Appendix D

Theta functions of given characteristic

A holomorphic function $T : \mathbb{C} \to \mathbb{C}$ is called a theta function with period $\tau$ and characteristic $(a_1,b_1;a_2,b_2)$ if it is almost periodic on the lattice, i.e. if it transforms according to

$$T(v + 1) = e^{a_1v + b_1} T(v), \quad \text{and} \quad T(v + \tau) = e^{a_2v + b_2} T(v)$$

We call $n := (a_1 - a_2)/2\pi i$ the degree of the function.

For example, the following functions are all theta functions with characteristic and degree

$$y^{1/2} : (0,i\pi;0,i\pi \tau) \quad 0$$
$$\vartheta_1(v) : (0,i\pi;-2\pi i,-i\pi(\tau + 1)) \quad 1$$
$$\vartheta_2(v) : (0,i\pi;-2\pi i,-i\pi \tau) \quad 1$$
$$\vartheta_3(v) : (0,0;-2\pi i,-i\pi \tau) \quad 1$$
$$\vartheta_4(v) : (0,0;-2\pi i,-i\pi(\tau + 1)) \quad 1$$
$$\vartheta_1(2v|2\tau) : (0,0;-4\pi i,-2\pi i\tau - i\pi) \quad 2$$
$$\vartheta_2(2v|2\tau) : (0,0;-4\pi i,-2\pi i\tau) \quad 2$$
$$\vartheta_3(2v|2\tau) : (0,0;-4\pi i,-2\pi i\tau) \quad 2$$
$$\vartheta_4(2v|2\tau) : (0,0;-4\pi i,-2\pi i\tau - i\pi) \quad 2$$

Note that characteristics add up when multiplying theta functions. Note also that $\vartheta_3(nv|n\tau)$ and $\vartheta_3(v|\frac{\tau}{n})$ are of degree $n$ and characteristic $(0,0;-2n\pi i,-n\pi \tau)$.

As another example, consider the character functions of the level $k$ and isospin $l$ representation of affine $su(2)$ algebra [ET2-88]:

$$\chi_k^l(y) := \frac{q^{(l+1/2)^2/(k+1/2)-1/8}}{\prod_{n \geq 1} (1-q^n)(1-y^2q^n)(1-y^{-2}q^{n-1})} \times \sum_{m \in \mathbb{Z}} \frac{q^{(k+2)m^2+(2l+1)m} (y^{2m(k+2)+2l} - y^{-2m(k+2)-2l-2})}{1-q^{2m(k+2)+2l}}$$

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This is a theta function of characteristic \((0,0; -4k\pi i, -2k\pi \tau)\) and degree \(2k\), i.e. it transforms like
\[
\chi_k^l(v + \tau) = q^{-k}y^{-2k}\chi_k^l(v).
\]

Each theta function can be multiplied by trivial theta functions (i.e. of degree 0) so that the resulting characteristic reads \((0,0; -2\pi in, b_2)\) where \(n\) the degree (an integer). For fixed \(b_2\), this is a vector space of dimension \(n\) as can be seen from the fact that contour integration around one lattice cell yields \(n\) zeros for \(T: P - Z = \oint T'/T = \oint \partial \log T = -n\). We denote this complex vector space by \(\mathcal{T}_{n,b_2}\). For \(b_2 = -n\pi i\tau\), it's spanned by \(\vartheta_3(nv|n\tau), y \vartheta_3(nv + \tau|n\tau), \ldots, y^{n-1} \vartheta_3(nv + (n-1)\tau|n\tau)\).

Thus for instance, all degree 2 theta functions of characteristic \((0,0; -4\pi i, -2\pi i\tau)\) should be expressible as linear combinations of \(\vartheta_1(v)^2\) and \(\vartheta_3(v)^2\) (or any two of the \(\vartheta_i(v)^2, i = 1, \ldots, 4\)) with \(\tau\)-dependent coefficients. This was the case for

\[
\chi_{1/2}^0(y) := \frac{q^{-1/24} \sum_{m \in \mathbb{Z}} q^{3m^2 + m} (y^{6m} - y^{-6m-2})}{\prod_{n \geq 1} (1 - q^n)(1 - y^2q^n)(1 - y^{-2}q^{n-1})} = \frac{\vartheta_3(2v|2\tau)}{\eta},
\]

\[
\chi_{1/2}^1(y) := \frac{q^{-5/24} \sum_{m \in \mathbb{Z}} q^{3m^2 + m} (y^{6m+1} - y^{-6m-3})}{\prod_{n \geq 1} (1 - q^n)(1 - y^2q^n)(1 - y^{-2}q^{n-1})} = \frac{\vartheta_2(2v|2\tau)}{\eta}.
\]

The right hand sides can be obtained by noting that these too belong to \(\mathcal{T}_{2,-2\pi i\tau}\) (and by checking the equalities at \(y = 1, q^{1/2}\) say). Alternatively, they are reproduced by the quintuple identity (C.0.13).

Similarly, any element of \(\mathcal{T}_{2,-2\pi i\tau}\) can be spanned by the N=4 characters \(\tilde{c}^{\text{NS}}_{0,\frac{1}{4}}\) and \(c^{\text{NS}}_0\), as was done with the NS orbits in (6.2.4).