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UNIDIMENSIONAL FACTOR MODELS IMPLY WEAKER PARTIAL CORRELATIONS THAN ZERO-ORDER CORRELATIONS

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In this paper we present a new implication of the unidimensional factor model. We prove that the partial correlation between two observed variables that load on one factor given any subset of other observed variables that load on this factor lies between zero and the zero-order correlation between these two observed variables. We implement this result in an empirical bootstrap test that rejects the unidimensional factor model when partial correlations are identified that are either stronger than the zero-order correlation or have a different sign than the zero-order correlation. We demonstrate the use of the test in an empirical data example with data consisting of fourteen items that measure extraversion.

Key words: factor models, partial correlations, zero-order correlations.

Unidimensional factor models (UFMs) are widely used throughout the social and behavioral sciences, and many of its implications have already been revealed. In this paper, we reveal yet another implication, namely that the partial correlation between two observed variables given any subset of other observed variables is always closer to zero than the zero-order correlation between these observed variables. As we will show, this implication adds to a series of implications generated by the work of, among others, Holland and Rosenbaum (1986), Ellis (2014) and Guttman (1940) and can be used to detect misfit of the UFM for the observed data.

In the remainder of this section we describe some of the implications of the UFM that have been revealed previously in order to situate our contribution. We then turn to the actual proof of our contribution, before discussing possible ways in which it can be used in future research. To illustrate the potential use of the result, we consider its use in a bootstrap test that can detect misfit of the UFM and can also indicate for which observed variables this misfit is obtained.

The basic assumption of UFMs is that the observed variables are conditionally independent given the latent factor. This condition is sometimes called *latent conditional independence* (Holland & Rosenbaum, 1986) but is more widely known as *local independence* in item response theory (IRT; Lord, 1980). The model implied covariance matrix of a UFM with local independence equals the sum of a diagonal matrix representing the residual covariance matrix and a matrix of rank one. This characteristic of the UFM has resulted in multiple implications for the covariance structure of the observed variables.

For example, building on Rosenbaum (1984)'s work on IRT models, Holland and Rosenbaum (1986) showed that nonnegative associations between observed variables in a UFM imply nonnegative conditional associations. When the observed variables are multivariate Gaussian, the result of Holland and Rosenbaum (1986) implies that nonnegative correlations imply nonnegative partial correlations. That is, for any set of multivariate Gaussian observed variables that load on a

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single factor, partial correlations cannot have the opposite sign of their corresponding zero-order correlation.

Ellis (2014) shows that in a unidimensional latent variable model for any pair of two binary observed variables given any subset of the other binary observed variables and their products, the partial correlation is nonnegative. This condition implies that the unconditional correlation matrix must show a certain ordinal pattern (Ellis, 2014). Yet another implication of the UFM is that the partial correlation between two observed variables given the other observed variables cannot equal zero (de Fátima Salgueiro, Smith, & McDonald, 2008). In addition, de Fátima Salgueiro et al. (2008) show that certain patterns of signs in the partial correlation matrix are incompatible with a UFM.

A final implication of the factor model that is of importance here is that Guttman claimed that as the number of observed variables increases to infinity the square of the multiple correlation coefficient of an observed variable on the other observed variables tends to the communality and all partial correlations between this observed variable and the other observed variables tend to zero (Guttman, 1940; 1953).

In this paper we present a related implication of the UFM: the partial correlation between two observed variables given any subset of the other observed variables is always closer to zero than the zero-order correlation between these observed variables. This, together with the implications mentioned previously that partial correlations cannot equal zero and cannot switch sign compared to the zero-order correlations, means that the UFM implies that the partial correlation between any two observed variables given any subset of the other observed variables lies between zero and the zero-order correlation between these two observed variables. This result implies that as the number of observed variables increases to infinity the partial correlations not only tend to zero but tend *monotonically* to zero.

1. Unidimensional Factor Models

We first define UFMs before presenting a proof. Let Σ denote the covariance matrix of $\mathbf{y}$, in which $\mathbf{y}$ denotes the vector of observed variables. We assume that Σ is nondegenerate and so is positive definite. Let $\boldsymbol{\lambda}$ denote the vector of factor loadings and Θ the residual covariance matrix. In a UFM all observed variables in $\mathbf{y}$ are a linear function of the same factor, η, and independent residuals, $\boldsymbol{\varepsilon}$:

$$y_i = \lambda_i \eta + \varepsilon_i. \tag{1}$$

We assume var$(\eta) = 1$. We also assume mutually uncorrelated residuals, that is, Θ is diagonal. In the following, we take Σ to be standardized and assume $\forall i$, $|\lambda_i| \in (0, 1)$, so that the observed variables are correlated with the factor but not perfectly correlated with the factor. The model implied covariance matrix of the observed variables is a function of the factor loadings and the residual covariance matrix:

$$\Sigma = \boldsymbol{\lambda}\boldsymbol{\lambda}' + \Theta. \tag{2}$$

Equation (2) implies that the covariance among observed variables is a function of their factor loadings. More precisely, because Θ is a diagonal matrix, the covariance between two variables y_i and y_j equals $\lambda_i \lambda_j$.

Consider three variables y_1, y_2 and z. The partial correlation between y_1 and y_2 given z can be expressed in terms of their zero-order correlations by (e.g., Chen & Pearl, 2014):

$$\rho_{y_1 y_2 \cdot z} = \frac{\rho_{y_1 y_2} - \rho_{y_1 z}\rho_{y_2 z}}{\sqrt{(1 - \rho^2_{y_1 z})(1 - \rho^2_{y_2 z})}} \tag{3}$$

Some correlation structures for three variables imply that the partial correlation is stronger than the correlation. For example, a negative $\rho_{y_2 z}$ in combination with a positive $\rho_{y_1 y_2}$ and $\rho_{y_1 z}$ results in a partial correlation $\rho_{y_1 y_2 . z}$ that is stronger than the zero-order correlation $\rho_{y_1 y_2}$. Langford, Schwertman, and Owens (2001) point out that the property of being positively correlated is not transitive: for three variables it is possible to have a correlation structure with one negative and two positive correlations. However, a structure with one negative and two positive correlations is not possible under a UFM as it is impossible to choose factor loadings such that they result in two positive correlations and one negative correlation. In fact, all possible correlation structures that result in some partial correlations that are stronger than their corresponding zero-order correlation appear to be impossible under a UFM. For example, for three variables y_1, y_2 and z that are all elements of $\mathbf{y}$, we can substitute each correlation in Eq. (3) with factor loadings:

$$\rho_{y_1 y_2 \cdot z} = \frac{\lambda_{y_1}\lambda_{y_2}(1-\lambda_z^2)}{\sqrt{1-\lambda_{y_1}^2\lambda_z^2}\sqrt{1-\lambda_{y_2}^2\lambda_z^2}} \tag{4}$$

Note that both the denominator $\sqrt{1-\lambda_{y_1}^2\lambda_z^2}\sqrt{1-\lambda_{y_2}^2\lambda_z^2}$ and $(1-\lambda_z^2)$ are positive and that $\sqrt{1-\lambda_{y_1}^2\lambda_z^2}\sqrt{1-\lambda_{y_2}^2\lambda_z^2} > (1-\lambda_z^2)$. It follows that $0 < \frac{1-\lambda_z^2}{\sqrt{1-\lambda_{y_1}^2\lambda_z^2}\sqrt{1-\lambda_{y_2}^2\lambda_z^2}} < 1$ and thus that the partial correlation $\rho_{y_1 y_2 \cdot z}$ is weaker than the zero-order correlation $\lambda_{y_1}\lambda_{y_2}$. While this example only serves as a proof for $p = 3$ variables, in the following section it is proved that for any p the UFM implies that the partial correlations are necessarily weaker than the zero-order correlations.

2. Weaker Partial Correlations Than Zero-Order Correlations

We start with providing the assumptions defined in the previous section.

Assumption.

1. $\boldsymbol{\Sigma} = \boldsymbol{\lambda}\boldsymbol{\lambda}' + \boldsymbol{\Theta}$, and $\boldsymbol{\Sigma}_{ii} = 1$ for all i
2. $\boldsymbol{\Theta}$ is diagonal and positive definite,
3. $|\lambda_i| \in (0, 1)$ for all i.

These assumptions imply the following lemma.

Lemma *Assume 1–3 above. For a set of p continuous random variables that load on one common factor, the partial correlation between any two of them, conditional on any nonempty subset of the other $p-2$ variables, is weaker than their marginal correlation.*

Proof The proof is for the partial correlation conditional on all $p-2$ remaining variables. That the result holds for any subset follows from restricting $\boldsymbol{\Sigma}$ to the subset of variables.

Let $\mathbf{y} = [y_1, y_2, \ldots, y_p]'$. Partition $\mathbf{y}' = (\mathbf{Y}', \mathbf{Z}')$, where $\mathbf{Y} = (y_1, y_2)'$, and $\mathbf{Z} = (y_3, \ldots, y_p)'$. Partition $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}_Y', \boldsymbol{\lambda}_Z')$ accordingly. The partitioned covariance matrix of $\mathbf{y}$ is

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{YY} & \boldsymbol{\Sigma}_{YZ} \\ \boldsymbol{\Sigma}_{ZY} & \boldsymbol{\Sigma}_{ZZ} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}_Y\boldsymbol{\lambda}_Y' + \boldsymbol{\Theta}_{YY} & \boldsymbol{\lambda}_Y\boldsymbol{\lambda}_Z' \\ \boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Y' & \boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Z' + \boldsymbol{\Theta}_{ZZ} \end{pmatrix},$$

where $\boldsymbol{\Theta}_{YY} = \mathrm{diag}(1-\lambda_1^2, 1-\lambda_2^2)$, and $\boldsymbol{\Theta}_{ZZ} = \mathrm{diag}(1-\lambda_3^2, \ldots, 1-\lambda_p^2)$. The partial correlation between y_1 and y_2, conditioned on all the other variables, can be computed from the partial covariance matrix

$$\begin{aligned}\boldsymbol{\Sigma}_{YY\cdot Z} &= \boldsymbol{\Sigma}_{YY} - \boldsymbol{\Sigma}_{YZ}\boldsymbol{\Sigma}_{ZZ}^{-1}\boldsymbol{\Sigma}_{ZY} \\ &= (\boldsymbol{\lambda}_Y\boldsymbol{\lambda}_Y' + \boldsymbol{\Theta}_{YY}) - \boldsymbol{\lambda}_Y\boldsymbol{\lambda}_Z'(\boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Z' + \boldsymbol{\Theta}_{ZZ})^{-1}\boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Y' \\ &= \boldsymbol{\lambda}_Y\boldsymbol{\lambda}_Y'[1 - \boldsymbol{\lambda}_Z'(\boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Z' + \boldsymbol{\Theta}_{ZZ})^{-1}\boldsymbol{\lambda}_Z] + \boldsymbol{\Theta}_{YY}.\end{aligned}$$

Reading off the partial variances $(\boldsymbol{\Sigma}_{YY\cdot Z})_{11}$, $(\boldsymbol{\Sigma}_{YY\cdot Z})_{22}$ and covariance $(\boldsymbol{\Sigma}_{YY\cdot Z})_{12}$, the partial correlation can be expressed as

$$\begin{aligned}\rho_{y_1y_2\cdot Z} &= \frac{\lambda_1\lambda_2(1-\gamma^2)}{\sqrt{\lambda_1^2(1-\gamma^2)+(1-\lambda_1^2)}\sqrt{\lambda_2^2(1-\gamma^2)+(1-\lambda_2^2)}} \\ &= \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2+(1-\lambda_1^2)/(1-\gamma^2)}\sqrt{\lambda_2^2+(1-\lambda_2^2)/(1-\gamma^2)}},\end{aligned} \tag{5}$$

where $\gamma^2 = \boldsymbol{\lambda}_Z'(\boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Z' + \boldsymbol{\Theta}_{ZZ})^{-1}\boldsymbol{\lambda}_Z > 0$ by the assumption that $\lambda_i \neq 0$. Note that $\rho_{y_1y_2\cdot Z} = \rho_{y_1y_2} = \lambda_1\lambda_2$ is the marginal correlation if and only if $\gamma^2 \equiv 0$. The Sherman–Morrison formula (Sherman & Morrison, 1950) shows that $\gamma^2 < 1$:

$$\begin{aligned}\boldsymbol{\lambda}_Z'(\boldsymbol{\lambda}_Z\boldsymbol{\lambda}_Z' + \boldsymbol{\Theta}_{ZZ})^{-1}\boldsymbol{\lambda}_Z &= \boldsymbol{\lambda}_Z'(\boldsymbol{\Theta}_{ZZ}^{-1} - \boldsymbol{\Theta}_{ZZ}^{-1}\boldsymbol{\lambda}_Z[1+\boldsymbol{\lambda}_Z'\boldsymbol{\Theta}_{ZZ}^{-1}\boldsymbol{\lambda}_Z]^{-1}\boldsymbol{\lambda}_Z'\boldsymbol{\Theta}_{ZZ}^{-1})\boldsymbol{\lambda}_Z \\ &= \frac{\boldsymbol{\lambda}_Z'\boldsymbol{\Theta}_{ZZ}^{-1}\boldsymbol{\lambda}_Z}{1+\boldsymbol{\lambda}_Z'\boldsymbol{\Theta}_{ZZ}^{-1}\boldsymbol{\lambda}_Z} < 1.\end{aligned}$$

As a consequence, $0 < 1-\gamma^2 < 1$, and hence, the denominator in (5) is greater than 1, and $\rho_{y_1y_2\cdot Z}$ shrinks to 0 relative to $\rho_{y_1y_2}$. □

The factor γ^2 in the proof can be interpreted as the multiple correlation coefficient (coefficient of determination) between the latent variable η and the remaining $p-2$ variables. The smaller this multiple correlation is, the closer the partial correlation is to the marginal correlation; the larger this multiple correlation, the closer to zero the partial correlation is. This is because a larger multiple correlation implies that the variables that are partialled out have strong factor loadings and thus conditioning on these variables pulls out much of the shared variance between y_1 and y_2.

In our proof, we defined Z as consisting of all $p-2$ variables other than Y. However, the vector of variables that are partialled out does not need to include *all* variables loading on the factor other than Y in order for the partial correlation to be necessarily weaker than the zero-order correlation. In fact, the proof also holds for any subset of Z. Given that the UFM also implies that the partial correlation cannot equal zero (de Fátima Salgueiro et al., 2008) and also cannot have a different sign as the zero-order correlation (Holland & Rosenbaum, 1986), we can conclude that the partial correlation implied by a UFM is bounded on two sides: the partial correlation between any two observed variables given a set of the other observed variables lies between zero and the zero-order correlation between these observed variables.

The proof implies that when a UFM is hypothesized to underlie some dataset, the identification of partial correlations stronger than their corresponding zero-order correlations is an indication of model misfit.

3. Application of the Result

In the previous section we proved that the UFM implies that the partial correlation between any two variables is the product of the zero-order correlation between these two variables and a constant between zero and one. However, while this should hold for the correlations in the population, sampling variability can result in partial correlations that are not between zero and the zero-order correlation in the sample, even though a UFM is the true data-generating model. For example, the upper triangle of the matrix in Table 1 represents a sample correlation matrix of which the true data-generating model is a UFM. The lower triangle represents the corresponding partial correlations. In this sample correlation matrix there are multiple partial correlations that are not between zero and the zero-order correlations (i.e., $r_{16}, r_{23}, r_{24}, r_{25}, r_{26}, r_{34}, r_{35}, r_{36}, r_{46}$). To decide whether the UFM should be rejected based on these observations, a test for statistical significance is needed. In the following we propose one possible version of such a test, an empirical bootstrap test that uses the result developed in this paper to identify misfit of the UFM. There are four observations we use for the test that provide evidence against a UFM:

1. The correlation and corresponding partial correlation are both positive, but the partial correlation is more strongly positive than the correlation.
2. The correlation and corresponding partial correlation are both negative, but the partial correlation is more strongly negative than the correlation.
3. The correlation is negative, while the corresponding partial correlation is positive.
4. The correlation is positive, while the corresponding partial correlation is negative.

In the bootstrap test that we lay out in the following, the goal is to identify correlations for which any of the four observations above holds, accounting for sampling variability.

These four points can be summarized in one implication of the UFM, that is, the UFM implies $0 < \rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} < 1$. If and only if observation (1) or (2) is the case $\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} > 1$. If and only if observation (3) or (4) is the case $\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} < 0$. Note that $0 < \rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} < 1$ if and only if $|2\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} - 1| < 1$, and so the latter statistic can be used to test for all four observations above.[1] The distribution of this ratio is not easily obtained. We therefore constructed a bootstrap procedure to test whether $|2\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} - 1| > 1$. Finding that $|2\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} - 1| > 1$ provides evidence against the UFM.

3.1. Bootstrap Test

We propose the following empirical bootstrap procedure to test the hypothesis that the data come from a UFM:

1. Sample n observations with replacement from the original data set
2. From the m-th bootstrap sample, for all $i, j = 1, \ldots, p, i \neq j$,
   (a) compute the correlations $r_{y_i y_j}$ and partial correlations $r_{y_i y_j \cdot Z_{ij}}$, where $Z_{ij} = \{y_k\}_{k \neq i,j}, |Z_{ij}| = p - 2$
   (b) Compute the statistics $s_{ij}^m = |2r_{y_i y_j \cdot Z_{ij}}/r_{y_i y_j} - 1|$
3. Repeat steps 1 through 2b M times (e.g., $M = 1000$)
4. Use the bootstrapped $\{s_{ij}^m\}_{m=1}^M$ to construct a one-sided 95% confidence interval (CI). Any pair of variables y_i, y_j for which the 95% CI only includes values greater than one indicates misfit of the UFM.

[1] Let $x = \rho_{y_i y_j \cdot Z}/\rho_{y_i y_j}$. It follows that $0 < x < 1 \iff 0 < (x - 0.5) + 0.5 < 1 \iff -0.5 < x - 0.5 < 0.5 \iff |x - 0.5| < 0.5 \iff 2|x - 0.5| < 1 \iff |2x - 1| < 1$.

TABLE 1.
Upper triangle of matrix represents sample correlation matrix of dataset with 60 observations that is simulated from UFM with both positive and negative factor loadings.

	V1	V2	V3	V4	V5	V6
V1	1	0.212	0.139	0.231	0.193	0.079
V2	0.129	1	0.296	0.170	0.010	0.042
V3	0.128	0.319	1	−0.157	0.065	−0.060
V4	0.207	0.210	−0.266	1	0.193	−0.101
V5	0.158	−0.074	0.084	0.166	1	−0.088
V6	0.122	0.071	−0.110	−0.137	−0.076	1

The absolute factor loadings are sampled from a uniform distribution over [0.05, 0.2]. The lower triangle of the matrix represents the corresponding partial correlations.

TABLE 2.
Lower bound of the CI obtained with the bootstrap test for each zero-order correlation and corresponding partial correlation in Table 1.

	V1	V2	V3	V4	V5	V6
V1	–	0.028	0.08	0.076	0.045	0.117
V2	–	–	0.216	0.129	0.065	0.084
V3	–	–	–	0.633	0.063	0.080
V4	–	–	–	–	0.068	0.103
V5	–	–	–	–	–	0.049
V6	–	–	–	–	–	–

The supplementary materials provide an R function named `onefactor.test()` that implements the above algorithm. The output of the test provides a 95% CI of the parameter $|2\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} - 1|$ for each of the $p(p-1)/2$ unique correlations and their corresponding partial correlation. The default significance level α is set to 0.05 corresponding to a 95% CI, but one can adjust α to account for multiple testing. The test constructs a CI for each pair of variables in the data and thus performs $p(p-1)/2$ tests.

Earlier we stressed the need for a test by arguing that sampling variability can result in partial correlations that are not between zero and the zero-order correlation even when a UFM underlies the data. The correlation matrix in Table 1, for example, corresponds to data that were simulated from a UFM with factor loadings close to zero. Sampling data from a UFM with small factor loadings result in a partial correlation matrix that is close to the marginal correlation matrix. (When factor loadings are exactly zero, the population correlation matrix and the population partial correlation matrix are both diagonal matrices.) For an illustration of the test, it is most interesting to sample data from a population in which the partial correlation matrix is close to the marginal correlation matrix, because in that situation sampling variability can easily lead to partial correlations that are not between zero and the zero-order correlation. As mentioned earlier, the sample correlation and sample partial correlation matrices included in Table 1 include many partial correlations that are stronger than the zero-order correlation. Applying the bootstrap test to the data corresponding to this correlation matrix did not result in any significant results (all CIs for $|2\rho_{y_i y_j \cdot Z}/\rho_{y_i y_j} - 1|$ included the value 1, see Table 2), and thus, the UFM was not rejected. We included a more detailed description of the results of this analysis in supplementary materials, as well as R code to replicate the example. Furthermore, the supplementary materials include (a) a similar illustration of the test on data simulated from a random correlation matrix, and (b) a simulation study that shows that the performance of the test in terms of controlling type I errors is adequate.

TABLE 3.
Upper triangle of this matrix represents the zero-order correlations of the extraversion item responses.

	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14
V1	1	0.34	0.36	0.37	0.41	0.45	0.42	0.45	0.22	0.56	0.60	0.25	0.37	0.45
V2	0.08	1	0.16	0.3	0.13	0.33	0.16	0.27	0.24	0.26	0.28	0.04	0.40	0.35
V3	0.07	-0.02	1	0.26	0.26	0.40	0.22	0.21	0.14	0.38	0.30	0.13	0.17	0.28
V4	0.04	0.11	0.04	1	0.17	0.49	0.32	0.23	0.28	0.31	0.36	0.14	0.28	024
V5	0.07	-0.08	0.09	-0.02	1	0.24	0.23	0.57	0.06	0.43	0.33	0.27	0.20	0.31
V6	0.11	0.08	0.22	0.28	-0.03	1	0.31	0.35	0.37	0.37	0.41	0.10	0.31	0.31
V7	0.10	-0.08	0.01	0.10	-0.03	0.03	1	0.28	0.24	0.39	0.40	0.31	0.31	0.27
V8	0.11	0.10	-0.06	-0.02	0.46	0.1	0.05	1	0.19	0.38	0.38	0.17	0.28	0.33
V9	-0.04	0.06	-0.01	0.07	-0.07	0.21	0.1	0.06	1	0.15	0.28	0.01	0.24	0.18
V10	0.19	0.03	0.14	0.03	0.16	0.05	0.11	0.01	-0.04	1	0.46	0.32	0.29	0.47
V11	0.32	0.02	0.03	0.08	0.05	0.05	0.13	0.05	0.11	0.11	1	0.16	0.33	0.36
V12	0.05	-0.05	-0.01	0.04	0.13	-0.06	0.21	-0.03	-0.06	0.14	-0.06	1	0.14	0.21
V13	0.05	0.23	-0.03	0.05	0.00	0.03	0.13	0.03	0.07	-0.02	0.05	0.01	1	0.41
V14	0.11	0.14	0.07	-0.03	0.05	0.01	-0.02	0.03	0.03	0.21	0.01	0.05	0.21	1

The lower triangle of this matrix represents the corresponding partial correlations. Each zero-order correlation with a gray background corresponds to a partial correlation with a gray background, and the combination refers to a pair for which the bootstrap test was significant. (The CI obtained with the bootstrap test did not include one.)

4. Empirical Illustration

In this section, we illustrate the use of the statistical procedure by applying it to an empirical example concerning extraversion. Extraversion is an interesting application because the appropriateness of the latent variable model for personality has been contested. For example, Eysenck (1983) claims that extraversion and neuroticism are strong candidates for referring to a real trait underlying the item responses and according to Eysenck 'there seems to be little doubt that personality traits have a firm basis in the individual's biological structure and functioning', while alternative theories of personality deny such an interpretation of the latent variables in psychometric models for personality (Cramer et al., 2012). Thus, it is interesting to evaluate to what extent the data are compatible with the unidimensional factor model, as Eysenck's theory would predict.

We used data from the 'Vijf PersoonlijkheidsFactoren Test' (Five Personality Factors Test) developed by Elshout en Akkerman (Elshout & Akkerman, 1975). The data consist of 8954 observations on 14 extraversion items on a seven-point Likert scale. For a more detailed description of the dataset, we refer to the paper of Smits, Dolan, Vorst, Wicherts, and Timmerman (2013) who made the data publicly available. We note that the exact wording of the items is not available due to copyright issues (Smits et al., 2013). We here focus on the extraversion factor but because the complete data for all five factors is publicly available anyone who is interested can perform the test on any of the five personality factors.

The upper triangle of Table 3 includes the zero-order correlations between the 14 extraversion items. The lower triangle of Table 3 includes the corresponding partial correlations. As can be seen

TABLE 4.
Lower bound of the CI obtained with the bootstrap test for each zero-order correlation and corresponding partial correlation between the extraversion items.

	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14
V1	-	0.31	0.39	0.57	0.49	0.38	0.33	0.34	1.03	0.20	0.00	0.35	0.54	0.36
V2	-	-	0.85	0.08	1.56	0.31	1.50	0.03	0.2	0.54	0.65	1.26	0.01	0.02
V3	-	-	-	0.44	0.11	0.00	0.61	1.18	0.68	0.09	0.57	0.62	0.91	0.29
V4	-	-	-	-	0.84	0.05	0.17	0.82	0.28	0.57	0.38	0.01	0.43	0.92
V5	-	-	-	-	-	0.88	0.99	0.53	1.76	0.14	0.51	0.00	0.69	0.48
V6	-	-	-	-	-	-	0.57	0.23	0.00	0.54	0.57	1.28	0.61	0.71
V7	-	-	-	-	-	-	-	0.44	0.00	0.29	0.21	0.21	0.00	0.86
V8	-	-	-	-	-	-	-	-	0.10	0.72	0.53	0.91	0.50	0.59
V9	-	-	-	-	-	-	-	-	-	1.01	0.04	1.67	0.18	0.38
V10	-	-	-	-	-	-	-	-	-	-	0.37	0.00	0.87	0.00
V11	-	-	-	-	-	-	-	-	-	-	-	1.22	0.47	0.70
V12	-	-	-	-	-	-	-	-	-	-	-	-	0.38	0.20
V13	-	-	-	-	-	-	-	-	-	-	-	-	-	0.00
V14	-	-	-	-	-	-	-	-	-	-	-	-	-	-

The values that have a gray background refer to a combination of a zero-order correlation and partial correlation for which the CI obtained with the bootstrap test does not include one.

from Table 3, all zero-order correlations between the items are positive. We used the bootstrap function '`onefactor.test()`' that is included in the supplementary materials, and set the number of bootstraps to 100.000. We used a Bonferroni correction to correct for multiple testing. The results of the bootstrap test on the extraversion data are summarized in Table 4 which includes the lower bounds of the CI's for each pair of a zero-order correlation and partial correlation between two variables. For example, consider the zero-order correlation and partial correlation between V3 and V12 (Table 3 shows that these are 0.13 and -0.01, respectively). The lower bound of the CI associated with this pair is 0.62, which is not larger than one, and therefore we conclude that although in the sample the sign for the partial correlation between V3 and V12 is different from the sign of the zero-order correlation, this is not a significant result. Cells with a gray background refer to pairs of variables for which the partial correlation is either significantly stronger than the corresponding zero-order correlation or has a different sign (i.e., the lower bound of the CI is larger than one). There are 10 pairs of variables for which the partial correlation had a different sign than the corresponding zero-order correlation. This means that the bootstrap test rejects the UFM based on the implication of UFMs that all partial correlations should be between zero and the zero-order correlation.

Because the bootstrap test is a local test of model fit, we can use the significant results in Table 4 to eliminate items that cause misfit. To select variables for elimination, we used the selection rule that the variable that is part of the most pairs of two variables for which the correlation and partial correlation are significant in the bootstrap test is removed. For example, Table 4 shows that V12 is part of four pairs of variables for which the correlation has a different sign than the partial correlation ($r_{2,12}$, $r_{6,12}$, $r_{9,12}$ and $r_{11,12}$). We thus started with eliminating the variables V9 and V12 because these both were part of four pairs of variables for which the bootstrap test was significant. As a result, the number of pairs that were significant decreased from 10 to 4 pairs. V2 and V5 were both part of two pairs of variables for which the bootstrap test was significant, and thus, we eliminated V2 and V5 as well. This resulted in a set of 10 variables (all variables except for V2, V5, V9 and V12) for which the bootstrap test does not reject the UFM.

5. Discussion

In this paper we proved that the UFM implies that partial correlations between observed variables given any subset of the other observed variables are always closer to zero than the zero-order correlation between these observed variables. To the best of our knowledge, this is a new result, which implies that the identification of partial correlations in the data that are significantly stronger than the zero-order correlation may cast doubt on whether the UFM is the data-generating model. To facilitate the use of this result in data analysis, we presented a bootstrap test to evaluate the tenability of the implication and illustrated the use of the test in an empirical example.

Most of the fit indices for the UFM available in the literature rely on distributional assumptions or asymptotic normality (e.g., the gamut of fit measures based on normal theory LRT or approximate asymptotic distributions; see Bollen and Ting (1993) for an overview). The main result of this paper only relies on the structure of the covariance matrix and not on properties of the statistical distribution such as the normal distribution. The proposed bootstrap method is essentially distribution free. Furthermore, as argued in Browne and Cudeck (1992), the factor model in itself is all but certain to be incorrect. Therefore, instead of focusing on the absolute fit of the factor model, it is more sensible to assess where the lack of fit occurs (Browne & Cudeck, 1992). Saris, Satorra, and Van der Veld (2009) show that global fit indices (FIs) vary in their sensitivity to different types of misspecifications so that a model with a small substantively irrelevant misspecification may be rejected because the FI is sensitive to such misspecifications, while this FI may support the acceptance of a model with an important misspecification due to its low sensitivity to such misspecifications. Claeskens and Hjort (2008) even argue that the ‘best model’ depends on the parameter of interest and propose the use of focused information criteria tailored to parameters of interest. The inequality of the main result of this paper allows for a local fit evaluation that indicates where misfit occurs. In addition, the procedure can be used to guide the selection of items in a test: Items that are consistently involved in violations of the proposition should be considered for removal (see Clark and Watson (1995) for a discussion of item selection).

We believe that the result presented in this paper has the potential to help decide whether a UFM is a plausible candidate to have generated the observed data. However, we also think that the result is not useful to test the dimensionality of the factor model when comparing different factor models. This is because we think it is likely that the implication presented in this paper generalizes to, for example, correlated factor models or hierarchical factor models. That is, partial correlations between zero and the zero-order correlation are a necessary condition for the UFM but not a sufficient condition as this property likely holds for other factor models as well. In a correlated factor model, for example, the correlation between two observed variables that load on different factors equals the product of their factor loadings and the correlation between the factors these observed variables load on. As such, the correlation matrix and partial correlation matrix implied by such models will not only be a function of the factor loadings but also of Ψ, the correlation matrix of the latent variables. Future research might reveal how Ψ in the function for the correlation and partial correlation influences the relative size of the partial correlation with respect to the correlation. Another interesting extension would be to investigate whether the result presented in this paper generalizes to other types of latent variable models such as IRT models, in which the test scores are not continuous. Finding such extensions of the result to other latent variable models implies that this result should not be used to compare different factor models with each other within the realm of factor modeling, but rather provides a first test of whether one should enter the gateway to the factor modeling world at all. We hope that future research explores how general the class of latent variable models is for which this property holds. For now we can conclude that the identification of partial correlations that are not between zero and the zero-order correlation provides evidence against a UFM.

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