Dual views of string impurities. Geometric singularities and flux backgrounds
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The objective of this chapter is to collect and expose several different geometric perspectives which can be used to describe supersymmetric 'compactifications' of string theory. The term 'compactification' is somewhat inappropriate, as most of the 'compactification' spaces discussed are not compact and also often singular. Such spaces are considered as local models of degenerate limits of smooth but not necessarily compact manifolds which can make up part of a string vacuum. In chapter 4 the physical motivation of such degenerate limits is discussed. Very briefly stated, it is possible that some cycles in a smooth manifold become small. Then some massive nonperturbative degrees of freedom of the compactified theory become light and make up physics which is localized at the degeneration of the manifold. This 'localized physics' can be decoupled in appropriate scaling limits. It depends on the local geometry near such a degeneration.

At present we are concerned with the geometry of such local models. Differential and algebraic geometric methods exist to characterize some of these. The various characterizations are interconnected in intriguing and insufficiently understood ways, and also connected to various descriptions of possible worldsheet conformal field theories, which are discussed in chapter 3. In this chapter the following topics are discussed.

If a space is part of a supersymmetric string vacuum, it must satisfy certain differential geometric requirements. For example, it might have to be Ricci flat and Kähler. Such requirements also hold for singular spaces. The four dimensional singular spaces which fit the bill are the hyper-Kähler surface singularities. These have various interchangeable descriptions, notably as quotient singularities, as hypersurfaces embedded in $\mathbb{R}^6 \cong \mathbb{C}^3$ and as metric cones.

These descriptions can be used to describe many higher dimensional singularities as well, where the focus will be on complex singularities. It is however not true, that any
Chapter 2 - Supersymmetry, Spinors and Holonomy

given singularity can be described in all of the above fashions. Metric cones are interesting because the differential geometric constraints on the cone lead to constraints on the base of the cone. Typically the base, also known as the link of the cone is a Sasaki-Einstein manifold. Sasaki manifolds have a circle isometry and the corresponding orbit space is Kähler. For a Sasaki-Einstein manifold, it is Kähler-Einstein.

Some examples of Kähler-Einstein spaces are homogeneous. A considerable number can be constructed as hypersurfaces where a weighted homogeneous polynomial vanishes in an appropriate weighted homogeneous space. Such examples often have orbifold singularities. The zero locus of such a polynomial in affine space is precisely a supersymmetric singularity. A class of very interesting polynomials are not precisely of the form for which the known proof is valid. These polynomials ‘define’ certain (Landau-Ginzburg) conformal field theories which also have a geometric (sigma model) interpretation.

The generic presence of a circle isometry that exists for a Sasakian manifold partly motivates the study of T-duality for complex supersymmetric singularities in chapter 4. Some ingredients in the description of such singularities return in an apparently quite different context in chapter 3, where they are used to construct abstract conformal field theories which describe supersymmetric string vacua. In particular, weighted homogeneous polynomials are quite generally used to construct superconformal field theories. Some specific choices of the polynomial correspond to conformal field theories which have a known interpretation as coset conformal field theories. The corresponding symmetric spaces are Kähler-Einstein.

2.1 Supersymmetry, Spinors and Holonomy

2.1.1 Supersymmetry and Differential Geometry

We are interested in supersymmetric vacua of string theory of the form

\[ \mathcal{M}_{10} = \mathbb{R}^{9-d,1} \times \mathcal{M}_d, \]  

in the absence of fluxes and with a constant dilaton. If \( \mathcal{M}_d \) is a smooth \( d \)-dimensional manifold, the low energy effective theory is the appropriate supergravity theory in this background. If this geometry is indeed a vacuum, the Ricci tensor of \( \mathcal{M}_{10} \) must vanish. To find the number of conserved supersymmetries in this background one considers the supersymmetric variations of all the fermionic fields. In the backgrounds of this form, these variations are parametrized by a spinor field. They are proportional to the spinor or to its covariant derivative. The number of conserved supersymmetries is equal to the number of covariantly constant sections of the spinor bundle over \( \mathbb{R}^{9-d,1} \times \mathcal{M}_d \), times the number of independent supersymmetry transformations that can be constructed out of one section, which is \( n = 1 \) for heterotic and \( n = 2 \) for Type II theories. The spinors can be decomposed into spinors over \( \mathcal{M}_d \) and spinors over \( \mathbb{R}^{9-d,1} \). The number of supersymmetry charges conserved by the background \( \mathbb{R}^{9-d,1} \times \mathcal{M}_d \) is

\[ s = n \, 2 \, \text{tr}^{-\frac{10-d}{2}} \ell, \]  

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where $n = 1$ for heterotic theories and $n = 2$ for Type II theories. The number $2 \lfloor \frac{10 - d}{2} \rfloor$ is the number of covariantly constant spinors on $\mathbb{R}^{d-1,1}$, and $\ell$ is the number of covariantly constant spinors on $\mathcal{M}_d$. So the condition for supersymmetry is that $\mathcal{M}_d$ has at least one covariantly constant spinor.

Manifolds which admit covariantly constant spinors are characterized by their holonomy. The holonomy group of a general $\mathcal{M}_d$ is $SO(d)$, but if its spinor bundle admits a covariantly constant section, the holonomy group has to be a proper subgroup $H \subset SO(d)$. After all, the covariantly constant spinor obviously transforms in the trivial representation of $H$, but this representation must be obtained by decomposing the spinor representation of $SO(d)$ into representations of $H$.

The possible subgroups that can appear are classified. If $\mathcal{M}_d$ is a product manifold, its holonomy group is the product group of the individual holonomy groups. If $\mathcal{M}_d$ is a simply connected Riemannian symmetric space it can be written as $G/H$ where $G$ is a Lie group of isometries that acts transitively and $H \subset G$ is the isotropy subgroup, which leaves a point fixed, then $\text{Hol}(\mathcal{M}_d) = H$. If $\mathcal{M}_d$ is a simply connected Riemannian symmetric space $G/H$, the holonomy group is $H$. This was shown long ago by Cartan. Finally, if $\mathcal{M}_d$ is a simply connected Riemannian manifold that is not a product manifold and non-symmetric, there is a list of possible holonomy groups, due to Berger. In addition to the generic case $SO(d)$, there are the cases listed in table 2.1.

The holonomy groups in table 2.1 imply certain parallel tensors, and hence certain geometric structures, see, for example [65]. If the holonomy is $U(n)$, it is possible to split the tangent bundle into a holomorphic and an antiholomorphic part. Such a split is effected by the complex structure $J(\cdot, \cdot)$ which is an endomorphism of the complexified tangent bundle of $\mathcal{M}$. To speak of holonomy, there must be a connection. It is always possible to choose a Hermitian metric $g$ compatible with $J$, i.e. $g(\cdot, \cdot) = g(J \cdot, J \cdot)$. From these two structures it is possible to construct a two-form $\omega(\cdot, \cdot) = g(J \cdot, J \cdot)$, using the property that $J^2 = -1$. This two-form is non-degenerate. If it is also closed, $\omega$ is symplectic and, by compatibility with $J$, Kähler; the Hermitian connection coincides with the Christoffel connection and it is the sum of a holomorphic one-form taking values in the endomorphisms of the holomorphic

<table>
<thead>
<tr>
<th>Dimension $d$ of $\mathcal{M}_d$ is $d$</th>
<th>Holonomy group $\text{Hol}(\mathcal{M}_d)$</th>
<th>Name of $\mathcal{M}_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 2n$</td>
<td>$U(n)$</td>
<td>Kähler</td>
</tr>
<tr>
<td>$d = 2n$</td>
<td>$SU(n)$</td>
<td>Calabi-Yau</td>
</tr>
<tr>
<td>$d = 4n$</td>
<td>$Sp(n)$</td>
<td>Hyper-Kähler</td>
</tr>
<tr>
<td>$d = 4n$</td>
<td>$Sp(n)Sp(1)$</td>
<td>Quaternionic Kähler</td>
</tr>
<tr>
<td>$d = 7$</td>
<td>$G_2$</td>
<td>$G_2$-Manifold</td>
</tr>
<tr>
<td>$d = 8$</td>
<td>$Spin(7)$</td>
<td>$Spin(7)$-Manifold</td>
</tr>
</tbody>
</table>

Table 2.1: Berger's list of possible reduced holonomy groups of simply connected irreducible non-symmetric Riemannian manifolds.
tangent bundle, in addition there is an entirely antiholomorphic equivalent. This means that under parallel transport (anti-)holomorphic tangent vectors remain (anti-)holomorphic, so the holonomy is contained in $U(n)$. Using this connection $J$ is covariantly constant, and so is $\omega$.

On a Kähler manifold one can construct the Ricci form from the Riemann tensor of the Kähler metric, using the complex structure: using the Dolbeault differentials $\partial$ and $\bar{\partial}$ it can be expressed as $\mathcal{R} = i\partial\bar{\partial} \log \sqrt{\det g}$. This is manifestly closed, but usually not exact, because $\det g$ is not a scalar. The cohomology class of the Ricci form is $2\pi$ times the first Chern class of the (tangent bundle of the) Kähler manifold. The Chern class is an analytic invariant: continuous changes of the metric do not alter the cohomology class of $\mathcal{R}$.

In addition to preserving some supersymmetry, the geometry of (2.1) should solve the equations of motion, which means that the Ricci tensor of $\mathcal{M}_d$ must vanish. The $U(1) \to U(n)$ part of the holonomy is generated by Ricci tensor. So if the Ricci tensor vanishes, the holonomy is $SU(n) \subset U(n)$. But given a Kähler manifold with Kähler form $\omega$ it is possible to deform this to $\omega'$ without altering the cohomology class of the Kähler form (the Kähler form cannot be exact, because that would be contradictory to it being non-degenerate). The new Kähler form $\omega'$ is such that its associated Ricci form is precisely the first Chern class. Yau's theorem implies that such a choice of $\omega'$ is always possible. So a Kähler manifold with $SU(n)$ holonomy admits a metric with vanishing Ricci tensor. The restrictions on a hyper-Kähler metric are so strong, that necessarily any such metric is Ricci flat.

The hyper-Kähler and Calabi-Yau manifolds, and singularities, will play a considerable rôle in the rest of this chapter. Some important reasons for this are the following. As complex manifolds, powerful tools from algebraic geometry are known to study such spaces. The Kähler structure of these manifolds appears naturally in $\mathcal{N} = 2$ superconformal models discussed in chapter 3. The properties of these models are used in chapter 4 to relate hyper-Kähler and Calabi-Yau singularities to other backgrounds of string theory.

### 2.1.2 Hyper-Kähler Surface Singularities

This section discusses the geometry of the best understood supersymmetric singularities: complex surface singularities which are hyper-Kähler. These are complex surfaces, so locally they look like $\mathbb{C}^2 \simeq \mathbb{R}^4$, have holonomy group $Sp(1) \simeq SU(2)$, with an isolated singularity. A great deal is known about these, both from a mathematical point of view and also from the perspective of string theory. Because so much is known about them, they take a special place. Some of the special properties they have are:

- They are classified;
- The classification is isomorphic to that of many other interesting objects in mathematics and string theory;
They have a number of different descriptions which illustrate descriptions of higher dimensional singularities;

- For the hyper-Kähler surface singularities all descriptions are interchangeable, unlike for higher dimensional ones;

- The hyper-Kähler singularities are a motivation and the clearest example of the T-duality for cones discussed in chapter 4.

One way to describe the hyper-Kähler surface singularities, is as quotients of \( \mathbb{C}^2 \). On a space of \( SU(2) \) holonomy there is a parallel holomorphic two-form. On the covering \( \mathbb{C}^2 \) such a two form can be taken as \( \omega = dz_1 \wedge dz_2 \). This two-form is preserved by \( SU(2) \) mixing the holomorphic coordinates. This group has a fixed point at the origin. Take an discrete subgroup \( \Gamma \subset SU(2) \). Then the quotient space \( \mathbb{C}^2 / \Gamma \) is a complex surface with a singularity at the origin and \( SU(2) \) holonomy, with the constant holomorphic two-form given by projection of \( dz_1 \wedge dz_2 \) on the covering space.

The discrete subgroups of \( SU(2) \) were classified in the nineteenth century by Klein and the quotient singularities \( \mathbb{C}^2 / \Gamma \) are also referred to as Kleinian singularities. The Kleinian singularities exhaust the hyper-Kähler surface singularities. There is a one-to-one correspondence of the subgroups \( \Gamma \subset SU(2) \) and Dynkin diagrams of simply laced Lie algebras. This motivates the name ‘ADE-singularities’ which is also commonly used. In fact, there is a huge web of connections, containing the topology of desingularizations of these singularities, the representation theory of \( \Gamma \subset SU(2) \) [60] and a lot of different areas of mathematics and physics, such as conformal field theory [17] and gauge theories [61].

From the description as quotients, one can obtain a different description. One can think of a point in \( \mathbb{C}^2 \) as the zero of a monomial

\[
z_0 \leftrightarrow (z - z_0) = 0. \tag{2.3}
\]

Such monomials are the prime divisors of polynomials with complex coefficients, and the algebraic structure of polynomials can be used to study geometry. An arbitrary divisor in the polynomial ring \( \mathbb{C}[z_1, z_2] \) is of the form

\[
\prod_{i=1}^{k} (z - z_i)^{\alpha_i},
\]

and can be viewed as the divisor

\[
\sum_{i=1}^{k} \alpha_i [z_i]
\]

in the sense of algebraic geometry.
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<table>
<thead>
<tr>
<th>$\Gamma \subset \mathbb{C}^2$</th>
<th>$F_\Gamma(z_1, z_2, z_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$z_1^n + z_2^2 + z_3^2$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$z_1^{n-1} + z_1z_2^2 + z_3^2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$z_1^4 + z_2^2 + z_3^2$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$z_1^3 + z_1z_2^3 + z_3^2$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$z_1^5 + z_2^3 + z_3^2$</td>
</tr>
</tbody>
</table>

Table 2.2: The hyper-Kähler surface singularities as quotients $\mathbb{C}^2/\Gamma$ and as surfaces $F_\Gamma^{-1}(0) \subset \mathbb{C}^3$.

From the point of view of algebraic geometry it is the polynomial ring $\mathbb{C}[z_1, z_2]$, generated by $z_1$ and $z_2$ which characterizes the space. Consider the $A_n$ singularity

$$A_n = \mathbb{C}^2/\Gamma,$$

$$\Gamma : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{n+1}}z_1, e^{-\frac{2\pi i}{n+1}}z_2). \quad (2.4)$$

Not every polynomial in $\mathbb{C}[z_1, z_2]$ is invariant under the action of $\Gamma$. The subset of $\Gamma$ invariant polynomials is generated by the three generators

$$u = z_1^{n+1},$$

$$v = z_2^{n+1},$$

$$x = z_1z_2, \quad (2.5)$$

which clearly satisfy the relation

$$uv = z_1^{n+1}. \quad (2.6)$$

So the divisors on $A_n$ are those polynomials in $\mathbb{C}[u, v, x]$ which vanish on the hypersurface defined by (2.6). Or, put differently, as far as algebraic geometry is concerned, the quotient singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$ is the hypersurface $z_1^{n+1} + z_2^2 + z_3^2 = 0$ in $\mathbb{C}^3$.

Similarly all the ADE-singularities\(^1\) have a description as surfaces $F_{ADE}^{-1}(0)$ in $\mathbb{C}^3$. The polynomials $F_{ADE}(z_1, z_2, z_3)$ are collected in table 2.2. Note that all the polynomials are weighted homogeneous, i.e. for each $F_\Gamma$ there exists a set of weights $a_i$ which are (positive) integers, such that

$$F(\lambda^{a_1}z_1, \lambda^{a_2}z_2, \lambda^{a_3}z_3) = \lambda^dF(z_1, z_2, z_3). \quad (2.7)$$

The description as a quotient singularity $\mathbb{C}^2/\Gamma$ also provides a third description, which is more differential geometric in nature. The space $\mathbb{C}^2\setminus\{0\}$ can be fibered by three-spheres.

\(^1\)The $D_{k+2}$ singularity can be obtained by a $\mathbb{Z}_6$ quotient of the $A_k$ singularity. The $A_k$ singularity is $\mathbb{C}^2/\mathbb{Z}_{k+1}$ where $\mathbb{Z}_{k+1}$ acts on the coordinates of $\mathbb{C}_2$ as $(z_1, z_2) \sim (e^{2\pi i/(k+1)}z_1, e^{-2\pi i/(k+1)}z_2)$. Quotienting further by $\mathbb{Z}_2 : (z_1, z_2) \sim (z_2, -z_1)$ yields a $D_k$ singularity.

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The metric $\text{d}s^2 = \text{d}z_1 \text{d}\bar{z}_1 + \text{d}z_2 \text{d}\bar{z}_2$ is written as $\text{d}s^2 = \text{d}r^2 + r^2 \text{d}\Omega^2$, i.e. a cone over the three sphere. As $SU(2)$ acts on $\mathbb{C}^2$ in a way that leaves invariant $r^2 = |z_1|^2 + |z_2|^2$, an ADE-singularity can be written as the metric cone

$$\mathbb{C}^2/\Gamma = \mathbb{R}_+ \times S^3/\Gamma,$$

$$\text{d}s^2 = \text{d}r^2 + r^2 \text{d}\Omega^2, \quad (2.8)$$

where $d\Omega^2$ is the line element on the smooth space $S^3/\Gamma$. The action of $\Gamma$ on $S^3$ is obtained from the action in the embedding $\mathbb{C}^2$. The spaces $S^3/\Gamma$ are simple examples of a more general class discussed in section 2.2, which can all be viewed as circle fibrations.

The base of each $A_k$ metric cone, $S^3/Z_{k+1}$ is a circle bundle over $S^2$, and in fact all circle bundles over the two-sphere are of this form (they are so-called lens spaces). One way to view the lens spaces $S^3/Z_{k+1}$, is as quotient spaces $(S^3 \times S^1)/U(1)$, see for example [80]. Let $S^3$ be parametrized by $z = (z_1, z_2) \in \mathbb{C}^2$ that satisfy the condition $|z_1|^2 + |z_2|^2 = 1$. Let $S^1$ be parametrized by $\sigma = e^{i\theta}$. The $U(1)$ equivalence relation identifies $(z_1, z_2) \sim (e^{i\phi} z_1, e^{i\phi} z_2)$ and $\sigma \sim e^{-i(k+1)\phi} \sigma$. By an equivalence transformation one can always set $\sigma = 1$, unless $k + 1 = 0$. This ‘gauge choice’ fixes the $U(1)$ action up to a $Z_{k+1}$ subgroup. So quotient space is $S^3/Z_{k+1}$. This is bundle over $S^2$, with the projection

$$\pi : S^3/Z_{k+1} \rightarrow S^2$$

$$z \mapsto \bar{v} = z^\dagger \bar{\sigma} z, \quad (2.9)$$

where $\bar{\sigma}$ indicates the three Pauli matrices. The vector $\bar{v}$ has unit length, because $|z_1|^2 + |z_2|^2 = 1$, and hence parametrizes $S^2$. When $k + 1 = 0$, the total space is the trivial bundle $S^2 \times S^1$, and when $k + 1 = 1$, the fiber bundle structure is the Hopf fibration $S^1 \rightarrow S^3 \rightarrow \mathbb{P}^1 \sim S^2$.

The bases of the $D_{k+2}$ metric cones can be considered in a similar fashion, as quotient spaces $(S^3 \times S^1)/(U(1) \times Z_2)$. The $U(1)$ part acts as it does in the $A_k$ case, the $Z_2$ acts as

$$Z_2 : ((z_1, z_2); s) \mapsto ((z_2, -z_1); s). \quad (2.10)$$

The $Z_2$ action also acts on the image of the projection $\pi$. The image is not the entire $S^2$, but rather $S^2/Z_2$, with antipodal points identified, i.e. the bases of the $D_{k+2}$ metric cones are circle bundles over the base $\mathbb{P}^2$.

The different descriptions each have their advantages, emphasizing different properties of the ADE-singularities. The algebraic geometric description as surfaces in $\mathbb{C}^3$ emphasizes the complex structure of the singularity. Actually, since these are hyper-Kähler spaces, they have three independent complex structures $I_1, I_2, I_3$ and $a_1 I_1 + a_2 I_2 + a_3 I_3$ is again a complex structure if the three real numbers $a_i$ satisfy $a_1^2 + a_2^2 + a_3^2 = 1$. So it is better to say that it emphasizes one particular complex structure out of the whole $S^3$'s worth. A deformation of the polynomial defining the hypersurface corresponds to a deformation of the complex geometry of the singularity.
Consider the example of an $A_1$ singularity, defined as $uv - x^2 = 0$ in $\mathbb{C}^3$. This can be deformed to $uv = (x + \epsilon)(x - \epsilon)$. The surface defined by this deformed equation no longer passes through $u = v = 0$, where the singular point was. Instead, the product of the moduli $|u|$ and $|v|$ is determined by the equation, and it vanishes at $x = \pm \epsilon$. Only the difference of phases of $u$ and $v$ is free. In the surface $uv = (x + \epsilon)(x - \epsilon)$ there is a two sphere which is a circle fibration over the line segment from $x = -\epsilon$ to $x = +\epsilon$. This is an example of a kind of deformation which can be applied to any polynomial which defines a hypersurface with an isolated singularity at the origin:

$$F(x_1, \ldots, x_n) \rightarrow F(x_1, \ldots, x_n) + \mu.$$  

(2.11)

This deformation will be considered in chapter 4.

It is possible to characterize all deformations of the ADE-singularities. The number of independent deformations actually equals the rank of the corresponding ADE Lie algebra. By successive deformations a singular surface can be ‘desingularized’ by blowing up two-spheres. Hyper-Kähler metrics on the resulting smooth non-compact manifolds are known [62, 63, 64]. The construction of these metrics makes use of the fact that the singular spaces are quotient singularities $\mathbb{C}^2/\Gamma$ and the McKay correspondence [60] which relates the representation theory of $\Gamma$ and the topology of the smoothed space. Far away from the origin, the smoothing does not change much and the smooth metrics asymptote to the metric cones $\mathbb{R}_+ \tilde{x}(S^3/\Gamma)$.

Crucially, in one description the differential geometry of a singularity is explicit but deformations of the singularity are not at all apparent: this is the metric cone description. In another description deformations are apparent, but there is no hyper-Kähler metric apparent: the description as surfaces in $\mathbb{C}^3$. The logical connection between these two descriptions, is the realization as quotient singularities. The deformation parameters in the polynomials are related to the representation theory of the quotient group.

In higher dimensions, not all descriptions of supersymmetric singularities are interchangeable. That is to say, there are supersymmetric singularities which are not quotient singularities. Such singularities may have descriptions as Ricci flat metric cones with the right holonomy, $SU(n)$ or $Sp(n/2)$, but whose base manifolds are not $S^{2n-1}/\Gamma$. It is not so clear how to deform such a metric conical singularity to a smooth space which still admits a Calabi-Yau or hyper-Kähler metric if there is no apparent hypersurface description $F^{-1}(0) \subset \mathbb{C}^{n+1}$. Nor is it immediately clear if there might be a hypersurface description. In fact, for a lot of interesting singularities there is no hypersurface description, like for example $\mathbb{C}^3/\mathbb{Z}_3$. Approaching the matter from the other direction, starting with a hypersurface singularity, it is often difficult to find a differential geometric description of it, like an explicit metric, or the group of isometries of the space.

These issues are discussed in the subsequent sections. Typical questions are the following. What are the conditions on a polynomial $F$ so that $F^{-1}(0)$ in $\mathbb{C}^{n+1}$ is a supersymmetric singularity which can be used as a string vacuum? What can be said about the geometry of a singularity defined by such a polynomial? If the singularities are not quotient singu-
larities, what is left of existing and conjectured correspondences in the spirit of the McKay correspondence, and what new correspondences are gained by leaving the set of quotient singularities? Some questions will be answered in the following sections, and some interconnections will be discussed. Together with the ingredients of chapter 3 these will be put to use in chapter 4.

2.2 METRIC CONES

An acceptable supersymmetric singularity of dimension $d = 2n$ which can serve as a string background must be Ricci flat and have a holonomy group which is contained in $SU(n)$. Take as such a singularity the metric cone $C(L)$,

$$C(L) = \mathbb{R}_+ \times L$$

$$ds^2_{2n} = dr^2 + r^2 ds^2_{2n-1}. \quad (2.12)$$

That is to say, it is the warped product of the manifold $L$ of dimension $2n - 1$ with the half line $r > 0$, with the above metric. The question is: what are the properties of the base manifold $L_{2n+1}$?

An answer was given by Bär [7], who studied metric cones of restricted holonomy. Essentially, one uses the canonical vector field on a metric cone, $r\partial/\partial r$, called the Euler vector field. With this vector field, the different special tensor fields on the cone can be mapped to special tensor fields on $L_{2n+1}$.

First, if the Ricci tensor of $C(L)$ vanishes, then $L$ is a positively curved Einstein manifold. We call a manifold Einstein if there is a constant number $\lambda$ such that the Ricci tensor $Ric$ and the metric tensor $g$ satisfy

$$Ric = \lambda g, \quad (2.13)$$

i.e. its scalar curvature is a constant. Only the sign of the Ricci curvature is really interesting, since the absolute value can be changed by rescaling $L$. Conversely, if $B$ is an Einstein manifold of positive curvature, it can always be appropriately scaled to make $C(L)$ a Ricci flat cone².

THE GEOMETRY OF $L$

Next, the restricted holonomy of $C$ gives rise to various parallel tensors on the cone. The Kähler form $\omega$ on $C(L)$ satisfies $d\omega = 0$ and $\wedge^n \omega \neq 0$. Contracting the Euler vector with $\omega$ yields a one-form $\eta$ on $L$. This one-form satisfies

$$\eta \wedge (d\eta)^{n-1} \neq 0, \quad (2.14)$$

²The rescaling is proportional to $n - 1$, with some constants of proportionality dependent on conventions, $n = d/2$ being the complex dimension of the cone.
everywhere on $L$. This equation states that $\eta$ is a contact form on $L$. A symplectic metric cone $C(L)$ has a base $L$ that is a contact manifold. In addition to the contact form, a contact manifold also has a unique vector field, dual to $\eta$: the Reeb vector field $\xi$. It satisfies

$$
\iota_{\xi} \eta = 1
$$

$$
\iota_{\xi} d\eta = 0.
$$

The Reeb vector field on $L$ is obtained from the complex structure $J$ on $C(L)$, by acting with $J$ on the Euler vector field of $C(L)$. The contact form $\eta$, Reeb vector field $\xi$ and an endomorphism $T$ of the tangent bundle $TL$ together define an almost contact structure on $L$. They satisfy

$$
\iota_{\xi} \eta = 1
$$

$$
T^2 = -id + \xi \otimes \eta.
$$

A compatible metric $g$ must satisfy

$$
g(T(\cdot),T(\cdot)) = g(\cdot,\cdot) - \eta(\cdot)\eta(\cdot),
$$

analogous to an almost Hermitean metric on an almost complex manifold.

On $B$ the endomorphism $T : TL \rightarrow TL$ is obtained as

$$
T(\phi) = -\nabla_{\phi} \xi,
$$

via the covariant derivative, where $\phi$ is any section of $TL$. The tensor fields $\xi, \eta, T$ and $g$ on $L$ form a special kind of metric contact structure because $L$ is the base of a metric cone $C(L)$ which is Kähler, i.e. on which the complex structure, Hermitean metric and symplectic form are compatible. This special kind of metric contact structure is called a Sasaki structure, and $L$ is a Sasaki manifold.

One definition of a Sasaki manifold, is precisely that the metric cone over a manifold is Kähler iff the manifold is Sasaki. An equivalent definition, see for example [8], is a Riemannian manifold $(M, g)$ with a Killing vector field of unit length $\xi$, and endomorphism $T$ defined as $T(\phi) = -\nabla_{\phi} \xi$ for any section $\phi$ of $TM$ that satisfies

$$
(\nabla_{\chi} T) \psi = g(\chi, \psi) - g(\xi, \psi) \chi,
$$

for all vector fields $\chi, \psi$.

If the cone $C(L)$ is hyper-Kähler, it has three independent complex structures which form a quaternion algebra. Analogously, $L$ inherits three related Sasakian structures and $L$ is a tri-Sasakian manifold. A good overview of the properties of (tri-) Sasaki manifolds used in this section and the next, is [8].

The Reeb vector field $\xi$ that any Sasaki manifold $L$ has (often called its characteristic vector field), gives rise to some important consequences. For one thing, it means that a
metric cone has a Killing vector field which degenerates at the apex \( r = 0 \). One might be tempted to perform a T-duality along this isometry, and we are tempted to do so in chapter 4. The vector field \( \xi \) is also very interesting from a purely geometric point of view. Note that because \( \xi \) is nonvanishing, its integral curves define a one-dimensional foliation of \( L \).

The space of leaves of this foliation turns out to be quite interesting. We call the space of leaves \( Z \). When the leaves are closed curves, so the Reeb vector field is a Killing vector field of a \( U(1) \) isometry, \( L \) is called quasi-regular. In this case \( Z \) is a Kähler space which can have finite quotient singularities. When \( Z \) is a smooth Kähler manifold, \( L \) is called regular. If \( Z \) has finite quotient singularities, \( L \) is called non-regular (\( L \) is called irregular if the leaves do not close).

Regularity is a very strong condition and many examples of Sasaki-Einstein manifolds are non-regular. Explicit metrics are rarely known, with the exception of homogeneous spaces. As we will see shortly, methods and results from algebraic geometry have provided means to prove the existence of (quasi-regular) Sasaki-Einstein metrics on a much larger class of spaces. However, these methods are not constructive, and they give only limited information about the differential geometry of the spaces. The spaces for which these methods apply, are described as specific kinds of affine hypersurfaces. This description is compatible in a natural way with our duality prescriptions discussed in chapter 4.

Recently explicit metrics have been found for many five and seven dimensional Sasaki-Einstein manifolds, including the first irregular ones [104, 73, 74], using a supergravity/string theory approach. Our present interest will be with quasi-regular Sasaki-Einstein manifolds, but within an adapted framework, irregular ones should be of great interest as well, especially for string theory. For example, they could be related to rather exotic irrational conformal field theories, through a gauge/gravity correspondence. We will not discuss these further. Rather, we focus of the geometry of the leaf-space \( Z \) of a quasi-regular Sasaki-Einstein manifold.

**THE GEOMETRY OF \( Z \)**

If each point in \( B \) has a neighborhood such that any leaf of the characteristic foliation intersects the transversal at most a finite number of times \( k \), then \( L \) is called quasi-regular. Equivalently \( B \) is quasi-regular if the leaves are compact. So all Sasaki manifolds which appear as compact bases of cones are quasi-regular. If \( k = 1 \), \( L \) is called regular. A quasi-regular \( L \) that is not regular, is called non-regular. Regularity is a very strong condition. The vast majority of compact Sasaki spaces is non-regular.

At this point we have seen that the particular structure of a metric cone, or the Euler vector field, led to geometric structures on the link \( L \leftrightarrow C(L) \). The metric Calabi-Yau cones have Sasaki-Einstein links, either regular or non-regular. The hyper-Kähler cones have tri-Sasaki links, which will be discussed in more depth later. Now focus on the Sasaki-Einstein manifolds\(^3\), and to be more specific, on the regular Sasaki-Einstein manifolds. It is

\(^3\)The curvature of a Sasaki-Einstein manifold is necessarily positive and hence it can always be used to construct
useful to consider the leaf space $Z$ of the foliation of $L$ by the Reeb vector field,

$$\pi : L \longrightarrow Z.$$  

The regular Sasaki structure ensures that $S$ is a smooth Kähler manifold, and the fact that $L$ is Sasaki-Einstein results in $Z$ being Kähler-Einstein. Moreover $Z$ is positively curved, $c_1(Z) > 0$: $Z$ is a Fano manifold with a smooth Kähler-Einstein metric.

Explicit realizations of Kähler-Einstein Fano manifolds are provided by Hermitian symmetric spaces. These are compact Kähler manifolds and Riemannian symmetric spaces, and positively curved. As an aside, as such these spaces are geometrically formal, that is to say, the wedge product of harmonic forms is again a harmonic form. It is proved in [66] that any geometrically formal Kähler manifold of non-negative Ricci curvature is Einstein. The Hermitian symmetric spaces play an important part in the construction of superconformal field theories 3.4. The harmonic forms on the Hermitian symmetric spaces are in one-to-one correspondence with $(c, c)$ primary operators in the conformal field theory. These special fields have the property that under the naive operator product, they form a nilpotent ring.

The Hermitian symmetric spaces are classified. Only spaces of which the dimension is not too large can be used to build metric cones for a superstring compactification. The Hermitian symmetric spaces are listed in table 2.3.

In dimension $d = 2$, the only Kähler-Einstein manifold with $c_1 > 0$ is

$$\mathbb{P}^1 \simeq SU(2)/U(1).$$  

In dimension $d = 4$, the manifolds with $c_1 > 0$ are known as del Pezzo surfaces, those which admit a Kähler-Einstein metric have been classified [38] and are collected in table 2.4. On the del Pezzos obtained by blowing up $\mathbb{P}^2$ at three to eight generic points, no explicit

4A manifold with $c_1 > 0$ is called a Fano manifold.
Chapter 2 - Metric Cones

<table>
<thead>
<tr>
<th>$L$, del Pezzo surface</th>
<th>Homogeneous, $G/H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}^2$</td>
<td>$SU(3)$</td>
</tr>
<tr>
<td>$\mathbb{P}^1 \times \mathbb{P}^1$</td>
<td>$SU(2) \times SU(2)$</td>
</tr>
<tr>
<td>$dP_n = \mathbb{P}^2 # \mathbb{P}^2$, $3 \leq n \leq 8$</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 2.4: Smooth del Pezzo surfaces admitting a Kähler-Einstein metric.

metrics are known. The del Pezzo surfaces $dP_1$ and $dP_2$ do not feature in the classification [38] of Tian and Yau. It is a well known fact in the mathematics community, that the del Pezzo surfaces $dP_1$ and $dP_2$ do not admit a Kähler-Einstein metric\(^5\).

In general there can be several Sasaki-Einstein circle bundles over a base $Z$

$$C(L) \rightarrow L \xrightarrow{\pi} Z \quad (2.19)$$

The first Chern class of the circle fibration $L$ must divide the first Chern class of $Z$ [36, 67] in order to get a smooth total space. In concreto this means that the possible regular Sasaki-Einstein manifolds are\(^6\)

i. $S^5 \rightarrow \mathbb{P}^2$,

ii. $S^5/\mathbb{Z}_3 \rightarrow \mathbb{P}^2$,

iii. $T^{1,1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$,

iv. $T^{1,1}/\mathbb{Z}_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$,

v. $S_n \rightarrow dP_n$.

The metric cone over $S^5$ is just $\mathbb{R}^6$ and therefore not interesting from the point of view of singularities. The manifold $T^{1,1} \cong SO(4)/SO(2) \cong (SU(2) \times SU(2))/U(1)$ is the link of the conifold. There is a natural interpretation why only the $\mathbb{Z}_3$ quotient of $S^5$ gives a regular Sasaki-Einstein space, from the perspective of quotienting $\mathbb{C}^3$ by a discrete subgroup $\Gamma \subset SU(3)$. $\mathbb{C}^3$ can be viewed as the total space of the tautological bundle over $\mathbb{P}^2$. The $U(1) \hookrightarrow SU(3)$ which acts only on the fiber but not on the base, acts on the homogeneous coordinates as $[z_1 : z_2 : z_3] \mapsto [\eta z_1 : \eta z_2, \eta z_3]$. The only nontrivial discrete subgroup

\(^5\)This is because their automorphism groups are not reductive. But a theorem of Matsushima says that a Kähler-Einstein manifold with $c_1 > 0$ must have a reductive automorphism group.

\(^6\)The spaces $T^{1,1}/\mathbb{Z}_2$ and $S^5/\mathbb{Z}_3$ are regular because the canonical class of $\mathbb{P}^1 \times \mathbb{P}^1$ is $2H$, twice the hyperplane class, and similarly $K_{\mathbb{P}^2} = 3H$. The other del Pezzo surfaces in the list have $K = H$. 

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\[ \Gamma \in U(1) \] that leaves the holomorphic three-form invariant is generated by \( e^{2\pi i/3} \). This precisely 'shortens' the fiber by a factor of three, and so increases the Chern class of the bundle by three.

In string theory, the case \( d = 6 \) is also interesting. The Kähler-Einstein Fano manifolds of dimension \( d = 6 \) have not been classified. The homogeneous manifolds are known,

i. \( \mathbb{P}^3 \),

ii. \( \mathbb{P}^2 \times \mathbb{P}^1 \),

iii. \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \),

iv. \( \widetilde{Gr}(5, 2) \),

v. \( F(1, 2|3) \),

where \( \widetilde{Gr}(5, 2) \) is the real Grassmannian \( SO(5)/(SO(3) \times SO(2)) \) and \( F(1, 2|3) \) is the flag manifold \( (SU(3) \times SU(2))/(SU(2) \times U(1) \times U(1)) \). There are homogeneous Sasaki-Einstein manifolds that are circle bundles over these spaces\(^7\). These manifolds are known from the study of compactifications of eleven-dimensional supergravity of the form \( AdS_4 \times \mathcal{M}_7 \) [68]. Some of these spaces are even tri-Sasakian. Examples of inhomogeneous Kähler-Einstein manifolds are \( \mathbb{P}^1 \times dP_n \).

About tri-Sasakian manifolds, more stringent results can be stated. These can be found in [8]. All homogeneous tri-Sasakian manifolds in any dimension are known and constructions exist which given one tri-Sasakian space yield others. At the base of these results lies the structure of tri-Sasakian manifolds. As Sasaki-Einstein manifolds, they can be seen as circle bundles over Kähler-Einstein spaces. But the tri-Sasakian structure allows them to be seen also as \( SU(2) \) fibrations over quaternionic Kähler manifolds. Also, the twistor space of the quaternionic Kähler manifold is the Kähler-Einstein manifold. A very good discussion is presented in [8].

Tri-Sasakian manifolds will not be further discussed here. Yet, they are very interesting for a number of reasons. Explicit geometric constructions of such manifolds exist, based on the hyper-Kähler quotient [62]. The hyper-Kähler cones preserve more supersymmetry than a generic Calabi-Yau cone and the structure as \( Sp(1) \) bundles might provide a way to consider non-abelian duality for hyper-Kähler cones in a spirit similar to that of T-duality in chapter 4. This, however remains a subject left entirely for future study.

**Summary**

Perhaps the main lesson from the description of supersymmetric singularities as metric cones, is that such cones generically have a \( U(1) \) isometry which degenerates at the apex of

\(^7\)A homogeneous Sasaki-Einstein manifold has a transitive group of isometries which preserve the Sasakian structure.
Chapter 2 - Hypersurfaces

<table>
<thead>
<tr>
<th>Metric Cone $C(L)$</th>
<th>$L$</th>
<th>$Z \simeq L/U(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>symplectic</td>
<td>contact</td>
<td>symplectic</td>
</tr>
<tr>
<td>Kähler</td>
<td>Sasaki</td>
<td>Kähler</td>
</tr>
<tr>
<td>Calabi-Yau</td>
<td>Sasaki-Einstein</td>
<td>Kähler-Einstein and Fano</td>
</tr>
<tr>
<td>hyper-Kähler</td>
<td>tri-Sasaki</td>
<td>Kähler-Einstein, Fano, twistor space of quaternionic Kähler</td>
</tr>
</tbody>
</table>

Table 2.5: Relation of geometries of some metric cones and associated spaces

the metric cone. This isometry is generated by the characteristic (or Reeb) vector field that any Sasaki manifold has. Some particular simple, exceptionally symmetric Sasaki-Einstein manifolds are $U(1)$ bundles over Hermitean symmetric spaces. The Hermitean symmetric spaces also appear in the construction of some particularly symmetric worldsheet conformal field theories which can be used to describe supersymmetric string compactifications, which appear in section 3.4.

The largest class of Sasaki-Einstein spaces fall outside this category. They are non-regular and thus $U(1)$ bundles over Einstein-Kähler spaces with isolated quotient singularities. Recently many such spaces were found, using algebraic geometric considerations. These constructions show that an orbifold Kähler-Einstein metric must exist on a large class of varieties, but does not explicitly construct such metric, not unlike the proof that certain varieties admit a Calabi-Yau metrics, based on algebraic geometric criteria. This construction can be used to construct supersymmetric cones as well, and it does so in terms of hypersurfaces defined by complex polynomials. These matters are discussed in section 2.3.

2.3 HYPERSURFACES

The description of singularities as hypersurfaces $C = F^{-1}(0) \subset \mathbb{C}^{n+2}$ provides a direct way to deform a singularity. By deforming the defining polynomial, a hypersurface may be completely smoothed. A deformation of the defining polynomial can be interpreted as a deformation of the complex structure of $C$. There is no simple way to smooth a singularity in a metric cone or quotient description. A smoothing operation normally has negligible effect asymptotically far away from the singular point, but does not fit with a global description in terms of a quotient or a metric cone that is also applicable near the smoothed singularity.

An asymptotic metric cone description is useful, as it provides a differential geometric picture with a characteristic Killing vector field on a Sasaki-Einstein link, which is generic for any supersymmetric metric cone. Hypersurface descriptions turn out to be not only useful to consider deformations of singular cones, but also to characterize Sasaki-Einstein manifolds in a way unlike those used in section 2.2. In particular, projective hypersurfaces,
Chapter 2 - Hypersurfaces

defined as the zero locus of a single weighted homogeneous polynomial in an appropriate weighted projective space, can be an algebraic geometric way to describe varieties that admit Kähler-Einstein metrics, possibly with orbifold singularities. Such varieties can be used to construct metric cones on non-regular Sasaki-Einstein manifolds, as $S^1$ bundles over the Kähler-Einstein base. Additionally, the links of projective hypersurfaces can be related to fiber bundles over a $S^1$ base. Topological properties of these bundles are related to the analytic properties of the hypersurface singularity. It is the object of this section to introduce these two viewpoints, both for hypersurfaces in $\mathbb{C}^3$ and in higher dimensions.

2.3.1 The ADE-singularities as Hypersurfaces

The ADE-singularities have descriptions as hypersurfaces $F_{ADE}^{-1}(0) \subset \mathbb{C}^3$. The polynomials $F_{ADE}$ are listed in table 2.2. These singularities are quite special, as discussed in section 2.1.2, for many reasons. For one, they also have descriptions as quotients $\mathbb{C}^2/\Gamma$ and hence also as metric cones. As quotient singularities, the McKay correspondence relates the homology of resolutions to the representation theory of the quotient groups, a point which has a beautiful string theoretic interpretation [61]. As surface singularities, both resolutions and deformations blow up two-cycles. The distinction between complex and Kähler deformations is not an invariant notion, because of the $Sp(1)$-family of complex structures on these hyper-Kähler surfaces. In higher dimensions, not all of these properties are simultaneously present in general.

The polynomials $F_{ADE}$ are weighted homogeneous, they satisfy (2.7),

$$F(\lambda^{a_1}z_1, \lambda^{a_2}z_2, \lambda^{a_3}z_3) = \lambda^d F(z_1, z_2, z_3).$$

So a hypersurface $C = F^{-1}(0)$ admits a $\mathbb{C}^* = \mathbb{R}_+ \times U(1)$ action, like a supersymmetric metric cone does. The link $L$ of a metric cone $C(L)$ is obtained as $L = C(L)/\mathbb{R}_+$. Analogously, one can fix the $\mathbb{R}_+$ scaling of $C = F^{-1} \subset \mathbb{C}^{n+2}$ by intersecting the hypersurface with a small sphere,

$$S_r^{2n+3} = \{z \in \mathbb{C}^{n+2} : \sum_{i=1}^{n+2} |z_i|^2 = r^2\},$$

$$C = \{z \in \mathbb{C}^{n+2} : F(z) = 0\},$$

$$L_r = C \cap S_r^{2n+3}. \quad (2.20)$$

which envelops an isolated singularity at the origin. For any hypersurface $C = F^{-1}(0)$ defined by a weighted homogeneous $F$ with an isolated singularity at the origin it makes sense to consider its link $L_r$ in this way and write $C(L)$.

One may ask to what extent this notion of a link is related to the link of a metric cone. The ADE-singularities have descriptions as metric cones, and one can compare the two notions. Let's call these the 'metric link' and the 'analytic link'. First of all, the metric links

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are $S^3/\Gamma$ and are Sasaki-Einstein manifolds. The base space of each $S^3/\Gamma$ is $S^3/U(1) \simeq (\mathbb{C}^2\setminus \{0\})/\mathbb{C}^* \simeq \mathbb{P}^1$. The analytic links can be viewed as $U(1)$ bundles over certain base spaces $Z(\Gamma)$. The space $Z(\Gamma)$ is characterized as the projective hypersurface $F^{-1}(0)$ in a weighted projective space defined by the weighted $\mathbb{C}^*$ action on the weighted homogeneous polynomial $F$.

The projective hypersurfaces $Z(\Gamma)$ are characterized using the adjunction formula. Recall the adjunction formula in ordinary projective space, see, for example [76]. It gives the canonical bundle of a hypersurface $\mathcal{P} = F^{-1}(0) \subset \mathbb{P}^m$. Such a hypersurface is the zero locus of a section of the line bundle $\mathcal{O}_{\mathbb{P}^m}(d)$, where $d$ is the degree of the homogeneous polynomial $F$ that defines the hypersurface $\mathcal{P}$. It can also be viewed as a submanifold of $\mathbb{P}^m$. There is the following short exact sequence,

$$0 \to \mathcal{TP} \to T\mathbb{P}^m|_{\mathcal{P}} \xrightarrow{\nabla F} \mathcal{O}(d)\mathbb{P}^m|_{\mathcal{P}} \to 0. \tag{2.21}$$

The meaning of this sequence is as follows, reading from left to right. The tangent bundle to $\mathcal{P}$ is a subbundle of the tangent bundle to the embedding $\mathbb{P}^m$, restricted to $\mathcal{P}$, so there is an inclusion map. The next arrow maps every tangent vector $X^i \nabla_i \in T\mathbb{P}^m|_{\mathcal{P}}$ to a section of $\mathcal{O}_{\mathbb{P}^m}(d)$, i.e. to a homogeneous polynomial of degree $d$. Its kernel is formed by vectors tangent to $\mathcal{P}$. The map that achieves this is the covariant gradient,

$$\nabla X F = X^i (F, \Gamma_i F).$$

The second term involves a connection $\Gamma_i$ on $\mathcal{O}_{\mathbb{P}^m}(d)$, but restricted to $\mathcal{P}$ it drops out, as $F = 0$ on $\mathcal{P}$ by definition. The vectors mapped to zero are the vectors tangent to $\mathcal{P}$ since by definition $\mathcal{P}$ is the surface of which $F$ has the constant value $F = 0$. The short exact sequence (2.21) implies for the determinant line bundles

$$\det T\mathbb{P}^m|_{\mathcal{P}} \simeq \det \mathcal{TP} \otimes \mathcal{O}_{\mathbb{P}^m}|_{\mathcal{P}}.$$ 

The determinant bundle of the cotangent bundle to a complex manifold is also called the canonical bundle $\mathcal{K}$, and its dual, the determinant bundle of the tangent bundle, is the anti-canonical bundle, denoted by $-\mathcal{K}$ or $\mathcal{K}^*$. The above expression implies that the canonical bundle of $\mathcal{P}$ is given by

$$\mathcal{K}_{\mathcal{P}} \simeq (\mathcal{K}_{\mathbb{P}^m} \otimes \mathcal{O}(d))|_{\mathcal{P}}. \tag{2.22}$$

This relation is the statement of the adjunction formula. As $\mathcal{K}_{\mathbb{P}^m} \simeq \mathcal{O}_{\mathbb{P}^m}(-m - 1)$, the adjunction formula can be written as

$$\mathcal{K}_{\mathcal{P}_d \subset \mathbb{P}^m} \simeq \mathcal{O}_{\mathbb{P}^m}(d - m - 1)|_{\mathbb{P}_d}, \tag{2.23}$$

For a degree $d$ hypersurface in $\mathbb{P}^m$.

The adjunction formula can be generalized to weighted projective hypersurfaces (see section 2.3.3). The ordinary projective space $\mathbb{P}^m$ is a special case, with all weights

$$a_1 = \ldots = a_{m+1} = 1.$$
Chapter 2 - Hypersurfaces

The adjunction formula applied to the complex curves \( Z_\Gamma \), written as zero loci of the ADE polynomials \( F_\Gamma \) in the appropriate weighted projective space gives the first Chern class of \( Z_\Gamma \). Hence gives its Euler characteristic, \( \chi = -2c_1 \), in terms of the first Chern classes of the embedding space and a that of the line bundle with section \( F_\Gamma \). The result is

\[
c_1(Z_\Gamma) = -d + \sum_{i=1}^{3} a_i = 1. \tag{2.24}
\]

For all ADE-polynomials, listed in table 2.2, the relation between weights and weighted degree is as in (2.24). Such hypersurfaces are called anticanonically embedded,

\[-K_{ADE} = O(1).\]

Many higher dimensional hypersurfaces are not anticanonically embedded, while their defining polynomial does define a supersymmetric affine hypersurface. Consequently, they are of importance for string theory. But from the mathematicians' point of view the anticanonically embedded ones have received special attention. It will turn out that the distinction between anticanonically embedded hypersurfaces and others also has a (slight) consequence for the string theory duality transformation. In particular, the worldsheet field theories employed in the formulation of the duality transformation describe exactly affine hypersurfaces of the 'anticanonical' kind, and particular cyclic quotients of surfaces which are not of the 'anticanonical' kind. These worldsheet models are discussed at the end of section 3.3.2 and in section 4.4.

2.3.2 **Topology of Affine Hypersurfaces**

This section is relatively disconnected from the rest. We discuss some aspects of affine hypersurface singularities, defined by a weighted homogeneous polynomial, in arbitrary dimension. So these results in particular hold for six and eight dimensional singularities, which are of interest in string theory.

The description as a hypersurface obscures any differential geometric data of the space. However, there is a remarkable connection between analytic properties of the polynomial defining the affine hypersurface and topological properties. The 'topological properties' conceptually split into two sorts. First, there is the topology of a resolution of the singularity. This is related to deformations of the defining polynomial; essentially this is a statement in the context of Morse theory.

Second, there is the topology of the 'base of the cone', the analogue of \( L \) for metric cones. Topological properties of \( L \), or rather its equivalent in the hypersurface context, are related to analytic properties of the defining polynomial as well. This may seem quite remarkable. This may seem quite remarkable, since \( L \), regarded as the 'base' very far from the apex of a cone, is quite insensitive to small deformations of the singular apex.
Sasaki and Milnor: Circle Fiber or Circle Base?

In higher dimensions, many interesting ‘supersymmetric’ hypersurface singularities are not anticanonically embedded, but the ones that are play a special rôle, as it can be proved that some admit Kähler-Einstein metrics. This requirements seems more of a technical condition in the proof than a fundamental necessity. We will return to the higher dimensional cases in the next section. In any case, the Kähler-Einstein base manifolds $Z$ of all ADE-hypersurfaces are $\mathbb{P}^1$, as $\chi = -2c_1 = -2$. This coincides with the base of the metric cone description $\mathbb{C}^2 / \Gamma \to (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^* \simeq \mathbb{P}^1$.

Can the links of the metric cones, $S^3 / \Gamma$ and the links of the hypersurface singularities $F^{-1}_\Gamma(0) \cap S^5$ also be identified? Given the weights $a_i$ of $F_\Gamma$ there is a natural Sasakian structure on $S^5 \subset \mathbb{C}^3$ with contact form $\eta_a$ and characteristic vector $\xi_a$ defined in terms of the coordinates $z_k = x_k + iy_k$ on $\mathbb{C}^3$,

$$\eta_a = \sum_{k=1}^{3} \frac{(x_k dy_k - y_k dx_k)}{a_k (x_k^2 + y_k^2)} \quad \text{(2.25)}$$

$$\xi_a = \sum_{k=1}^{3} a_k \left( x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k} \right).$$

This Sasakian structure is in general non-regular. It generalizes to $S^{2n+3}$ spheres for any $n$. This restricts to a Sasakian structure on $L_\Gamma = S^5 \cap F^{-1}_\Gamma(0)$, and the question is to find a metric on $L_\Gamma$ that is not only compatible with this Sasakian structure, but that is also Sasaki-Einstein, i.e. the $U(1)$ action above should be an isometry and it should be the action of a characteristic vector field on a Sasakian manifold. Analytic sufficient conditions can be found, discussed in a more general case in the next section, which are met by the ADE-hypersurfaces. Much like the proof of existence of Calabi-Yau metrics, it is not constructive. But from the hypersurface, some topological information about the analytic link can be found.

The link of a weighted homogeneous hypersurface singularity can be viewed not only as a circle bundle over a projective variety, such as $\mathbb{P}^1$ in the case of the ADE-hypersurfaces. It can also be seen as the ‘boundary’ of a fiber bundle with a relatively complicated fiber, but with $S^1$ for a base. The topology of the link is studied via the topology of its complement $S^{2n+3} \setminus L$. This approach is essentially similar to the study of one-dimensional knots and links via their embedding in $S^3$, related to complex curve singularities

$$\mathbb{C}^2 \supset C \to L \subset S^3.$$ 

Topological information about the link is related to topological information about its complement, which in turn is related to analytic information about the hypersurface.

More specifically, deformations of the defining polynomial of a hypersurface correspond to smoothings of the singular point. Such smoothings do not change the asymptotic form
of the hypersurface toward infinity. The link of a weighted homogeneous hypersurface is obtained by intersecting it with a sphere that contains the singular point, and may be large. The deformations of the singularity occur inside the enveloping sphere and may not affect the asymptotic geometry near the sphere. Yet, the possibility of these analytic deformations far inside, which can smooth out the singularity, have a consequence for the topology of the link as well. The connection between singularity theory and topology is a very interesting matter and only a very small part will be discussed, in the context of not only ADE-hypersurfaces but also higher dimensional cases. A nice starting point, containing many classic references is [72].

A polynomial $F : \mathbb{C}^{n+2} \to \mathbb{C}$ defines an affine hypersurface $\mathcal{M} = F^{-1}(0)$. This hypersurface is singular where the $dF = 0$, in other words, at the critical points of $F$, where in addition $F = 0$. We assume that $F$ has isolated critical points. Around such a critical point $F$ can be expanded as

$$F(z_1, \ldots, z_{n+2}) = \sum_{i=1}^{n+2} z_i^{k_i} (\alpha_k^{(i)} + \alpha_{k_i+1} z_i + \ldots),$$

(2.26)

and the multiplicity of the critical point is

$$\mu = \sum_{i=1}^{n+2} (k_i - 1).$$

(2.27)

For a weighted homogeneous polynomial,

$$F(\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+2}} z_{n+2}) = \lambda^d F(z_1, \ldots, z_{n+2})$$

(2.28)

this number is determined by the weights and the weighted degree,

$$\mu = \sum_{i=1}^{n+2} \frac{d - a_i}{a_i}.$$  

(2.29)

The number $\mu$ is called the Milnor number of the hypersurface.

A polynomial $F$ with a degenerate critical point can be deformed, $F \to \tilde{F}$ so that $\tilde{F}$ has $\mu$ non-degenerate critical points. The Milnor number can also be expressed as the dimension of the following quotient ring

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}[z_1, \ldots, z_{n+2}]}{\partial F}.$$  

(2.30)

The 'numerator' is the polynomial ring generated by all variables in $F$ and the 'denominator' is the ideal generated by the first derivatives of $F$, known as the Jacobian ideal of $F$. This quotient ring is also the $(c, c)$ ring of a $\mathcal{N} = (2, 2)$ Landau-Ginzburg model, as discussed in section 3.3.1. Every $(c, c)$ state corresponds to a critical point of $F$ and to count
the multiplicity correctly, one can deform $F \to \tilde{F}$ so that the degenerate critical point of $F$, at the origin, splits into $\mu$ non-degenerate critical points, $z_i \in \mathbb{C}^{n+2}$, of $\tilde{F}$, such that $\|z_i\| \leq r$. These critical points are mapped to $\mu$ critical values, $\tilde{F}(z_i) = \zeta_i \in \mathbb{C}$.

The function $\tilde{F}$ is continuous with non-degenerate critical points which maps the ball $B_r = \{\|z\| \leq r\} \subset \mathbb{C}^{n+2}$ containing all $\mu$ critical points into the disk $D_\rho = \{|z| \leq \rho\} \subset \mathbb{C}$ containing all critical values. Such a function is a Morse function and it can be used to extract topological information about the hypersurface, see for example [72]. Define

$$
\Gamma = \tilde{F}^{-1}(D_\rho) \cap B_r,
\phi = \tilde{F}^{-1}(\zeta),
$$

where $\zeta$ is a generic point. The function $\tilde{F}$ can be used to find the relative homology [72]

$$
H_k(\Gamma, \phi; \mathbb{Z}) \simeq \left\{ \begin{array}{ll}
0 & \text{if } k \neq n+2 \\
\mathbb{Z}^\mu & \text{if } k = n+2.
\end{array} \right.
$$

The function $\tilde{F}$ can be used to explicitly visualize a basis of $H_{n+2}(\Gamma, \phi; \mathbb{Z})$. Choose a point $\delta$ on the boundary of the disk $D_\rho$. Non-intersecting paths from $\delta$ to the critical values $\zeta_i$ are the images of homology cycles in the deformed hypersurface. These cycles shrink as critical points move together, see figure 2.1.

When the deformation is turned off completely, $\tilde{F} \to F$, all critical points coincide at the origin, and $F^{-1}(\zeta)$ is smooth, except when $\zeta = 0$, in which case the hypersurface has its only singularity isolated at the origin $z = 0$.

Both the cone $C = F^{-1}(0)$ and its complement $\mathbb{C}^{n+2\setminus C}$ admit a $\mathbb{C}^*$ action. One can divide out the $\mathbb{R}_+$ part by intersecting with $S^{2n+3}_\rho = \partial B_r$. Using the fact that $F'$ has no critical points outside the origin, it can be shown that $L = F^{-1}(0) \cap S^{2n+3}$ and $M = S^{2n+3}_\rho \setminus L$ are smooth manifolds. $M$ can be viewed as a fiber bundle with base $U(1)$. The projection map $M \to U(1)$ is given by

$$
\pi : M \to U(1),
$$

$$
z \mapsto \frac{F(z)}{|F(z)|}.
$$

29
The fiber is a $2n + 2$-dimensional manifold, $\Phi \simeq \Phi_\theta \simeq \pi^{-1}(e^{i\theta})$, known as the Milnor fiber. And the total space

\[
\begin{array}{ccc}
\Phi & \hookrightarrow & M \\
\pi & & \downarrow \\
S^1 & & 
\end{array}
\]

(2.34)

is the Milnor fibration [71]. Clearly the complement of the $M$ in the sphere, or $\partial M = M \setminus M$, is the link $L$.

It was shown by Milnor [71] that $\Phi \simeq \partial M$ and also, taking a Morsification $\tilde{F}$ of $F$ which has $\mu$ nondegenerate critical points inside a ball $B_r$ and $\mu$ corresponding critical values inside a disk $D_\rho$, that

\[
H_k(\Gamma, \phi; \mathbb{Z}) \simeq \begin{cases} 
0 & k \neq n + 1 \\
\mathbb{Z}^\mu & k = n + 1,
\end{cases}
\]

(2.35)

taking $\Gamma = \tilde{F}^{-1}(D_\rho) \cap B_r$ and $\phi = \tilde{F}^{-1}(e^{i\theta}) \cap B_r$. Furthermore he showed that this $\Gamma$ is contractible. Using this together with the long exact sequence for relative homology groups, $\phi \subset \Gamma$,

\[
\ldots \partial H_k(\phi) \hookrightarrow H_k(\Gamma) \xrightarrow{\partial} H_k(\Gamma, \phi) \xrightarrow{\partial} H_{k-1}(\phi) \xrightarrow{\partial} \ldots ,
\]

(2.36)

it is found that the homology of the Milnor fiber is given by

\[
H_k(\phi; \mathbb{Z}) \simeq \begin{cases} 
0 & k \neq n + 1 \\
\mathbb{Z}^\mu & k = n + 1.
\end{cases}
\]

(2.37)

This means that the Milnor fiber is homotopy equivalent to a bouquet of $(n + 1)$-spheres,

\[
L \simeq S^{n+1} \vee \ldots \vee S^{n+1}.
\]

(2.38)

The number of spheres in the bouquet is the Milnor number $\mu$. A bouquet of spheres $S^{n+1} \vee S^{n+1} \vee \ldots \vee S^{n+1}$ is the topological space obtained by taking the union of the topologies of the separate copies of $S^{n+1}$ and identifying a marked point on each sphere to a single point, like in figure 2.2.

The total space $M$ of the Milnor fibration is obtained by gluing the Milnor fibers over the circle in an appropriate way, using a homeomorphism

\[
h : \Phi \to \Phi,
\]

(2.39)

known as the characteristic map,

\[
M = (\Phi \times [0, 2\pi]) / \sim,
\]

(0, $\Phi) \sim (2\pi, h(\Phi))$.

(2.40)
Chapter 2 - Hypersurfaces

Figure 2.2: Three $S^1$'s glued into a bouquet

Figure 2.3: Simplified version of a Milnor fibration. The link is a bouquet of three circles, a point on each of the three circles in the fiber is identified, see figure 2.2. The base space is the large circle direction. Traversing the base, the fibers are glued together in a non-trivial fashion.
An attempt to illustrate this point of view of the Milnor fibration in made in figure 2.3.

The topology of the Milnor fiber does not yet clarify the topology of the link. Note that the for a \((2n + 2)\)-dimensional hypersurface \(C = F^{-1}(0) \subset \mathbb{C}^{n+2}\), the link is a manifold of dimension \(\dim(L) = 2n + 1\), the complement of the Milnor fibration in \(S^{2n+3}\), which has a \((2n + 2)\)-dimensional Milnor fiber. The homeomorphism \(h : \Phi \to \Phi\) induces a linear map

\[ h_* : H_{n+1}(\Phi; \mathbb{C}) \to H_{n+1}(\Phi; \mathbb{C}). \tag{2.41} \]

This map can be used to construct the exact sequence [70], using the fiber bundle structure\(^8\) of \(M\) and \(\partial M = L\),

\[ 0 \to H_{n+1}(L; \mathbb{Z}) \to H_{n+1}(\Phi; \mathbb{Z}) \xrightarrow{I - h_*} H_{n+1}(\Phi, \mathbb{Z}) \to H_n(L; \mathbb{Z}) \to 0. \tag{2.42} \]

This implies that \(H_{n+1}(L, \mathbb{Z}) = \text{Ker}(I - h_*)\) is a free Abelian group. And \(H_n(L; \mathbb{Z}) = \text{Coker}(I - h_*)\). This may have torsion, but its free part is isomorphic to \(\text{Ker}(I - h_*)\) as well. The kernel of \(I - h_*\) is determined from the characteristic polynomial

\[ \Delta(t) = \det(t I - h_*). \tag{2.43} \]

There is an algorithmic way [70] to determine \(\Delta(t)\) in terms of the \(a_i\) and \(d\) of a weighted homogeneous polynomial like (2.28) on page 28, and from that, the Betti numbers \(b_{n+1}(L)\) and \(b_n(L)\). This recipe is as follows.

For the Milnor fibration associated with a hypersurface \(F^{-1}(0)\) defined by \(F\) as in (2.28), the homeomorphism \(h\) can be chosen to act on the coordinates as

\[ h : (z_1, \ldots, z_{n+2}) \mapsto (e^{\frac{2\pi i a_1}{d}} z_1, \ldots, e^{\frac{2\pi i a_{n+2}}{d}} z_{n+2}). \tag{2.44} \]

In order to write down \(\Delta(t)\), it is convenient to introduce different notation. Define \(r_i = d/a_i\), and write these as fractions of relatively prime pairs \(r_i = s_i/t_i\). Associate divisors to polynomials as follows,

\[ \text{divisor } \prod_{i=1}^k (t - \alpha_i) = \langle \alpha_1 \rangle + \ldots + \langle \alpha_k \rangle. \]

A divisor, like the one denoted on the right hand side of the above equation, can be regarded as a formal linear combination of points in \(\mathbb{C}\). More clearly, a divisor is an element of a free Abelian group\(^9\). Each generator \(\langle \alpha_i \rangle\) of this group is in one-to-one correspondence with a point in \(\mathbb{C}\), which can be regarded as the zero of a complex monomial function \(t - \alpha_i\).

\(^8\)In particular the Wang sequence is used, for fiber bundles over odd-dimensional spheres.

\(^9\)One could even say the divisors form the group ring \(\mathbb{ZC}\), which is formally a better way to think of them. The 'special' divisors \(\mathbb{E}_n\) are then considered not to form a subgroup, but a genuinely different group ring: \(\mathbb{QC}\) (the coefficients of the \(\langle \eta_n \rangle\) are rational numbers).
Each \(\langle \alpha_i \rangle\) generates a subgroup isomorphic to \(\mathbb{Z}\). The group operation in this group can be denoted as addition, and one can concisely write

\[
\langle \alpha_1 \rangle + \langle \alpha_1 \rangle = 2\langle \alpha_1 \rangle.
\]

We can introduce some additional structure, multiplication, on a subgroup, if we realize that the \(\alpha_i\) are also complex numbers, not just labels for geometric points. We restrict to a special subgroup of divisors. Define

\[
E_n = \frac{1}{n} \text{divisor } (t^n - 1) = \frac{1}{n} \sum_{i=0}^{n-1} \langle \eta_n \rangle^k,
\]

where \(\eta_n\) is a primitive \(n\)-th root of unity. Now a multiplication rule for these special divisors is proposed, inspired by complex multiplication of roots of unity. The \(E_k\) form a ring with multiplication rule

\[
E_k E_l = E_{[k,l]},
\]

where \([k, l]\) denotes the least common multiple of \(k\) and \(l\). With this notation the divisor of \(\Delta(t)\) associated to the Milnor fibration of \(F^{-1}(0)\) as in (2.28) reads

\[
\text{divisor} \Delta = \prod_{k=1}^{n+2} (r_k E_{s_k} - 1).
\]

The Betti numbers \(b_{n+1} = b_n\) of the link \(L = F^{-1}(0) \cap S^{2n+3}\) are equal to the number of factors of \((t-1)\) in \(\Delta(t)\) [70].

Recapitulating, the weights and degree of a weighted homogeneous polynomial \(F\) determine the Milnor number \(\mu\) of the hypersurface \(F^{-1}(0)\). This number counts the number of deformations of the singularity or in other words, the multiplicity of the critical point at the singularity. As such, it is related to Landau-Ginzburg models, counting the number of \((c, c)\) primary states (see section 3.3.1). But \(\mu\) also gives the dimension of the middle integral homology of the Milnor fiber \(\Phi \to M \to S^1\); \(\Phi \simeq S^{n+1} \cup \ldots \cup S^{n+1}\). The total space of the Milnor fibration \(M\) is obtained by gluing \(\Phi\) along the base, twisting it by the characteristic map \(h\). The boundary of \(M\) is the link \(F^{-1} \cap S^{2n+3}\). Its Betti numbers \(b_n(L) = b_{n+1}(L)\) are determined, employing the \(h\), in terms of the weights and degree of \(F\). The link itself is a circle fibration over a projective variety \(S^1 \to L \to Z\).

For the A-type hypersurfaces, \(Z \simeq \mathbb{P}^1\), which admits a Kähler-Einstein metric of positive curvature. This is in agreement with the observation that the \(F_{ADE}\) in table 2.2 are precisely those weighted homogeneous polynomials that satisfy,

\[
\sum_{i=1}^{3} a_i \geq d + 1.
\]

The \(F_{ADE}\) even saturate this inequality.
Chapter 2 - Hypersurfaces

One can consider other weighted homogeneous hypersurfaces $F^{-1}(0)$, as 'cones' in $\mathbb{C}^3$ or projective surfaces in a weighted projective space $(\mathbb{C}^3\setminus\{0\})/\mathbb{C}^*[a]$. Notably, one might consider projective hypersurfaces with $c_1 \leq 0$. The corresponding cones will not be suitable to serve as supersymmetric compactifications by themselves, only the ADE-cones do. Yet there are still some interesting points to note.

The simplest of ADE-hypersurfaces are those of Brieskorn-type: the $A_n$-series together with $E_6$ and $E_8$. These are of the form $z_1^{r_1} + z_2^{r_2} + z_3^{r_3} = 0$. Intersected with $S_{r_1=1}^2 \subset \mathbb{C}^3$ these define the Brieskorn manifolds $M(r_1, r_2, r_3)$. The three dimensional Brieskorn manifolds were studied by Milnor [69]. He demonstrated that $M(r_1, r_2, r_3)$ are homogeneous spaces which fall into three categories, depending on the canonical class of the corresponding projective hypersurface.

\[
\begin{align*}
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &> 1 \quad &c_1 = 1, \quad (2.47) \\
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &= 1 \quad &c_1 = 0, \quad (2.48) \\
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &< 1 \quad &c_1 < 1. \quad (2.49)
\end{align*}
\]

In the cases (2.47) the homogeneous spaces $M(r_1, r_2, r_3)$ are of the form $SU(2)/\Gamma$, as familiar from the quotient description. In the case (2.49) the spaces $M(r_1, r_2, r_3)$ are $PSL(2;\mathbb{R})/\Gamma$, quotients of the universal cover of the projective version of $SL(2;\mathbb{R})$ by discrete subgroups. The case (2.48) is different, there $M(r_1, r_2, r_3) \simeq G/H$ where $G$ is the Heisenberg group, with elements the matrices

\[
[a, b, c] = \begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}, \quad a, b, c \in \mathbb{R}, \quad (2.50)
\]

and $H$ are subgroups where $a, b, c \in k\mathbb{Z}$ for some integer $k$, see [69].

The polynomials which define Brieskorn manifolds of type (2.48) are

\[
\begin{align*}
F_{E_6}(z_1, z_2, z_3) &= z_1^3 + z_2^3 + z_3^3 \quad (+\alpha z_1 z_2 z_3), \\
F_{E_7}(z_1, z_2, z_3) &= z_1^2 + z_2^4 + z_3^4 \quad (+\alpha z_1^2 z_2^2), \\
F_{E_8}(z_1, z_2, z_3) &= z_1^2 + z_2^3 + z_3^5 \quad (+\alpha z_1^4 z_2).
\end{align*}
\]

The conformal field theories defined by these polynomials, as Landau-Ginzburg models (see section 3.3.1) have $\hat{c} = 1$, with the Brieskorn polynomials ($\alpha = 0$) corresponding to cft's with enhanced symmetry. The polynomials in (2.51) define tori in the appropriate weighted projective spaces, as is seen from the adjunction formula\textsuperscript{10}. The links of the singularities are circle bundles over tori, in these cases, and are homogeneous spaces.

\textsuperscript{10}In fact, this enhanced symmetry of the cft can be interpreted as the tori being at the self-dual radius. This will not be discussed. The connection between Landau-Ginzburg models, which a priori have no geometric interpretation, and sigma models, is discussed in section 3.3.
There is an interesting correspondence between the polynomials in (2.51) that define curves with trivial anticanonical class and thus cannot be used to make supersymmetric cones directly, and the del Pezzo surfaces $dP_6$, $dP_7$ and $dP_8$, that not only can be used to construct supersymmetric cones (as metric cones over regular Sasaki-Einstein manifolds), but also have descriptions as projective hypersurfaces, but are not homogeneous.

2.3.3 Kähler-Einstein Hypersurfaces

The 4d supersymmetric singularities are classified, have different but equivalent descriptions, and are related, via the ADE classification, to an enormous number of apparently very different objects that appear in mathematics. Each different description of one singularity highlights different aspects. For example, the metric cone shows there is a $U(1)$ isometry, degenerating at the apex. The quotient description relates singularities to homogeneous spaces. It also relates metric cones and hypersurfaces to one another, at least in the case of the complex surface singularities.

In the hypersurface description possible deformations are more apparent. In addition, important for our purposes, the defining polynomials of hypersurfaces play a rôle in worldsheet conformal field theories describing strings moving on a hypersurface and also T-dual spaces. Finally, many weighted homogeneous hypersurfaces give rise to Sasaki-Einstein manifolds, mostly non-regular ones. This section deals with the relation between hypersurfaces and metric cones in dimension $d > 4$.

Affine Calabi-Yau Hypersurfaces

The condition on a metric cone to be part of a supersymmetric string vacuum, i.e. a Calabi-Yau cone, is that its link is Sasaki-Einstein. Is there an analogous condition on hypersurfaces? The answer is: “yes”. Consider an affine hypersurface $C = F^{-1}(0) \subset (\mathbb{C}^{n+2} \setminus \{0\})$ defined by a weighted homogeneous polynomial,

$$F(\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+2}} z_{n+2}) = \lambda^d F(z_1, \ldots, z_{n+2}),$$

(2.52)

with a singularity only at the origin. If the weights $a_i$ and the weighted degree $d$ of $F$ are such that

$$J = -d + \sum_{i=1}^{n+2} a_i > 0,$$

(2.53)

then $C$ is Calabi-Yau $[27]$.

Note that the condition (2.53) is different from the Calabi-Yau condition for hypersurfaces in a projective space. Such hypersurfaces are Calabi-Yau iff $J = 0$, as a consequence of the adjunction formula and Yau’s proof of the conjecture of Calabi. But (2.53) deals with affine hypersurfaces, not projective ones. Nevertheless, since $F$ is a weighted homogeneous polynomial, one may consider the hypersurface $\mathcal{P} = F^{-1}(0)$ in an appropriate weighted
projective space. Such hypersurfaces, which satisfy (2.53) are called Fano. In terms of the first Chern class, \( c_1 > 0 \) for a Fano manifold.

Such a projective hypersurface is Kähler, since it is embedded holomorphically in a weighted projective space. It can be positively curved, as \( c_1 > 0 \). So maybe it can be the leaf space of a Sasaki-Einstein manifold. But this is only possible if the hypersurfaces admits a positive Kähler-Einstein metric (possibly with orbifold singularities).

One important question is: “What are necessary and sufficient conditions that such a \( P \) admit a positive Kähler-Einstein metric?” And a following question is: “Can a Sasaki-Einstein manifold be constructed from a \( P \) that admits such a metric, and if so, how?”

The latter question can be answered affirmatively. Given a hypersurface that has a Kähler-Einstein with positive scalar curvature, and at worst cyclic orbifold singularities, a Sasaki-Einstein manifold can be constructed, using the \( \mathbb{C}^* \) action on the weighted homogeneous polynomial \( F \) [33, 32, 37]. The answer to the former question is a lot more involved. It is possible to find sufficient conditions, that \( P \) admit a Kähler-Einstein metric with at worst cyclic quotient singularities, but part of these conditions is likely to be too strict [39, 34, 35]. Many hypersurfaces which are interesting from the perspective of string theory do not satisfy all of these sufficient conditions.

**Weighted Projective Basics**

First, let us recall some basic definitions and properties of weighted projective spaces; see, for example [75]. Weighted projective spaces \( \mathbb{P}[a_1, \ldots, a_{n+2}] \) are generalizations of ordinary projective spaces \( \mathbb{P}^{n+1} = \mathbb{P}[1, \ldots, 1] \). Points in \( \mathbb{C}^{n+2} \setminus \{0\} \) are identified by the weighted \( \mathbb{C}^* \) action,

\[
(z_1, \ldots, z_{n+2}) \sim (\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+2}} z_{n+2}),
\]

where \( \lambda \in \mathbb{C}^* \). Unlike ordinary projective spaces, weighted projective spaces can have singularities. These are seen in the affine coordinate patches where \( z_i \neq 0 \). In such a patch, one can set \( z_i = 1 \) by a weighted \( \mathbb{C}^* \) transformation. The coordinates is such a patch are \( \zeta_j^{(i)} = z_j / z_i \). If the weight \( a_i \) of the coordinate \( z_i \) is larger than one, then a \( \mathbb{Z}_{a_i} \) subgroup of the weighted \( \mathbb{C}^* \) action leaves invariant \( z_i = 1 \), but does act on the other coordinates:

\[
(z_1, \ldots, z_i = 1, \ldots, z_{n+2}) \mapsto (\eta^{a_i} z_1, \ldots, z_i = 1, \ldots, \eta^{a_{n+2}} z_{n+2}),
\]

where \( \eta \) is a primitive \( a_i \)-th root of unity. So the affine coordinate patches where \( z_i \neq 0 \) can have cyclic quotient singularities. These singularities occur at the so-called vertices \( P_i \) of the weighted projective space. The vertex \( P_i \) is the point \( \{z_j = 0\} \), \( j \neq i \). The singularity at \( P_i \) is said to be of type \( \frac{1}{a_i}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+2}) \). A hat over an element means that that element is omitted from the list. If some of the weights have common factors, there may also be singular lines, planes etc. The singular lines occur at edges \( P_i P_j \) (i.e \( z_k = 0, i \neq k \neq j \)) and are of type \( \frac{1}{\gcd(a_i, a_j)}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_{n+2}) \), an so on.
Clearly the weight vectors \((a_1, \ldots, a_{n+2})\) and \((ka_1, \ldots, ka_{n+2})\) correspond to isomorphic weighted projective spaces, for any integer \(k\). So one can assume that all \(a_i\)'s are relatively prime. In fact, there are further isomorphisms between weighted projective spaces, and every weighted projective space is isomorphic to a weighted projective space, as says a theorem by Delorme\(^{11}\). A well formed projective space \(\mathbb{P}[a_1, \ldots, a_{n+2}]\) has a weights such that
\[
\gcd(a_1, \ldots, a_i, \ldots, a_{n+2}) = 1 \quad 1 \leq i \leq n + 2.
\] (2.54)

A hat over an element means that the element is omitted. In a well formed projective space, the affine coordinate charts \((z_i \neq 0)\) have \(\mathbb{Z}_a\) quotient singularities. Some examples of some weighted projective spaces are
\[
\mathbb{P}[p, q] \simeq \mathbb{P}[1, 1] \quad \forall p, q
\]
\[
\mathbb{P}[6, 10, 15] \simeq \mathbb{P}[6, 2, 3] \simeq \mathbb{P}[3, 1, 3] \simeq \mathbb{P}[1, 1, 1]
\] (2.55)

A hypersurface in a weighted projective space inherits singularities from the embedding space if it passes through vertices, singular lines, etc. In general a hypersurface cannot avoid all vertices. It can avoid all vertices if
\[
a_i \mid d \quad \forall i.
\] (2.56)

A hypersurface with singularities that are all due to the singularities of \(\mathbb{P}[a_1, \ldots, a_{n+2}]\) alone\(^{12}\) is called quasi-smooth. Its singularities are all cyclic quotient singularities. Mathematicians know how to deal with such ‘mild’ sorts of singularities, and objects familiar from the algebraic geometry in ordinary projective spaces can be generalized [75]. In particular there is an adjunction formula if a hypersurface does not contain any singularities of codimension 2. Such a hypersurface is called well formed. A hypersurface \(\mathcal{P} = F^{-1}(0)\), defined by a polynomial of weighted degree \(d\) in \(\mathbb{P}[a_1, \ldots, a_{n+2}]\) is called ‘well formed’ if the following conditions are satisfied,
\[
\mathbb{P}[a_1, \ldots, a_{n+2}] \text{ is well formed, and} \\
gcd(a_1, \ldots, a_i, \ldots, a_{n+2}) \mid d \quad \forall i.
\] (2.57)

The adjunction formula gives the canonical class of the \(\mathcal{P}\) in terms of the weights \(a_i\) and the weighted degree \(d\) of \(F\), which can be seen as a section of the sheaf \(\mathcal{O}_P(d)\). The adjunction formula tells us
\[
K_\mathcal{P} \simeq \mathcal{O}(d - \sum_{i=1}^{n+2} a_i).
\] (2.58)

\(^{11}\)Consider a weighted projective space \(\mathbb{P}[a_1, \ldots, a_{n+2}]\). It can be shown that this space is isomorphic to \(\mathbb{P}[a_1, a_2/g, a_3/g, \ldots, a_{n+2}/g]\), where \(g = \gcd(a_2, \ldots, a_{n+2})\). Making use of this equivalence at most \(n + 2\) times produces a well formed projective space.

\(^{12}\)It is to say, that there are no singularities due to the way the hypersurface is embedded, i.e \(F = dF = 0\) has no solutions in \(\mathbb{P}[a_1, \ldots, a_{n+2}]\).
Chapter 2 - Hypersurfaces

Table 2.6: Smooth del Pezzo hypersurfaces admitting a Kähler-Einstein metric.

<table>
<thead>
<tr>
<th>Surface $F^{-1}(0) \subset \mathbb{P}[a_1, a_2, a_3, a_4]$</th>
<th>$F = 0$</th>
<th>$\mathbb{P}[a_1, a_2, a_3, a_4]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}^2$</td>
<td>$z_1 + z_2 + z_3 + z_4 = 0$</td>
<td>$\mathbb{P}[1, 1, 1, 1]$</td>
</tr>
<tr>
<td>$\mathbb{P}^1 \times \mathbb{P}^1$</td>
<td>$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$</td>
<td>$\mathbb{P}[1, 1, 1, 1]$</td>
</tr>
<tr>
<td>$dP_6$</td>
<td>$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$</td>
<td>$\mathbb{P}[1, 1, 1, 1]$</td>
</tr>
<tr>
<td>$dP_7$</td>
<td>$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$</td>
<td>$\mathbb{P}[1, 1, 1, 2]$</td>
</tr>
<tr>
<td>$dP_8$</td>
<td>$z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$</td>
<td>$\mathbb{P}[1, 2, 3, 3]$</td>
</tr>
</tbody>
</table>

A well formed hypersurface is Fano iff $J \equiv -d + a_1 + \ldots + a_{n+2} > 0$. Such hypersurfaces stand a chance of having positive Kähler-Einstein metrics, thus providing a connection with metric cones.

**HYPERSURFACES ADMITTING KÄHLER-EINSTEIN METRICS**

Which quasi-smooth hypersurfaces admit a Kähler-Einstein metric? A general answer is not known, but there are many examples, in various dimensions. First of all, there are the complex curves defined by the ADE polynomials, in table 2.2. As discussed earlier, all the ADE polynomials define a $\mathbb{P}^1$ hypersurface, which of course admits a Kähler-Einstein metric. Next, we know from section 2.2 which smooth complex surfaces admit positive Kähler-Einstein metrics. These are $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and the del Pezzo surfaces $dP_n$ for $3 \leq n \leq 8$. Of these, the ones that can be realized as hypersurfaces in weighted projective space are listed in table 2.6.

In addition to these smooth surfaces, there are many more quasi-smooth cases. Quasi-smoothness and well-formedness, see (2.54) and (2.57), impose conditions on the weights and degree similar to the smoothness condition (2.56). These conditions are not quite strong enough to determine all surfaces. It is possible to determine all surfaces and three-folds that satisfy one more condition, which is that they be anticanonically embedded,

$$J \equiv -d + a_1 + \ldots + a_{n+2} = 1,$$

(2.59)

All the conditions impose a set of linear relations among the weights $a_i$, which were organized in such a way [35, 34] that all solutions were found using a computer program.

The authors of [35, 34] also discuss the existence of Kähler-Einstein orbifold metrics on these hypersurfaces. The criteria that are used are sufficient but not necessary. Many

---

13 The conditions are the following, see [34] 2. Quasi-smoothness requires that for every $i$ there exist a $j$ and a monomial $z_i^{m_i} z_j$ of weighted degree $d$. The case $i = j$ gives the smoothness condition (2.56). Well-formedness furthermore requires that if $\gcd(a_i, a_j) > 0$, then there must be a monomial $z_i^{b_i} z_j^{b_j}$ of weighted degree $d$. Also, if every hypersurface of weighted degree $d$ contains a coordinate axis $z_k = z_l = 0$, then a general such hypersurface must be smooth along it, or have only a singularity at the vertices. This is the case if for all $i, j$ there is either a monomial $z_i^{b_i} z_j^{b_j}$ of degree $d$ or a pair of monomials $z_i^{d_i} z_j^{d_j} z_k$ and $z_i^{d_i} z_j^{d_j} z_l$ of degree $d$.  

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Hypersurfaces which are very interesting from the point of view of string theory are not anticanonically embedded. For example, the hypersurfaces defined by

$$ F(z_1, \ldots, z_{n+2}) = H(z_1, \ldots, z_n) + z_{n+1}^2 + z_{n+2}^2 $$

are not, except for those defined by the $A_k$ polynomials of table 2.2. Yet such polynomials have a special rôle in chapter 4.

In fact, from the point of view of string theory the single essential condition on an affine hypersurface is

$$ \exists = -d + \sum_{i=1}^{n+2} a_i > 0, $$

which ensures that it is Calabi-Yau, assuming that the only singularity is at the origin. Actually, it ensures that the cone without the apex at the origin is Calabi-Yau. For string theory one would also like that there are deformations of the singular hypersurface to a smooth one and that the smooth hypersurfaces as well as the singular limit are Calabi-Yau. This is indeed the case [77]. It would be interesting to know to what extent (2.53) is sufficient for the existence of a Kähler-Einstein metric (with singularities) on the projective hypersurface that it defines, and what additional conditions are necessary and sufficient.

A sufficient condition, based on [39] and [35, 34] and references therein, is given in [78]. They consider a Brieskorn hypersurface $F^{-1}(0)$, i.e. one defined by a polynomial of the form

$$ F = \sum_{i=1}^{n+2} z_i^{r_i}, $$

with $F = dF = 0$ only at the origin. $F$ has weighted degree

$$ d = R = \text{lcm}\{a_i\}. $$

The weighted homogeneous action on the coordinates $z_i$ is

$$ (z_1, \ldots, z_{n+2}) \simeq (\lambda^{R/r_1} z_1, \ldots, \lambda^{R/r_{n+2}} z_{n+2}). $$

Actually, they consider any deformation of such a hypersurface by a polynomial

$$ f(z_1, \ldots, z_{n+2}) $$

of weighted degree $d$,

$$ \tilde{F} = F + f, $$

provided that the intersections with any number of hyperplanes $z_i = 0$ are smooth away from the origin. The condition of [78] that a hypersurface admit a Kähler-Einstein orbifold metric of positive scalar curvature, is

$$ 1 < \sum_{i=1}^{n+2} \frac{1}{r_i} < 1 + \frac{n+1}{n} \min_{i,j} \left\{ \frac{1}{r_i}, \frac{1}{b_i b_j} \right\}. $$

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Here the $b_i$ are somewhat complicated expressions, in terms of the $a_j$,

\[ C^j \equiv \text{lcm}\{r_1, \ldots, r_j, \ldots, r_{n+2}\}, \]
\[ b_j \equiv \text{gcd}\,(r_j, C^j). \]

The lower bound is a necessary condition. It is the requirement that the hypersurface be Fano. The upper bound is a sufficient condition. It derives from certain estimates that guarantee the existence of a Kähler-Einstein metric [39]. These will not be discussed. The estimates are related to those used to find smooth Kähler-Einstein metrics on del Pezzo surfaces [38]. Essentially, it comes down to the question if a particular nonlinear partial differential equation has a solution, similar to the reformulation of the Calabi conjecture in the proof of Yau.

**NOT ANTICANONICALLY EMBEDDED: KÄHLER-EINSTEIN?**

These estimates discussed above are not sharp enough to determine if a Kähler-Einstein metric exists on many interesting hypersurfaces. For example

\[ z_1^{r_1} + z_2^{r_2} + z_3^{r_3} + z_4^{r_4} = 0 \]

does not satisfy (2.64). Unfortunately no sharper criteria are known to determine if a Kähler-Einstein orbifold metric exists. It would be especially interesting to find a way to determine if such metrics exist for hypersurfaces of the form

\[ F(z_1, \ldots, z_{n+2}) = H(z_1, \ldots, z_n) + z_{n+1}^2 + z_{n+2}^2, \]

which are important in chapter 4. However, if one has a hypersurface in weighted projective space that does have a Kähler-Einstein metric with at worst cyclic quotient singularities, then there is always a Sasakian-Einstein metric on the link $L = F^{-1}(0) \cap S^{2n+3}$ of the corresponding affine hypersurface [33]. Basically, the weighted projective $\mathbb{C}^*$ action restricts to a weighted $S^1$ action on $S^{2n+3} \subset \mathbb{C}^{2n}\setminus\{0\}$, and also on the link. This weighted $S^1$ action is that of a characteristic vector field of a Sasakian manifold. There is a Sasakian structure on the link with such a characteristic vector field that also has a compatible metric that is an Einstein metric.

### 2.4 SUMMARY AND CONTEXT

**WHAT HAVE WE DONE?**

Various spaces have been discussed which can feature as part of a supersymmetric string vacuum of the form

\[ \mathbb{R}^{9-2m,1} \times C_{2m}. \]


All $\mathcal{C}_m$ must preserve some supersymmetry and have a metric with a vanishing Ricci tensor. Also, $\mathcal{C}_m$ are non-compact and have an isolated singularity. There are numerous different ways to describe such spaces, among those discussed the most prominent two are metric cones and (weighted homogeneous) affine hypersurfaces.

Any particular exponent of a space $\mathcal{C}_m$ may have a description in both of these ways, in just one of the two, or in neither of them. Either way of describing a $\mathcal{C}_m$ emphasizes some characteristics of the space. A metric cone has a characteristic $S^1$ isometry which degenerates at the apex. This isometry is interesting for T-duality of such a space.

But possible deformations of the singularity are obscured in the description as a metric cone. On the other hand, a description as a hypersurface manifests some possible deformations, to be specific, deformations of the complex structure. Some such deformations can even smooth out a singularity completely, without affecting the asymptotic form of the space.

As we have seen, the number of such deformations is indicated by the Milnor number of the singularity. But this number also describes aspects of the topology of the hypersurface away from the singularity. It does so in two different ways. First, the hypersurface $\mathcal{C}_m$ cuts out a link in a $S^{2m-1}$ surrounding the singular point. This link is a fiber bundle with a circle fiber. The Milnor number roughly speaking indicates how far the fibration is from being trivial. Second, the complement of the link is a fiber bundle with a circle as a base. The fiber is a special manifold, the Milnor fiber and the Milnor number determines its complete homology. Finally, in a somewhat different context, the Milnor number counts the number of ground states in certain superconformal field theories, as discussed in section 3.3.1.

So these two descriptions, metric cones and affine hypersurfaces highlight different aspects and obscure others. Is it possible to construct one description from the other? A connection between metric cones and hypersurfaces is clearly present in some cases, most notably the $\mathcal{C}_4$ ADE singularities. In those instances, there is a direct connection via the quotient description $\mathbb{C}^2/\Gamma_{ADE}$. In higher dimensional cases, if there is a connection at all, it is more indirect.

In specific cases, a connection can be established. The most obvious similarity between the metric cones and the hypersurfaces, is that both admit a special $\mathbb{C}^*$ action. For metric cones, this comes partly from the definition, the $\mathbb{R}_+$ scaling, and partly from the requirement of supersymmetry, the $S^1$ of the characteristic isometry of a Sasakian base. The Sasakian base of a supersymmetric metric cone is itself a circle bundle over a Kähler manifold (possibly with quotient singularities). On the other hand, a weighted homogeneous polynomial, such as defines the affine hypersurfaces under consideration, also defines a hypersurface in a weighted projective space. Such a hypersurface is Kähler.

If it is Kähler-Einstein, then the affine hypersurface is Calabi-Yau, and it can be viewed as a metric cone. There is also a sufficient condition, due to Tian and Yau, that an affine hypersurface be Calabi-Yau. If is phrased in terms of the scaling weights $a_i$ and the weighted degree $d$ of the defining polynomial: $J \equiv -d + \sum a_i > 0$. Some of these Calabi-Yau hypersurfaces $\mathcal{C}$ certainly give rise to Kähler-Einstein $\mathcal{C}/\mathbb{C}^*$ and can thus be viewed as metric
Chapter 2 - Summary and Context

cones, with a $S^1$ isometry. It is not known what the minimal sufficient conditions are, for this to be the case. It would be interesting to know such conditions, so that metric cones and hypersurfaces can be related.

From the point of view of the T-duality of chapter 4 and further string applications, there are many affine hypersurfaces (or actually, discrete quotients of hypersurfaces, see section 4.4) which are not known to be connected to Kähler-Einstein hypersurfaces with the present status of mathematical knowledge.

**WHY ARE WE DOING THIS?**

Ultimately the interest of the connection of metric cones and hypersurfaces might be motivated from the T-duality of Calabi-Yau singularities, in chapter 4, which, where 'understood', relates almost all objects which have an ADE classification. A broad question would be: "If, as it seems, such a T-duality holds for a wider range of singularities, what objects does it relate, and how can these objects be interpreted in string theory, particularly from a stringy geometric point of view?"

But this met get ahead of the ideas presented to this point. Let us put hypersurfaces and metric cones in some perspective. Both metric cone and hypersurface descriptions emphasize certain objects which are important in another context, that is not discussed much in this chapter, but becomes more important in later ones. These objects have to do with worldsheet descriptions of string backgrounds. The weighted homogeneous polynomials that describe hypersurfaces, also describe Landau-Ginzburg conformal field theories. These can be used to build worldsheet conformal field theories that do not have a direct target space interpretation. However, in some cases, Landau-Ginzburg models are related to a target space.

Often Landau-Ginzburg models can be considered to describe string backgrounds that are compact Calabi-Yau hypersurfaces in weighted projective space. Or rather, a Landau-Ginzburg (-orbifold) describes a “Kähler” deformation of such a background to a non-geometric ‘phase’. It may be that a similar connection exists to non-compact Calabi-Yau hypersurfaces in affine space. A very different geometric interpretation of a Landau-Ginzburg model exists in a much more limited collection of cases. Sometimes a Landau-Ginzburg model has an interpretation as a coset model, and a coset model may have a geometric target space interpretation when the levels of the Kač-Moody algebras are large, so that stringy modifications to ordinary geometric concepts are small. In particular, the coset models that preserve the same amount of supersymmetry as the $C_{2m}$ of this chapter, are so-called Hermitean symmetric space coset models. Even if the levels are large so that there is a classical geometric target space interpretation, the target space of the Hermitean symmetric space coset models is very different from the geometry of the Hermitean symmetric spaces, which feature in the present chapter as particular examples of Kähler-Einstein manifolds. From these, Sasaki-Einstein manifolds can be built and from these, metric cones $C_{2m}$. 

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