Gromow-Witten Invariants and Elliptic Genera

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Appendix D

Theta functions of given characteristic

A holomorphic function \( T : \mathbb{C} \to \mathbb{C} \) is called a theta function with period \( \tau \) and characteristic \((a_1, b_1; a_2, b_2)\) if it is almost periodic on the lattice, i.e. if it transforms according to

\[
T(v + 1) = e^{a_1 v + b_1} T(v), \quad \text{and} \quad T(v + \tau) = e^{a_2 \tau + b_2} T(v)
\]

We call \( n := (a_1 \tau - a_2)/2\pi i \) the degree of the function.

For example, the following functions are all theta functions with characteristic and degree

\[
\begin{align*}
y^{1/2} : & \quad (0, i\pi; 0, i\pi \tau) \quad 0 \\
\vartheta_1(v) : & \quad (0, i\pi; -2\pi i, -i\pi (\tau + 1)) \quad 1 \\
\vartheta_2(v) : & \quad (0, i\pi; -2\pi i, -i\pi \tau) \quad 1 \\
\vartheta_3(v) : & \quad (0, 0; -2\pi i, -i\pi \tau) \quad 1 \\
\vartheta_4(v) : & \quad (0, 0; -2\pi i, -i\pi (\tau + 1)) \quad 1 \\
\vartheta_1(2v|2\tau) : & \quad (0, 0; -4\pi i, -2\pi i\tau - i\pi) \quad 2 \\
\vartheta_2(2v|2\tau) : & \quad (0, 0; -4\pi i, -2\pi i\tau) \quad 2 \\
\vartheta_3(2v|2\tau) : & \quad (0, 0; -4\pi i, -2\pi i\tau) \quad 2 \\
\vartheta_4(2v|2\tau) : & \quad (0, 0; -4\pi i, -2\pi i\tau - i\pi) \quad 2 \\
\vartheta_i(v)^2 : & \quad (0, 0; -4\pi i, -2\pi i\tau) \quad 2, \quad i = 1, \ldots, 4
\end{align*}
\]

Note that characteristics add up when multiplying theta functions. Note also that \( \vartheta_3(nv|n\tau) \) and \( \vartheta_3(v|\frac{\tau}{n}) \) are of degree \( n \) and characteristic \((0, 0; -2n\pi i, -n\pi i\tau)\).

As another example, consider the character functions of the level \( k \) and isospin \( l \) representation of affine \( su(2) \) algebra [ET2-88]:

\[
\chi_k^l(y) := \frac{q^{(l+1/2)^2/(k+1/2)-1/8}}{\prod_{n \geq 1} (1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n-1})} \times \sum_{m \in \mathbb{Z}} q^{(k+2)m^2 + (2l+1)m} \left( y^{2m(k+2) + 2l} - y^{-2m(k+2) - 2l} \right)
\]
This is a theta function of characteristic \((0, 0; -4k\pi i, -2k\pi \tau)\) and degree \(2k\), i.e. it transforms like
\[
\chi_k^I(v + \tau) = q^{-k} y^{-2k} \chi_k^I(v).
\]
Each theta function can be multiplied by trivial theta functions (i.e. of degree 0) so that the resulting characteristic reads \((0, 0; -2\pi in, b_2)\) where \(n\) the degree (an integer). For fixed \(b_2\), this is a vector space of dimension \(n\) as can be seen from the fact that contour integration around one lattice cell yields \(n\) zeros for \(T\): \(P - Z = \oint T'/T = \oint \partial \log T = -n\). We denote this complex vector space by \(\mathcal{T}_{n, b_2}\). For \(b_2 = -n\pi i\tau\), it's spanned by \(\vartheta_3(nv|\eta \tau), y \vartheta_3(nv + \tau|\eta \tau), \ldots, y^{n-1} \vartheta_3(nv + (n - 1)\tau|\eta \tau)\).

Thus for instance, all degree 2 theta functions of characteristic \((0, 0; -4\pi i, -2\pi i\tau)\) should be expressible as linear combinations of \(\vartheta_1(v)^2\) and \(\vartheta_3(v)^2\) (or any two of the \(\vartheta_i(v)^2, i = 1, \ldots, 4\)) with \(\tau\)-dependent coefficients. This was the case for the N=4 massless NS characters (6.2.1), for \(\vartheta_2(v)^2\) or \(\vartheta_4(v)^2\) as in (C.0.15), or for the level 1 \(su(2)\) theta functions:

\[
\chi_0^0(y) := \frac{q^{-1/24} \sum_{m \in \mathbb{Z}} q^{3m^2 + m} (y^{6m} - y^{-6m-2})}{\prod_{n \geq 1} (1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n+1})} = \frac{\vartheta_3(2v|2\tau)}{\eta} \\
\chi_1^1(y) := \frac{q^{-5/24} \sum_{m \in \mathbb{Z}} q^{3m^2 + m} (y^{6m+1} - y^{-6m-3})}{\prod_{n \geq 1} (1 - q^n)(1 - y^2 q^n)(1 - y^{-2} q^{n+1})} = \frac{\vartheta_2(2v|2\tau)}{\eta}.
\]

The right hand sides can be obtained by noting that these too belong to \(\mathcal{T}_{2,-2\pi i\tau}\) (and by checking the equalities at \(y = 1, q^{1/2}\) say). Alternatively, they are reproduced by the quintuple identity (C.0.13).

Similarly, any element of \(\mathcal{T}_{2,-2\pi i\tau}\) can be spanned by the N=4 characters \(\tilde{c}^{NS}_{0, \frac{1}{2}}\) and \(c_0^{NS}\), as was done with the NS orbits in (6.2.4).