Gromow-Witten Invariants and Elliptic Genera

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Citation for published version (APA):
Appendix B

Momentum on a Lattice

B.1 Lattice $\Gamma_{1,1}$

Motivation

The quantised momentum on a lattice of signature $(1,1)$ is easiest to understand in the CFT context of free bosons on a torus. The torus is specified by the periods $1, \tau$ on the complex plane. The fermionic coordinates satisfy $\psi(w + 1) = \pm \psi(w + \tau) = \pm \psi(w)$, where the sign ambiguity in both periods allows for four different spin structures. The bosonic coordinates are assumed periodic, $X = X + 2\pi r$, i.e. compactified on the one-dimensional target space $S^1$, a circle of radius $r$. That is, they are maps $X : T^2 \to S^1$. Their compact coordinate allows for an infinite number of configurations (not just 4 as for the fermions), labelled by $n, n' \in \mathbb{Z}$: $X(w + 1) = X(w) + 2\pi r n$ and $X(w + 1) = X(w) + 2\pi r n'$.

There are two approaches to calculating the partition function $F$:

- Hamiltonian formulation: the partition function is given by a trace over the Hilbert space:
  \[ \text{tr } q^{L_0 - \frac{c}{24}} q^{\bar{L}_0 - \frac{c}{24}} \]

- Lagrangian formulation: the partition function is given by the functional integral
  \[ \int e^{-S} = \frac{1}{\eta^2} \sum_{n, m \in \mathbb{Z}} q^{\frac{1}{2} n^2} q^{\frac{1}{2} m^2} \]

In the latter, we have introduced the left and right components of the momentum vector $p = (p_L, p_R)$:

\[ p_L := \frac{m}{2r} + nr = \frac{p}{2} + \omega \]
\[ p_R := \frac{m}{2r} - nr = \frac{p}{2} - \omega \]
where the scalar momentum $p = m/r$ and winding $\omega = nr$ are quantised in terms of integers $m$ and $n$.

Comparison of the two approaches shows that the Hilbert space consists of sectors labelled by $m, n$, each sector generated by the state $|m, n\rangle$ playing the role of the ground state, i.e. $\alpha_l|m, n\rangle = 0, \forall l > 0$.

In the $|m, n\rangle$ sector, the bosonic Fock subspace $\mathcal{F}_l$ generated by the creation operator $\alpha_{-l}$ contains the states $|m, n\rangle$, $\alpha_{-l}|m, n\rangle$, $\alpha_{-l}^2|m, n\rangle$, $\ldots$. When the Virasoro operator $L_0 = \sum_{l \geq 1} \alpha_{-l} \alpha_l + \frac{1}{2} \alpha_0^2$ acts on them, we obtain $q^{L_0} \alpha_{-l}|m, n\rangle = q^j \alpha_{-l}|m, n\rangle$ for $j > 0$, and $q^{L_0}|m, n\rangle = q^{\alpha_0^2/2}|m, n\rangle$ for $j = 0$. Thus the $j > 0$ part yields the expected $\eta^{-1}$ of the bosonic partition function:

$$\text{tr} \otimes \mathcal{F}_l \; q^{L_0} = \prod_{l \geq 1} \text{tr} \mathcal{F}_l \; q^{L_0} = \prod_{l \geq 1} (1 + q^l + q^{2l} + \ldots) = \prod_{l \geq 1} (1 - q^l)^{-1} = \frac{q^{1/24}}{\eta},$$

independently of the sector $|m, n\rangle$. So the remaining $j = 0$ part, i.e. $q^{\alpha_0^2/2}$, must account for the remaining $q^{\alpha_0^2/2}$ of the Lagrangian approach. Hence we identify $\alpha_0 = p_l$ and $\alpha_0 = p_R$, that is

$$L_0|m, n\rangle = \frac{1}{2} \alpha_0^2|m, n\rangle = \frac{1}{2} (\frac{m}{2r} + nr)^2|m, n\rangle,$$

and similarly for $\bar{L}_0$. Then we see that the state $|m, n\rangle$ has energy eigenvalue $\frac{m^2}{2r^2} + n^2 r^2$ under $H = L_0 + \bar{L}_0$, and (integer) momentum eigenvalue $mn$ under $P = L_0 - \bar{L}_0$. Thus $|0, 0\rangle$ is the actual ground state.

**Lattice and Inner Product**

We can rewrite the momentum vector in the basis \{k, \bar{k}\}, where $k := (\frac{1}{2r}, \frac{1}{2r})$, $\bar{k} := (r, -r)$ generate the even self-dual Lorentzian lattice of signature (1, 1), denoted by $\Gamma_{1,1}$:

$$p = (p_L, p_R) = m(\frac{1}{2r}, \frac{1}{2r}) + n(r, -r) = \binom{m}{n}$$

The inner product associated to $\Gamma_{1,1}$ reads $(x, y) \cdot (x', y') = xx' - yy'$. This implies for the vertical vectors $\binom{m}{n} \cdot \binom{m'}{n'} = mn' + m'n$. The basis vectors satisfy $k^2 = \bar{k}^2 = 0, k \cdot \bar{k} = 1$, while the square momentum is $p^2 = p_L^2 - p_R^2 = 2mn$.

**B.2 Lattice $\Gamma_{2,2}$**

**Motivation**

When the bosonic coordinate $X$ is compactified on a torus instead of circle (as in the previous case of $\Gamma_{1,1}$), we have two radii $r_1, r_2$ to care about. So we have to sum over four integers $m_1, m_2, n_1, n_2$ to take all configurations of $X$ into account, i.e. we end up with a Siegel theta function for the $\Gamma_{2,2}$ lattice. Each of $p_L, p_R$ is now a vector in $\mathbb{R}^2$. 
In the previous section, the Narain sum over \( \Gamma_{1,1} \) only depended on one real modulus \( r \), reflecting the fact that the moduli space of \( \Gamma_{1,1} \) lattices is one-dimensional — equal to \( O(1,1)/O(1) \times O(1) \), i.e. \( O(1,1)/\mathbb{Z}_2 \times \mathbb{Z}_2 \). In general, we write \( O(p,q) \) for the Lorentz group of linear maps preserving the metric \( \text{Diag}(+,...,+,+,-,...,-) \), or equivalently \( \text{Aut}(\Gamma_{p,q}) \). Now however, we have more than just the two moduli \( r_1, r_2 \) for the choice of a \( \Gamma_{2,2} \) lattice: the moduli space \( O(2,2)/O(2) \times O(2) \) is four real-dimensional, and then one still has to mod out by \( O(2,2,\mathbb{Z}) \) to account for lattices related by such a transformation.

In fact, this moduli space is none but the Grassmannian \( \text{Gr}(2,2) \) of all choices of time-like 2-planes in \( \mathbb{R}^{2,2} \); a choice of such a plane amounts to a split of \( \mathbb{R}^4 \cong \mathbb{R}^{2,2} \) into the orthogonal direct sum of two real euclidean spaces \( \mathbb{R}_L^2, \mathbb{R}_R^2 \), and the time-like (negative norm) plane itself corresponds to \( \mathbb{R}_T^2 \). Whatever the point of view, we expect four real moduli, or two complex ones.

### Lattice and inner product

As advertised, we let \( p = (p_L, p_R) \in \mathbb{R}^{2,2} \) such that \( p^2 = p_L^2 - p_R^2 \) with \( p_L^2, p_R^2 > 0 \) and \( (p_R)^2 = -p_R^2 < 0 \). Then

\[
p = \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} m_2 + n_2 \\ m_1 - n_1 \\ m_2 - n_2 \\ m_1 + n_1 \end{pmatrix} \in \mathbb{R}_L^2 \oplus \mathbb{R}_R^2,
\]

or \( p = (-m_1, -n_1, m_2, n_2) \) in basis \( \{e_1, f_1, e_2, f_2\} \), where \( e_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \frac{1}{i} \\ 1 \\ 0 \end{pmatrix} \), \( f_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \frac{1}{i} \\ 1 \\ 0 \end{pmatrix} \), \( e_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \), \( f_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \). One easily checks that \( e_1^2 = f_1^2 = e_1 \cdot e_2 = f_1 \cdot f_2 = e_1 \cdot f_2 = e_2 \cdot f_1 = 0 \), while \( e_1 \cdot f_1 = -1 \) and \( e_2 \cdot f_2 = 1 \).

That is, we interpret our lattice as \( \Gamma_{2,2} = \Gamma_{1,1}(-1) \oplus \Gamma_{1,1}(1) = (e_1, f_1)_\mathbb{R} \oplus (e_2, f_2)_\mathbb{Z} \), where \( \Gamma_{1,1}(-1) \) denotes the lattice \( \Gamma_{1,1} \) but with bilinear form given by \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Hence \( p^2 = 2m_2n_2 - 2m_1n_1 \).

More generally, for any basis \( \{e_1, f_1, e_2, f_2\} \) of \( \Gamma_{1,1}(-1) \oplus \Gamma_{1,1}(1) \), we can create the split \( \mathbb{R}_L^2 \oplus \mathbb{R}_R^2 \) by choosing two vectors in \( \mathbb{R}^4 \) and setting \( \mathbb{R}_R^2 \) to be their span. Let these be \( u_1, u_2 \) for the complex null vector \( u = u_1 + iu_2 := (T, U, 1, TU) \) with two complex moduli \( T, U \) such that \( T_2, U_2 > 0 \). Then \( u_1^2 = u_2^2 = -2T_2U_2 < 0 \) and \( u_1 \cdot u_2 = 0 \), and we can project the vector \( p = (-m_1, -n_1, m_2, n_2) \) onto \( \mathbb{R}_R^2 \) via \( p_{R,PR} = \frac{p_{u_1}}{u_1^2} u_1 + \frac{p_{u_2}}{u_2^2} u_2 \). Thus

\[
p_R^2 = -\left( \begin{pmatrix} 0 \\ p_{u_1} \\ p_{u_2} \end{pmatrix} \right)^2 = -\left( \frac{p \cdot u_1}{u_1^2} \right)^2 - \left( \frac{p \cdot u_2}{u_2^2} \right)^2 = -\frac{|p \cdot u|^2}{u^2} = \frac{1}{2T_2U_2} |n_2 + m_2TU + n_1T + m_1U|^2
\]

Conversely, given any \( p = (p_L, p_R) = (m_1, n_1, m_2, n_2) \), we can arrive at the above expression for \( p_R^2 \) by viewing \( \mathbb{R}_L^2 \oplus \mathbb{R}_R^2 \cong \mathbb{C}_L \oplus \mathbb{C}_R \) (as in [MM-98], say),
i.e. viewing $p_L$ and $p_R$ as complex numbers, and choosing the basis to be

$$e_1 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} U \\ U \end{pmatrix}, \quad f_1 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} T \\ U \end{pmatrix}, \quad e_2 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} \bar{T}U \\ U \end{pmatrix}, \quad f_2 := \frac{1}{\sqrt{2T_2U_2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with associated symmetric inner product is $(x_L \cdot x_R) = \text{Re}(x_L \bar{y}_L - x_R \bar{y}_R)$. One easily checks that $e_i \cdot f_j$, etc, are as above. Hence $p^2 = |p_L|^2 - |p_R|^2 = 2m_2n_2 - 2m_1n_1$. Note that we could keep the convention $p = (-m_1,-n_1,m_2,n_2)$ - i.e. with minus signs as in the $\mathbb{R}_L^2 \oplus \mathbb{R}_R^2$ view – if we add extra minus signs in the definitions of $e_1, f_1$. Explicitly, the vector $p \in \mathbb{C}_L \oplus \mathbb{C}_R$ has components

$$p_L = \frac{1}{\sqrt{2T_2U_2}} \left( m_1U + n_1\bar{T} + m_2\bar{T}U + n_2 \right)$$

$$p_R = \frac{1}{\sqrt{2T_2U_2}} \left( m_1U + n_1T + m_2TU + n_2 \right)$$

such that $|p_R|^2 = \cdots = p_R^2$ above.

**Generalisation**

In fact, there are many different conventions about signs, bases, real or complex,... but they agree on the relevant structure. E.g., swapping $e_1$ and $f_1$ is immaterial, swapping the indices 1 and 2 yields an additional minus sign, since then $e_1 \cdot f_1 = -1$ and $e_2 \cdot f_2 = 1$, etc. But all versions somehow agree that $p^2 = 2m_2n_2 - 2m_1n_1$.

From here, it is straightforward to generalise to lattices of signature $(s+2,2)$: write $u = (\bar{u},T,U,1,TU - \frac{\bar{u}^2}{2})$, with $\bar{u}^2 < 2T_2U_2$ and $T_2,U_2 > 0$. This is the general form of a vector $u$ with $u^2 = 0$ and $u \cdot \bar{u} < 0$ (i.e. $u_1^2 = u_2^2 < 0$ and $u_1 \cdot u_2 = 0$), modulo any complex multiple. This means that $u_1, u_2$ specify a plane in $\mathbb{R}^{s+2,2}$. See [HM-95] or [N-98] for example.

Note that $(\bar{u},T,U) \in \mathbb{R}^{s+1,1} \otimes \mathbb{C}$, that is $\bar{u} \in \mathbb{C}^s$ has Euclidean inner product. Note also that $u \cdot \bar{u} = 2(\text{Re}(\bar{u},T,U))^2$; if this is to be negative, $\text{Re}(\bar{u},T,U)$ must be a light-like vector. Assume it is in the forward light cone (u and $\bar{u}$ yield same plane), so that $(\bar{u},T,U) \in \mathbb{R}^{s+1,1} + i \mathbb{C}^{s+1,1} =: \mathcal{H}^{s+1,1} = G_\tau(s+2,2) = O(s+2,2)/O(s+2) \times O(2)$ as in section 4.2. This is the Teichmüller space; one still needs to mod out by the left action of the arithmetic subgroup $O(s+2,2;\mathbb{Z})$ to arrive at the moduli space of $\Gamma_{s+2,2}$ lattices.