbb{Z}_2 \times \mathbb{Z}_2$. In general, we write $O(p,q)$ for the Lorentz group of linear maps preserving the metric $\text{Diag}(+,...,+,-,...,-)$, or equivalently $\text{Aut}(\Gamma_{p,q})$. Now however, we have more than just the two moduli $r_1, r_2$ for the choice of a $\Gamma_{2,2}$ lattice: the moduli space $O(2,2)/O(2) \times O(2)$ is four real-dimensional, and then one still has to mod out by $O(2,2,\mathbb{Z})$ to account for lattices related by such a transformation. In fact, this moduli space is none but the Grassmannian $\text{Gr}(2,2)$ of all choices of time-like 2-planes in $\mathbb{R}^{2,2}$; a choice of such a plane amounts to a split of $\mathbb{R}^4 \cong \mathbb{R}^{2,2}$ into the orthogonal direct sum of two real euclidean spaces $\mathbb{R}^2_1, \mathbb{R}^2_2$, and the time-like (negative norm) plane itself corresponds to $\mathbb{R}^2_T$. Whatever the point of view, we expect four real moduli, or two complex ones.

### Lattice and inner product

As advertised, we let $p = (p_L, p_R) \in \mathbb{R}^{2,2}$ such that $p^2 = p^2_L - p^2_R$ with $p^2_L, p^2_R > 0$ and $(p^2_R)^2 = -p^2_R < 0$. Then

$$p = \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} m_2 + n_2 \\ m_1 - n_1 \\ m_2 - n_2 \\ m_1 + n_1 \end{pmatrix} \in \mathbb{R}^2_L \oplus \mathbb{R}^2_R,$$

or $p = (-m_1, -n_1, m_2, n_2)$ in basis $\{e_1, f_1, e_2, f_2\}$, where $e_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $f_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $e_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $f_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. One easily checks that $e_1^2 = f_1^2 = e_1 \cdot f_1 = 0$, while $e_1 \cdot f_1 = -1$ and $e_2 \cdot f_2 = 1$.

That is, we interpret our lattice as $\Gamma_{2,2} = \Gamma_{1,1}(-1) \oplus \Gamma_{1,1}(1) = (e_1, f_1)_Z \oplus (e_2, f_2)_Z$, where $\Gamma_{1,1}(-1)$ denotes the lattice $\Gamma_{1,1}$ but with bilinear form given by $(-1, 1)$. Hence $p^2 = 2m_2n_2 - 2m_1n_1$.

More generally, for any basis $\{e_1, e_2, f_1, f_2\}$ of $\Gamma_{1,1}(-1) \oplus \Gamma_{1,1}(1)$, we can create the split $\mathbb{R}^2_L \oplus \mathbb{R}^2_R$ by choosing two vectors in $\mathbb{R}^4$ and setting $\mathbb{R}^2_R$ to be their span. Let these be $u_1, u_2$ for the complex null vector $u = u_1 + iu_2 := (T, U, 1, TU)$ with two complex moduli $T, U$ such that $T_2, U_2 > 0$. Then $u_1^2 = u_2^2 = -2T_2U_2 < 0$ and $u_1 \cdot u_2 = 0$, and we can project the vector $p = (-m_1, -n_1, m_2, n_2)$ onto $\mathbb{R}^2_R$ via $\left( \begin{pmatrix} 0 \\ p_R \end{pmatrix} \right) = \frac{p\cdot u_1}{u_1^2}u_1 + \frac{p\cdot u_2}{u_2^2}u_2$. Thus

$$p^2_R = \left( \begin{pmatrix} 0 \\ p_R \end{pmatrix} \right)^2 = -\left( \frac{p\cdot u_1}{u_1^2} \right)^2 - \left( \frac{p\cdot u_2}{u_2^2} \right)^2 = -\frac{|p\cdot u|^2}{u^2} = \frac{1}{2T_2U_2} |n_2 + m_2TU + n_1T + m_1U|^2$$

Conversely, given any $p = (p_{RL}) = (m_1, n_1, m_2, n_2)$, we can arrive at the above expression for $p^2_R$ by viewing $\mathbb{R}^2_L \oplus \mathbb{R}^2_R \cong \mathbb{C}_L \oplus \mathbb{C}_R$ (as in [MM-98], say),
i.e. viewing $p_L$ and $p_R$ as complex numbers, and choosing the basis to be
\[ e_1 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} U \end{pmatrix}, \quad e_2 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} T \end{pmatrix}, \quad f_1 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} \bar{T} \end{pmatrix}, \quad f_2 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} \bar{T}U \end{pmatrix}, \]
with associated symmetric inner product is $(\frac{x_L}{x_R}) \cdot (\frac{y_L}{y_R}) = \text{Re}(x_L\bar{y}_L - x_R\bar{y}_R)$. One easily checks that $e_i \cdot f_j$, etc, are as above. Hence $p^2 = |p_L|^2 - |p_R|^2 = 2m_2n_2 - 2m_1n_1$. Note that we could keep the convention $p = (-m_1, -n_1, m_2, n_2)$ - i.e. with minus signs as in the $\mathbb{R}_L^2 \oplus \mathbb{R}_R^2$ view - if we add extra minus signs in the definitions of $e_1, f_1$. Explicitly, the vector $p \in \mathbb{C}_L \oplus \mathbb{C}_R$ has components
\[ p_L = \frac{1}{\sqrt{2T_2U_2}} (m_1U + n_1\bar{T} + m_2\bar{T}U + n_2) \]
\[ p_R = \frac{1}{\sqrt{2T_2U_2}} (m_1U + n_1T + m_2TU + n_2) \]
such that $|p_R|^2 = \cdots = p_R^2$ above.

**Generalisation**

In fact, there are many different conventions about signs, bases, real or complex,... but they agree on the relevant structure. E.g., swapping $e_i$ and $f_i$ is immaterial, swapping the indices 1 and 2 yields an additional minus sign, since then $e_1 \cdot f_1 = -1$ and $e_2 \cdot f_2 = 1$, etc. But all versions somehow agree that $p^2 = 2m_2n_2 - 2m_1n_1$.

From here, it is straightforward to generalise to lattices of signature $(s + 2, 2)$: write $u = (\bar{u}, T, U, 1, TU - \frac{u_2}{2})$, with $u_2^2 < 2T_2U_2$ and $T_2, U_2 > 0$. This is the general form of a vector $u$ with $u^2 = 0$ and $u \cdot \bar{u} < 0$ (i.e. $u_1^2 = u_2^2 < 0$ and $u_1 \cdot u_2 = 0$), modulo any complex multiple. This means that $u_1, u_2$ specify a plane in $\mathbb{R}^{s+2,2}$. See [HM-95] or [N-98] for example.

Note that $(\bar{u}, T, U) \in \mathbb{R}^{s+1,1} \otimes \mathbb{C}$, that is $\bar{u} \in \mathbb{C}^s$ has Euclidean inner product. Note also that $u \cdot \bar{u} = 2(\Im(\bar{u}, T, U))^2$; if this is to be negative, $\Im(\bar{u}, T, U)$ must be a light-like vector. Assume it is in the forward light cone ($u$ and $\bar{u}$ yield same plane), so that $(\bar{u}, T, U) \in \mathbb{R}^{s+1,1} + i \mathcal{C}^{s+1,1} =: \mathcal{H}^{s+1,1} = Gr(s + 2, 2) = O(s + 2, 2)/O(s + 2) \times O(2)$ as in section 4.2. This is the Teichmüller space; one still needs to mod out by the left action of the arithmetic subgroup $O(s+2, 2; \mathbb{Z})$ to arrive at the moduli space of $\Gamma_{s+2, 2}$ lattices.