Quantum Hall spin liquids
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CHAPTER 3

SLOWLY ROTATING SPIN-1
CONDENSATES

In this chapter we study the ground states of spin-1 atoms in a slowly rotating trap. This is the regime where quantum fluctuations are weak and conventional mean-field theory is expected to give good results. By truncation of the one-particle spectrum to the lowest Landau level (LLL), we are able to obtain much stronger results, however. At the SU(3) point of the interaction (see below), the exact many-body ground states are found for small angular momenta.

The results in this chapter are not directly related to quantum Hall spin-1 liquids, but the model is largely identical to that in chapter 4. Here we will take into account a finite rotation frequency ω while in the high rotation limit it will be set to ω0, the trap frequency. Many differences with the scalar case are found in the slow rotation regime. It is expected that this regime is experimentally the most easily accessible.

The interaction of spin-1 bosons at low densities contains two contributions: a spin-independent interaction, with strength c0, and a spin-dependent interaction, strength c2. The dimensionless parameter γ = c2/c0 introduces many features in the phase-diagram of rotating spin-1 bosons. We consider generic values of γ in the repulsive regime c0 > 0, and also focus on the special case γ = 0, where the interaction has an SU(3) symmetry.[70]

This chapter is organized as follows. In section 3.1 we define the model by discussing LLL truncation in a disc, sphere or torus geometry, specifying the interaction Hamiltonian, and make remarks on the general symmetry properties for later use. In section 3.2, we study the phase diagram by direct numerical diagonalization. In section 3.3 exact quantum ground state wavefunctions and energies for a slowly rotating (angular momentum L ≤ N) system in the c2 = 0 limit are presented. For nonzero c2, we use a LLL mean field treatment to study the slowly-rotating system (in section 3.4) and the various skyrmion and vortex lattices (in section 3.5).

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1 This chapter is largely based on a collaboration with J.W. Reijnders, K. Schoutens and N. Read, see [70, 71]
Before we come to the details of our analysis, we briefly summarize some of the results in the literature for rotating bosons, with either a single component (scalar case) or several components, such as spin 1 (vector case).

**NON-ROTATING CONDENSATES AND TOPOLOGICAL EXCITATIONS**

The case of vector BEC without rotation has been investigated, most often by mean-field theory using the spin-1 Gross-Pitaevskii equations, or with the further approximation of neglecting the kinetic energy term (the Thomas-Fermi approximation). There are two regimes [34], as mentioned already. In the ferromagnetic regime $c_2 < 0$, the ground state has maximum possible spin, $S = N$. Such a spin state can be constructed by condensing all the bosons into a single-particle state with $S_z = +1$, or as a spin-rotation of this. In the opposite antiferromagnetic regime, $c_2 > 0$, the ground state has minimal spin. Such a spin state can be constructed by condensing all the bosons in the same $S_z = 0$ single-particle spin state, or by taking a spin-rotation of this (notice that this is distinct from all of the ferromagnetic states). This “polar” state breaks spin-rotation symmetry, even though the expectation value of the total spin, or of the spin density, is very close to zero. The distinct ordered states, that can be mapped onto each other by the (broken) symmetries of spin rotation and phase rotation, are labelled by points in an order parameter manifold (or target space). For the ferromagnetic case, this manifold is $SO(3)$ [34], while for the antiferromagnetic or polar case it is $S^1 \times S^2 / \mathbb{Z}_2$ [34, 103]. These ordered states possess excitations that can be described as topological defects in the order, either with a singularity at a point surrounded by a “core”, or without a singularity at all. A brief description of the different defects is given; those that carry non-zero vorticity are relevant to the rotating case which we discuss next.

**VORTICES AND SKYRMIONS**

If the number of bosons in a rotating trap is sufficiently large, the effect of slow rotation can be studied in a mean field framework. For a single species of bosons with repulsive interactions, the rotation is accommodated through the creation of singular vortices, with vanishing particle density at the vortex cores. As the rotation rate is increased from zero, there is a critical frequency at which a single vortex first appears in the system [97], followed by additional vortices at still higher rotation. With a spin degree of freedom, the system has several components in which to store the angular momentum. There is the possibility that a vortex core for one spin component is filled by another spin component, leading to core-less vortices or “skyrmions”. In such configurations, the total particle density is nowhere zero, and there is a smooth spin texture. Mean field states of this type for rotating spin-1 bosons have been investigated theoretically by solving the spin-full Gross-Pitaevskii (GP) equations [101, 37, 53, 52]. For attractive interactions, in contrast, the BEC remains in a compact blob, without any vortices, all the way up to the maximum rotation frequency (the trap frequency) [97].
LATTICES OF VORTICES AND SKYRMIONS

When several vortices are present in a rotating scalar boson condensate with repulsive interactions, they line up in a triangular (Abrikosov) vortex lattice [1, 16]. In a vector BEC one expects to find similar lattices, built from the coreless vortices just described. The details of all this depend crucially on the relative strength $\gamma = \epsilon_2/\epsilon_0$ of the spin-dependent interaction. For $\gamma = 0$, where the $SU(3)$ symmetry between the different spin components is not broken, the lattice that is expected upon rotation is composed of three intertwined triangular lattices. The vortex cores do not overlap, so that the density is (almost) uniform. This lattice has been shown to be independent of the strength of the interaction by Kita et al. [45]. The vortex lattice shows a rich phase diagram, however, when the interaction is spin-dependent. For a range of positive values of $\gamma$, a square lattice composed of $\pi$-disclinations has been predicted [45].

3.1 LLL MODEL HAMILTONIAN AND ITS SYMMETRY

In this section we describe the truncation of the space of single-particle states to those in the LLL, and then explain the use of different geometries (sphere, torus) once this truncation has been made. Then we give the form of the interaction Hamiltonian that will be assumed, and some analysis of the symmetries of the model, with particular reference to certain limits and different geometries.

TRUNCATION TO THE LOWEST LANDAU LEVEL

In a rotating frame of reference the Hamiltonian for $N$ trapped, weakly-interacting spin-1 bosons is

$$H = \sum_i \left[ \frac{\omega_0}{2} (-\nabla_i^2 + r_i^2) - \mathbf{\omega} \cdot \mathbf{L}_i \right] + H_{\text{int}}.$$  

(3.1)

Here $\omega$ is the frequency of the rotation drive, $\mathbf{L}_i$ the angular momentum of the $i$-th particle and $H_{\text{int}}$ the interaction Hamiltonian, which we discuss below. We have set $\hbar$ and the harmonic oscillator length $l \equiv (\hbar/m_\text{b}\omega_0)^{1/2}$ of the trap (with $\omega_0$ the trap frequency and $m_\text{b}$ the boson mass) equal to one. Modes in the direction of the rotation axis are frozen out, leaving us effectively with a two-dimensional (2D) system. The energy eigenvalues of the single-particle part of the Hamiltonian are then $E_{n,m} = (2n + m + 1)\omega_0 - m\omega$, with $n \geq 0$ the Landau level index and $m \geq -n$ the $z$-component of angular momentum, labelling the states within each Landau level.

We consider the model in which the single-particle states are restricted to the lowest ($n = 0$) Landau level (LLL) [96]. This is valid when the interactions are sufficiently weak, as we will explain momentarily. The normalized LLL wavefunctions are $\phi_{n,m}(z)\xi^\alpha$ with the orbital part $\phi_{n,m}(z) \propto z^m e^{-|z|^2/2}$ ($z = x + iy$), and $\xi^\alpha$ a three-component complex vector representing the spin state; here $\alpha$ labels the
eigenstates of the z-component of the spin $S_z$ for each particle, $\alpha = \uparrow, \downarrow$. [Later in the paper it will be convenient also to use the basis of Cartesian components for spin 1, labelled by $\mu = x, y, z$.] When we use second quantization, we will denote the boson creation and annihilation operators for these single-particle states by $b_{\alpha \mu}^\dagger$, $b_{\alpha \mu}$, and the corresponding occupation numbers by $n_{\alpha \mu} = b_{\alpha \mu}^\dagger b_{\alpha \mu}$. Also, we sometimes use the field operator $\varphi_\alpha(z) = \sum_\mu b_{\alpha \mu} \phi_\mu(z)$. The single particle contributions to the Hamiltonian add up to $\left(\omega_0 - \omega\right) L$, with $L = \sum_i L_{zi}$ the z-component of total angular momentum. We will refer to this geometry as the disc in view of the form of the fluid states (for repulsive interactions) which tend to form a disc or "pancake", because of the centrifugal force. Note that we must have $\omega \leq \omega_0$, otherwise the system becomes unstable.

To study the bulk properties of the quantum ground states, we will eliminate boundary effects by using instead two other geometries and taking the limit $\omega \to \omega_0$. In a spherical geometry [29], the orbital part of the LLL single-particle wavefunctions is $\phi_m(z) \propto z^m \left(1 + \left(|z|/2R\right)^2\right)^{1+N_v/2}$, where $z$ represents position on the sphere by stereographic projection to the plane, and $R$ is the radius of the sphere. The number of orbitals is restricted by the vorticity $N_v$ penetrating the sphere, $0 \leq m \leq N_v$. The $N_v + 1$ single-particle orbitals form a representation of orbital angular momentum equal to $N_v$, see Ref.[29]. In the limit $R \to \infty$, keeping $N_v/R^2$, $N$ and $z$ constant, the single-particle wave functions on the sphere reduce to those for the disc as above. The total angular momentum on the sphere is characterized by quantum numbers $L$ for the magnitude, and $L_z$ for the z-component. In terms of $L$ which has eigenvalues $L = \sum_i m_i$ as before, $L_z = \frac{1}{2} N N_v - L$. We emphasize that our definition of $L$ when used for the sphere does not have its usual meaning, but is related to the $z$-component in such a way that the $N_v \to \infty$ limit agrees with the plane.

The final geometry we use is the torus. Here the single-particle wavefunctions take the form $\phi(z) \propto f(z)e^{-y^2}$ in the Landau gauge, with $f$ a quasiperiodic holomorphic function. With $N_v$ flux quanta, $f$ has $N_v$ zeros in the unit cell. There are exactly $N_v$ independent solutions, of the form $f(z) = \prod_{i=1}^{N_v} \vartheta_1(z - z_i|\tau)$, with $\tau$ describing the geometry of the unit cell and $z_i$ the zeros of $f$. The use of $\vartheta$-functions ensures that $\phi$ is periodic. Many-body states can be classified by their Haldane momentum [30].

**INTERACTION HAMILTONIAN**

In a model description, the Hamiltonian describing the 2-body interactions of a system of $N$ spin-1 bosons is a contact interaction, and contains spin-independent ($H_n$) and spin-dependent ($H_s$) terms, of strengths $c_0$, $c_2$ respectively

$$H_{\text{int}} = H_n + H_s = 2\pi \sum_{i<j}^N \delta^{(2)}(r_i - r_j) \left[c_0 + c_2 S_i \cdot S_j\right]. \quad (3.2)$$

Here $c_0 = (g_0 + 2g_2)/3$, $c_2 = (g_2 - g_0)/3$, $g_S = 4\pi a_s^2\hbar^2/a_s/m_b$ and $a_s$ ($S = 0, 2$) the 2D s-wave scattering phase shift in the spin-$S$ channel [61, 34]. A factor $2\pi$ has been
extracted for later convenience. One can obtain these parameters by integrating over the third direction. Assuming, for example, harmonic confinement with quantum length $l_\perp$ in the $z$-direction, one finds $a_{2D}^D = a_{3D}^D / \sqrt{2\pi l_\perp}$ when $l_\perp \ll l$. For the sphere, the coordinates $r$ in this Hamiltonian take values on the surface of the sphere, with radius $R$.

In mean-field approximation of a Bose-Einstein condensate, we can define a local order parameter $\phi = \langle b \rangle = \sqrt{n} \zeta$, with $\zeta$ a normalized three-component complex vector. In the $x,y,z$ representation, the spin is given by $S = i\phi \times \phi$. Assuming a finite density everywhere, we can find the topological excitations. The vorticity carried by such an excitation can be calculated as

$$\Phi = \int \frac{d^2 \mathbf{r}}{2\pi} \epsilon^{ij} \partial_i \zeta \cdot \partial_j \zeta$$

(3.3)

$$= \int \frac{d\theta}{2\pi} \zeta \cdot \partial_\theta \zeta$$

(3.4)

It is easy to see that the above expression gives the correct result for a vortex in a one-component condensate.

In the polar regime, $|\langle S \rangle| = 0$, the order parameter $\zeta(r)$ can be written as $\zeta = e^{i\varphi} \mathbf{m}$, with $\mathbf{m}$ a three-component real vector of unit length, $\mathbf{m} \cdot \mathbf{m} = 1$. The vorticity simplifies to

$$\Phi_{\text{polar}} = \int \frac{d\theta}{2\pi} e^{-i\varphi(\theta)} \partial_\theta e^{i\varphi(\theta)}$$

$$= \frac{1}{2\pi} (\varphi(2\pi) - \varphi(0))$$

(3.5)

A skyrmion texture in $\mathbf{m}$, with $\varphi = \text{constant}$, does not carry any vorticity, but the $\pi$-disclination does. As $r \to \infty$, the $\pi$-disclination is given by $\varphi = \theta/2$, $\mathbf{m} = (\cos(\theta/2), \sin(\theta/2), 0)$ (or a global rotation and/or phase shift). The vorticity is then equal to $\frac{1}{2}$.

In the ferromagnetic regime, there is a vortex (corresponding to $\pi_1(SO(3)) = \mathbb{Z}_2$) with vorticity 1. As implied by the homotopy result, vortices with larger vorticity are unstable. Indeed, a ferromagnetic condensate supports skyrmions with vorticity $\frac{1}{2}$ [34].

It is noteworthy that such topological defects can exist in the lowest Landau level. The notion of $r \to \infty$ needs to be adjusted, however, as the size of a defect is fixed to the oscillator length $l$. This can be naturally done by compatifying the plane to a sphere.

The use of the LLL reduced Hamiltonian is justified when the interactions are weak. Physical quantities evaluated in the full model differ from those in the LLL model by relatively small corrections when $\nu \ll \omega_0$. Here $\nu$ is the typical filling factor (expectation of the occupation numbers, summed over $\alpha$ or $\mu$) of the single-particle states. Notice that this condition becomes much less stringent as $\omega \to \omega_0$ in the repulsive regime, as then the particles spread out into a pancake, and the filling factor $\nu$ becomes of order 1.
Finally then, the LLL Hamiltonian in the rotating frame which we wish to analyze is

$$H_\omega = (\omega_0 - \omega)L + H_{\text{int}}.$$  \hfill (3.6)

Note that we use precisely this definition in the case of the sphere as well as for the disc. It will be useful also to know the ground states of $H_{\text{int}}$ for each $L$.

**SU(3) Symmetry Analysis for $c_2 = 0$**

In general, the only symmetry in spin space of the Hamiltonians $H_{\text{int}}$ and $H_\omega$ is spin-rotation symmetry $SO(3)_\text{spin}$. This implies that spin states will come in multiplets of spin $S$ with degeneracy $2S + 1$ (with $S$ integer since the particles have spin 1). However, at $c_2 = 0$, the interaction Hamiltonian reduces to the spin-independent interaction $H_n$. In this case the spin-rotation symmetry is enlarged from $SO(3)_\text{spin}$ to $SU(3)_\text{spin}$. It will be useful to understand what this implies about the spin multiplets in a finite size system.

For $c_2 = 0$, the spectrum will contain degenerate spin multiplets labelled by $SU(3)$-quantum numbers $(p, q)$. These tuples are the Dynkin indices labelling irreducible representations of dimension $\dim_{(p,q)} = \frac{1}{2}(p + 1)(q + 1)(p + q + 2)$. Since $SO(3)$ is embedded in $SU(3)$, each multiplet can be decomposed into a set of $SO(3)$ multiplets. These $SO(3)$ spin quantum numbers can be deduced by using branching rules for $SU(3) \rightarrow SO(3)$. The fundamental branching rule states that a $(p, 0)$ or $(0, p)$ multiplet contains $S = p, p - 2, p - 4, \ldots, 1$ (0) for $p$ odd (resp., even). Using the fusion rule

$$ (p, 0) \otimes (0, q) = (p, q) \oplus (p - 1, q - 1) \oplus \cdots \oplus (p - q, 0), \hfill (3.7)$$

which is valid for $p \geq q$, general branching rules can be derived. A multiplet $(p, q)$ with $q$ odd and $p \geq q$ decomposes in $SO(3)$ multiplets with highest weights $S$ according to the branching rule

$$ (p, q) \rightarrow \bigoplus_{i=0}^{(p+q-2)\over 2} \bigoplus_{S=2i+1}^{(p+q-2)\over 2} S, \hfill (3.8)$$

For $q$ even we find

$$ (p, q) \rightarrow \left( \bigoplus_{i=0}^{(p+q-2)\over 2} \bigoplus_{S=2i+1}^{(p+q-2)\over 2} S \right) \oplus \left( \bigoplus_{j=2i+1}^{\frac{q}{2}} 2j \right), \quad p \text{ odd} \hfill (3.9)$$

$$ (p, q) \rightarrow \left( \bigoplus_{i=0}^{(p+q-2)\over 2} \bigoplus_{S=2i+2}^{(p+q-2)\over 2} S \right) \oplus \left( \bigoplus_{j=0}^{\frac{q}{2}} 2j \right), \quad p \text{ even.} \hfill (3.10)$$

Note that the highest $SO(3)$-spin in an $SU(3)$-multiplet $(p, q)$ is always $S = p + q$, and the lowest $S = 0$ or 1.
ORBITAL SYMMETRY IN SPHERICAL GEOMETRY

In the plane geometry, $H_{\text{int}}$ is invariant under translations and rotations in the plane. When working on the sphere, this symmetry group is replaced by the rotation group $SO(3)_{\text{orb}}$ (strictly, we should say $SU(2)_{\text{orb}}$ whenever $N_v$ is odd) of the sphere. In the limit $R \to \infty$ described above, this symmetry becomes translations and rotations of the plane. When taking this limit, we also hold $L$ fixed, and hence many-particle states of definite $SO(3)_{\text{orb}}$ quantum numbers $(\vec{L}, \vec{L}_z)$ become in the limit infinite-dimensional multiplets of the Euclidean group of the plane. States within each multiplet differ only in the state of the center of mass variable (which has coordinate $z_c = \sum_i z_i/N$). Thus, if $\psi_L$ is an eigenfunction of $H_{\text{int}}$ at certain angular momentum $L$, then there exists a whole "tower" of states $\psi_{L+L'} \propto z_c^{L'} \psi_L$ with the same interaction energy at angular momentum $L + L'$.

We remark that in situations where only a few quantum orbitals are available to the bosons, the spectrum is largely determined by symmetry considerations. Particular examples are the spectrum for $N_v = 2$ on the sphere, where the exact $N$-body energies are given in terms of Casimir invariants of the orbital and spin-symmetries (see eq. (3.13) below), and the case with $N_v = 4$ on the torus, where the topological degeneracy eq. (4.11) below pertaining to particular quantum liquid states is recovered from the $SU(3)$ spin-symmetry.

3.2 MAIN FEATURES OF THE PHASE DIAGRAM

In this section we make a first pass through the phase diagram with numerical results on moderate sizes. First we consider the ground states of $H_{\text{int}}$ in the disc geometry for each $L$, then use this to find the ground states of $H_{\omega}$ as a function of $\omega$. All this is done for general values of $c_0$, $c_2$.

GLOBAL STRUCTURE OF THE PHASE DIAGRAM

First we point out that the magnitude $(c_0^2 + c_2^2)^{1/2}$ only sets the overall energy scale, so it can be divided out. Thus the phase diagram can be thought of as a circle, in which a point on the circle represents a ray in the $c_0$-$c_2$ plane. We wish to examine this for each $L$, or later for each $\omega$. In Figure 3.1 the $c_0$-$c_2$ plane is shown with certain special directions $(c_0 = 0, c_2 = 0, g_0 = 0, g_2 = 0)$ that will be important later picked out.

For $L = 0$, the ground state has total spin $S = N$ for $c_2 < 0$ (ferro regime) and $S = 0$ (1) for $N$ even (odd) for $c_2 > 0$ (anti-ferro regime). These states are the way that the broken symmetry states described in Sec. 4.1 (the ferromagnetic and polar states respectively) appear in a finite size study. For $c_2 = 0$, there is a single $SU(3)$ multiplet of spin states, decomposing into one $SO(3)$ multiplet of each spin $S = N$, $N - 2$, ... The transition at $c_2 = 0$ can thus be viewed as levels crossing, with a larger degeneracy on the line $c_2 = 0$. As $L$ increases, these two phases at $c_2 \neq 0$ survive in part of the phase diagram, as compact drops of fluid, with the center of
mass carrying all the angular momentum. Meanwhile, the positive $c_0$ axis gradually opens into a region that contains other phases. By the time $L$ is $\geq N$, the $c_0-c_2$ plane contains the three regions labelled I-A, I-B, and II in figure 3.1.

The ground states in regions I-A and I-B are similar to the one in the "attractive" regime in the scalar case [97]. The orbital part of the ground state wavefunction is of the form $\Psi(z_i) \propto z_i^{k}$. In region I-A ($c_0 < 0, c_2 > 0$), the spin state is the same spin-singlet as for the $L=0$ ground state, and the energy [35] becomes $[c_0 N(N-1)/2 - Nc_2]$. In region I-B ($c_2 < 0, c_0 < -c_2$), the spin state is ferro, $S = N$, giving energy [97] $(c_0 + c_2)N(N-1)/2 < 0$. At $c_2 = 0, c_0 < 0$, the spin states again form the $SU(3)$ multiplet. In the remaining "repulsive" region II, the ground state is in general not a common eigenstate of the $c_0$ and $c_2$ parts of the interaction, and the ground state energy depends non-linearly on the ratio $\gamma = c_2/c_0$. Note that we have now located the repulsive region more precisely than in our previous characterization of it simply as $c_0 > 0$. Most of the following analysis focuses on region II only, which can be parametrized by $\gamma = c_2/c_0$ alone.

**Finite size results in region II as a function of $\omega$**

In figure 3.2 we show the ground state quantum numbers $(L,S)$ in region II for $N = 6$ bosons as a function of the rotation frequency $\omega$. As the phase diagram for each $\omega$ is a circle (which in region II can be parametrized by $\gamma = c_2/c_0$, or by $\phi = \arctan \gamma$), we are free to plot $\omega$ radially. The parameters are shown in units of $\omega_0$ and with $c_0 = 0.25$, but notice that the ground state quantum numbers can only depend on the dimensionless ratios of energies $(\omega_0 - \omega)/c_0$ and $c_2/c_0$, so that the structure shown is actually present (though with the radial variable rescaled and
3.2. MAIN FEATURES OF THE PHASE DIAGRAM

Figure 3.2: Ground state quantum numbers \( (L, S) \) in region II for \( N = 6 \) spin-1 particles in the planar (disc) geometry, as a function of the driving frequency \( \omega \) (plotted radially) and the ratio \( \gamma \) (corresponding to the angle with respect to the horizontal axis). The special directions \( g_2 = 0, c_2 = 0, g_0 = 0, c_0 = 0 \) are shown as double-dotted radial lines. The inset shows a cut along the \( c_2 = 0 \) direction, with the angular momentum given on the vertical axis and the (degenerate) spin values \( S \) marked at each of the steps. In this figure, the parameters \( c_0, c_2, \) and \( \omega \) are in units of \( \omega_0 \), and the value \( c_0 = 0.25 \) is used in the main figure as well as in the inset. For additional discussion, see the main text.
shifted) for all parameter values (unless \( c_0 \) is too large). The dashed rays are the lines \( c_2 = -c_0, c_2 = 0 \) and \( c_2 = c_0/2 \), and the outer dashed circle is the locus of \( \omega = \omega_0 \). The ground state angular momentum \( L \) and spins \( S \) at \( c_2 = 0 \) and as a function of \( \omega \), are shown in the inset. The degenerate spin values at \( L \leq N \) are seen to correspond to the following irreducible \( SU(3) \) multiplets: \( (p,q) = (6,0) \) for \( L = 0 \), \( (p,q) = (4,1) \) for \( L = 1 \), \( (p,q) = (2,2) \) for \( L = 2 \), \( (p,q) = (1,1) \) for \( L = 3 \) and \( (p,q) = (0,0) \) for \( L = 6 \). Note also that for \( c_2 < 0, c_0 > -c_2 \) the ground state spin gradually decreases from \( S = 6 \) at \( \omega = 0 \) to \( S = 0 \).

3.3 EXACT GROUND STATES

For larger sizes, a brute-force numerical approach is not feasible, so we develop other approaches. In this section we determine the exact ground state energies and wavefunctions for slow rotation (angular momentum up to the boson number, \( L \leq N \)), for \( c_0 > 0 \) and \( c_2 = 0 \), exploiting the \( SU(3) \) symmetry described in the section 3.1. Some of the ground states we find were described in Ref. [36]. We analyze a system of \( N \) spin-1 bosons in spherical geometry with \( N_v \) quanta of vorticity, with the disc geometry emerging as the limit \( N_v \to \infty \). We remark that for \( N \) sufficiently large, it becomes natural to discuss low energy properties in terms of mean field configurations that break the various symmetries and whose energy is slightly higher than that of the exact quantum ground state; this will be discussed in section 3.4.

EXACT EIGENSTATES OF \( H_n \)

The ground state spectrum for \( c_2 = 0 \) and \( L \leq N \) can be understood by exploiting the \( SU(3) \) symmetry of the Hamiltonian \( H_n \). In our analysis we proceed as follows. We consider two series of eigenstates of \( H_n \), in which (roughly speaking) the bosons occupy at most the lowest three orbitals. Among these eigenstates, we identify the exact quantum ground states on the disc and the sphere, as a function of the angular momentum. This then allows us to compute the \( \omega \) dependence of the ground state angular momentum for general \( TV \) at \( c_2 = 0 \).

We write the first series of eigenstates as \( |p,q,n\rangle \). These states contain doublets and triplets of spin-1 bosons that are fully antisymmetric in spin indices, and in the orbital indices (guaranteeing the overall symmetry that is required). The different numbers of single bosons, doublets and triplets correspond uniquely to the values of \( N \) and the quantum numbers \( (p,q) \) of the corresponding \( SU(3) \) multiplets. The triplets, which appear \( n \) times, are singlets under \( SU(3) \), and so do not affect the overall \( SU(3) \) representation. The highest spin component \( (S^z = p + q) \) of the corresponding \( SU(3) \) multiplet takes the following form (up to normalization)

\[
|p,q,n\rangle \propto |e_1 \cdot B\rangle^n |e_2 \cdot (B_0 \times B)^{\dagger}\rangle[n |B_0 \cdot (B_1 \times B_2)^{\dagger}\rangle^n |0\rangle,
\]

with \( e_1 = (1,0,0) \), \( e_2 = (0,0,1) \) and \( B_\alpha = (b^{\dagger}_{0,\alpha}, b^{\dagger}_{1,\alpha}, b^{\dagger}_{2,\alpha}) \). Clearly, the total
number of bosons is \( N = p + 2q + 3n \). The energies corresponding to eq. (3.11) are

\[
E_{p,q,n}^I/c_0 = \alpha_1^{N_v}n(n-1) + \alpha_2^{N_v}q(q-1) + \frac{1}{2}p(p-1) \\
+ \alpha_3^{N_v}np + \frac{3}{2}qp + \alpha_4^{N_v}nq,
\]

(3.12)

with

\[
\alpha_1^{N_v} = 3\frac{11N_v^2 - 20N_v + 6}{4(2N_v - 3)(2N_v - 1)} \\
\alpha_2^{N_v} = \frac{5N_v - 2}{2(2N_v - 1)} \\
\alpha_3^{N_v} = \frac{7N_v - 4}{2(2N_v - 1)} \\
\alpha_4^{N_v} = \frac{5\alpha_3^{N_v}}{2}.
\]

This energy is for spherical geometry, and it depends on the number \( N_v \) of flux quanta. For \( N_v \to \infty \) eq. (3.12) gives the energy in a disc geometry; \( N_v = 2 \) gives the energy on a sphere with 3 orbitals. On the basis of exact diagonalization studies for \( N = 6, 9, 12, 15, 18 \) particles we claim that on the disc for \( L \leq N/2 \), the ground state multiplet is precisely \( |p, q, 0\rangle^I \), with \( p = N - 2L, q = L \).

On the sphere with \( N_v = 2 \), we have obtained a much stronger result [74], namely a closed form result for all eigenvalues of \( H_n \). It turns out that these energies can be given in terms of the number \( N \) of bosons, the total angular momentum \( L \) and the \( (p, q) \) labels of the \( SU(3) \) representation, according to

\[
E_{p,q}^{N_v=2}/c_0 = \frac{5}{18}N(N - 1) + \frac{1}{6}T_{p,q}^2 + \frac{1}{6}L(L + 1),
\]

(3.13)

where \( T_{p,q}^2 = (p^2 + q^2 + pq)/3 + p + q \) is the quadratic Casimir operator for \( SU(3) \) in the representation \( (p, q) \). Specializing this expression to the states in series I, by eliminating \( N \) in favor of \( n \) and using the fact that \( \tilde{L} = p + q \), reproduces the result in eq. (3.12) for \( N_v = 2 \).

Analyzing the ground state on the disc for \( L > N/2 \), we identified a second series of states \( |p, q, n\rangle^II \). One can think of the type II states as having the \( p \) single bosons in \( m = 1 \) rather then \( m = 0 \), so that now \( c_1 = (0, 1, 0) \). That is not quite correct for the energy eigenstates, as we will explain below, but it does give the correct quantum numbers. The states in series I, II share the property of having \( p \) single bosons and \( q \) doublets, leading to \( SU(3) \) Dynkin labels \( (p, q) \). It may be illuminating to display the structure of the states in terms of diagrams similar to Young tableaux as in figure 3.3. For the orbital structure of the highest-weight states in either series I or II, the lengths of the three rows represent the number of bosons in the orbitals \( m = 0, 1, 2 \) respectively (in the rough point of view, which will be corrected below), while the differences \( p, q \) and \( n \) in the lengths correspond to the \( SU(3) \) structure. Essentially, these diagrams are ordinary Young tableaux for the states, with but the first two rows exchanged in the case of series II.

For the case of the type II states, the following correction must be made to obtain the energy eigenstates. In the case of scalar bosons, it is known[96, 13, 87] that the ground state configuration at \( L = p \) of \( p \) bosons is a vortex located at their center of mass, with wavefunction \( \prod_i(z_i - z_c) \) with \( z_c = \sum z_i/p \). This state is not
entirely restricted to the \( m = 1 \) orbital, as there are components in which other orbitals in the range \( 0 \leq m \leq p \) are occupied as well. The \( p \) bosons in the state \( |p, q, n\rangle^{II} \) form such a vortex. This complication makes it difficult to write down the closed form expression for the states in series II; based on numerical analysis for small \( N \) and mean field results for large \( N \) (see section 3.4), we do propose the following closed form expression for the corresponding energy on the disc

\[
E_{p,q,n}^{II}/c_0 = \frac{33}{16} n(n-1) + \frac{5}{4} q(q-1) + \frac{1}{4} p(p-2) \\
+ \frac{25}{8} nq + \frac{11}{8} np + qp.
\]

Note that the \( p \)-independent terms in this formula are identical to those for type I states with \( N_v = \infty \). The state \( |p,0,0\rangle^{II} \), has energy \( p(p-2)/4 \), which is exactly the ground state energy of a rotating scalar BEC at \( L = p = N \). This justifies the interpretation of the polarized subsystem with \( p \) bosons forming a vortex at the center of mass. However, it turns out that \( |p,0,0\rangle^{II} \) will never be the lowest energy configuration for a rotating spin-1 system.

Among the type I/II states the following are special. First, \( |p,0,0\rangle^{I} \) is the non-rotating ground state, corresponding to the \( (p,0) \)-multiplet. Second, \( |0,q,0\rangle \) gives a wavefunction composed of anti-symmetrized pairs of bosons, a Boson-Doublet-Condensate (BDC) or \((0,q)\)-multiplet. Third, \( |0,0,n\rangle \) is composed of 3-body singlets. It is a condensate of triplets or boson-triplet-condensate \([70]\) (BTC); we shall see that it forms the ground state at \( L = 3n = N \). The BTC-state can be regarded as a symmetrized version of the core-less vortices observed in mean field studies (see section 3.4 for more on this).

More generally, the type I/II states are examples of "(multi-) fragmented" condensates \([60]\), see also \([35]\), in the sense that they contain several macroscopically occupied elements in the density matrix. For instance, for the BTC- and for (any component of) the BDC-state we have \( \langle n_{ma}\rangle_{BDC} = (1 - \delta_{m,2})(1 - \delta_{a,1})q/2, \)
\( \langle n_{ma}\rangle_{BTC} = n/3. \) Since the spin is fixed in these states, \( (\Delta n_{\alpha})^2 = (\langle n_{\alpha} \rangle - \langle n_{\alpha} \rangle)^2 = 0, \) where \( n_{\alpha} = \sum_{m=0,1,2} n_{ma}. \) However, within each spin component, the fluctuations of the boson number between orbitals is of the order of the system size:

\( (\Delta n_{ma})_{BDC}^2 = q(q+2)/12, (\Delta n_{ma})_{BTC}^2 = n(n+3)/18. \) This is an indication that, as in the case of the singlet ground state at \( L = 0 \) in the antiferromagnetic regime.
EXACT GROUND STATES

The ground state of a rotating gas with N spin-1 bosons in the LLL and a spin-independent ($c_2 = 0$) interaction is formed by a sequence of type I or II states lying on a certain path in $(p, q)$-space as $L$ increases. To find the ground state in a rotating frame of reference, we need to find the ground state of $H_{\omega}$, eq. (3.6), instead. Since this Hamiltonian contains only two energy scales, the ground state angular momentum per particle $L/N$ can be written as a function of the ratio $(\omega - \omega_0)/(Nc_0)$. For finite boson number this function consists of a sequence of steps, as can be seen in figure 3.2. It turns out that (thanks to our judicious choices of factors of $N$) the limit $N \to \infty$ with $L/N$ and $(\omega - \omega_0)/(c_0N)$ fixed of this function exists, and this is the most convenient information to display. In the following we determine the path of the ground states in $(p, q)$-space as a function of $L$, and the $L(\omega)/N$ behavior of the ground states in this limit for both the sphere ($N_v = 2$) and the disc ($N_v = \infty$) in the regime $L/N \leq 1$. 

Figure 3.4: $L/N$ as $N \to \infty$ of the ground state on the disc (bold line) and on the sphere with $N_v = 2$ (dashed line), as functions of $\tilde{\omega} = (\omega - \omega_0)/(c_0N)$ at $c_2 = 0$. The horizontal lines mark the values $L/N = 1/3$ and $L/N = 2/3$. The cusps in both curves indicate the point where the $m = 2$ quantum orbital is first used in the ground state. 

and the related "polar" mean field state [35], it may be best to think of these states as broken symmetry states [2]. That is the approach we will take in section 3.4.
GROUND STATES ON THE SPHERE AT \( N_v = 2 \)

On the sphere, our notion of rotation is such that the \( SO(3)_{\text{orb}} \)-angular momentum \( \hat{L} \) decreases as the system rotates faster and faster. With three orbitals \( (N_v = 2) \) we have \( \hat{L}_z = N - L \) (see section 3.1). (We consider \( N_v = 2 \) because this case can just accommodate \( L \leq N \).) At \( \hat{L} = N \), we know already that the \( |N, 0, 0\rangle \) multiplet forms the ground state. As \( \hat{L} \) starts to decrease, again a type I state has the lowest energy; the \((p, q)\)-path is parametrized by \( (2\hat{L} - N, N - \hat{L}) \). Bosons are gradually added to the \( m = 1 \) orbital and form anti-symmetrized pairs with the remaining ones. The point up to which this continues can be found by comparing the energies of \( |2L - N, N - L, 0\rangle \) and \( |2L - N + t, N - \hat{L} - t, t/3\rangle \). After minimizing with respect to \( t \) this yields the critical \( SU(3) \)-indices \( (p, q)_{c} = (N/3, N/3) \). At this point, with \( \hat{L} = 2N/3 \), ground states with a nonzero \((n > 0)\) number of triplets become energetically favorable. In the remaining region, \( 2N/3 \geq L \geq 0 \), type I states are the ground states following the path \((p, q) = (\hat{L}/2, \hat{L}/2) \). Eventually this terminates on the BTC at \( \hat{L} = 0 \). \( L/N \) of the ground state as a function of the rotation drive \( \omega \) shows a cusp at \( \hat{L}/N = 2/3 \) \((L/N = 1/3)\), as is shown in figure 3.4.

GROUND STATES ON THE DISC

For a system on the disc \((N_v = \infty)\), the results are rather different. We will again present the ground states in order of increasing \( L \). At \( L = 0 \), the \( |N, 0, 0\rangle \)-multiplet forms the ground state as we know. For \( L \leq N/2 \) the ground state is formed by a type I state with \( n = 0 \) and \( SU(3) \)-quantum numbers \((p, q) = (N - 2L, L) \). This state terminates on the BDC at \( L = N/2 \). In this range, increasing \( L \) leads, as on the sphere, to more bosons occupying the \( m = 1 \) orbital, forming anti-symmetrized pairs with the ones in the \( m = 0 \) orbital. For \( L \geq N/2 \), the type II states have the lowest energy. As \( L \) increases, bosons move from the \( m = 0 \) into the \( m = 1 \) orbital, decreasing the number of doublets, and giving type II states at \((p, q) = (2L - N, N - L) \). Comparing the energies of \(|tN, (1 - t)N/2, 0\rangle \) and \(|(t - s)N, (1 - t)N/2, sN/3\rangle \), we can determine the point where it becomes favourable for triplets to enter the ground state. We find a critical angular momentum \( L = (1 - t_c)N \) with \( t_c \sim 1/3 - 3/N \), which approaches \( L = 2N/3 \) for \( N \) large. For \( L \geq 2N/3 \) the number of triplets is gradually increasing as \( L \) grows. Minimizing \(|2L - N - s, N - L, s/3\rangle \) with respect to \( s \), we find that the ground state is now the type II state with \( s(N, L) = 3L - 2N \), giving \((p, q, n) = (N - L, N - L, L - 2N/3) \). For \( L = N \) the ground state is the BTC with \( p = q = 0, n = N/3 \).

To summarize the above results, for \( 0 \leq L \leq N/2 \) the ground state is given by \( |N - 2L, L, 0\rangle \) and for \( N/2 \leq L \leq 2N/3 \) by \( |2L - N, N - L, 0\rangle \). In the remaining range \( 2N/3 \leq L \leq N \) the number of 3-body singlets is nonzero, and the ground state is given by \( |N - L, N - L, L - 2N/3\rangle \). Minimizing the energy in a rotating frame of reference leads to the \( L(\omega)/N \)-dependence of the ground states for \( N \rightarrow \infty \) which is depicted in figure 3.4. In this figure, the curve shows a cusp at the point where the \( m = 2 \) orbital first enters the ground state configuration, which is at \( L/N = 2/3 \) for the disc with \( N \) large. A signature of this cusp in an experimental
3.3. Exact ground states

\[ \begin{array}{cccccc}
S \backslash L & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & & & & & \\
1 & & 1 & & & & \\
2 & & & 1 & & & \\
3 & & & & 1 & & \\
4 & & & & & 1 & \\
5 & & & & & & 1 \\
6 & & & & & & \\
\end{array} \quad \begin{array}{cccccc}
S \backslash L & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & & & & \\
2 & 1 & 2 & 2 & 1 & 1 & & & \\
3 & 1 & 2 & 2 & 2 & 1 & 1 & & \\
4 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & \\
5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
6 & & & & & & & & \\
\end{array} \]

Figure 3.5: Degeneracies of zero-energy ground states on the sphere at \( \gamma = -1 \) for \( N = 6, N_v = 2 \) (top left), \( N = 5, N_v = 3 \) (top right) and \( N = 6, N_v = \infty \). All multiplicities refer to highest weight states of the orbital \( SO(3) \) symmetry.

The behavior at the phase boundaries I-A/II and I-B/II (see figure 3.1) deserves special attention. At the boundary I-B/II, where \( \gamma = c_2/c_0 = -1 \), the Hamiltonian simplifies as \( g_2 = 0 \) and only the contact interaction which projects onto the spin-singlet channel remains. As a result of this, large degeneracies occur. For example, all fully-polarized states \( (S = N) \) have zero energy. We have not obtained analytic expressions for these degeneracies. That they are not due to the specific geometry was observed on the torus. The zero energy states are not sensitive to changes in the geometry. As examples of these degeneracies, we have in figure 3.5 tabulated the \( L, S \) quantum numbers of the zero energy states for \( N = 6, N_v = 2 \) and for \( N = 5, N_v = 3 \) on the sphere, and for \( N = 6 \) particles in the disc geometry.

The system might be a change in the expansion rate (the rate of change of the outer radius of the drop with respect to \( \omega \)) if the angular momentum exceeds \( 2N/3 \). We shall see that the cusp survives in the anti-ferromagnetic regime, \( c_2 > 0 \).

It is important to contrast all this with the well-known behavior of scalar bosons in a rotating trap [96, 97]. In the latter case there is a jump from \( L/N = 0 \) to \( L/N = 1 \) (for all \( N \)) when one vortex enters the system, whereas for spin-1 bosons we find (at \( N \to \infty \)) a continuous \( L(\omega)/N \) curve with a discontinuous slope.
CHAPTER 3. SLOWLY ROTATING SPIN-1 CONDENSATES

3.4 LLL MEAN FIELD THEORY

At low rotation rates, the typical boson occupation numbers \( \langle n_{\alpha\alpha} \rangle \) of the occupied \( (n_{\alpha\alpha} \neq 0) \) single-particle states are large compared with 1. In this situation, a mean field (or classical) approach to the problem is generally expected to be quantitatively accurate. In such an approach, the boson operators are replaced by expectation values, which are complex c-numbers: \( b_{\alpha\alpha} \rightarrow \langle b_{\alpha\alpha} \rangle \), and the second-quantized Hamiltonian is then minimized with respect to both the magnitude and phase of these numbers to find the ground states. In essence the resulting state is a Bose condensate with the bosons condensed in one linear combination of the single-particle states. This typically involves breaking the orbital and spin symmetries, as well as particle number conservation. (States with definite values of the good quantum numbers such as \( N, S, L \) can be obtained afterwards by applying a projection to the mean field quantum state \([2]\).) In the case of very low rotation, where \( L \leq N \), we have seen that (neglecting the subtleties that arose for type II states) the states essentially involve only the \( m = 0, 1, \) and 2 states, so that the basis set for the mean field calculation is particularly small. In these cases, the mean occupation numbers of the single particle states are of order \( N \), and their energies exceed the exact ground state energy (which is of order \( N^2 \)) by an amount of order \( N \). We refer to \([44, 39]\) for an extensive account on mean field theory for the case of scalar bosons, including the order \( N \) correction related to quantum fluctuations.

In this section we pursue this mean field calculation for this regime. This gives us easy access to the ground states at large \( N \) for \( c_2 \neq 0 \) in region II. In the following section, we study instead the mean field states at larger rotation, which can be assumed to be states in which the translational and rotational symmetry group of the plane is broken to that of a lattice.

In terms of the complex numbers \( b_{\alpha\alpha}, b_{\alpha\alpha}^* \), the energy becomes a quartic polynomial and the ground state can be found by minimizing this polynomial with respect to these variables. This is done here with the mean boson number \( \langle N \rangle = \sum_{\alpha\alpha} b_{\alpha\alpha}^* b_{\alpha\alpha} \) and angular momentum \( \langle L \rangle = \sum_{\alpha\alpha} m b_{\alpha\alpha}^* b_{\alpha\alpha} \) fixed at the values \( N, L \), respectively. The spin is not constrained at all. For \( c_2 = 0 \) the Hamiltonian on the sphere takes the form

\[
H_n = c_0 \sum_{\alpha\beta} \sum_{m_1 \cdots m_4} V_{m_1m_2m_3m_4} b_{m_1\alpha}^* b_{m_2\beta}^* b_{m_3\alpha} b_{m_4\beta},
\]

with matrix elements

\[
V_{m_1 \cdots m_4} = \frac{1}{2} \sqrt{\binom{N_v}{m_1} \binom{N_v}{m_2} \binom{N_v}{m_3} \binom{N_v}{m_4}} \delta_{m_1+m_2, 2N_v} \delta_{m_3+m_4, 2N_v}.
\]

This exhibits the dependence on the spatial orbitals. For \( c_2 \neq 0 \), the matrix elements in the additional term consist of \( V_{m_1 \cdots m_4} \) multiplied by matrix elements of \( \mathbf{S}_i \cdot \mathbf{S}_j \).
which depend on the $\alpha_1, \ldots, \alpha_4$ labels of the bosons. These matrix elements can be found in standard quantum-mechanics texts.

The fact that mean-field configurations break the various symmetries implies that the minima of the mean field energy form orbits under the action of these same symmetries. On a disc, and at $c_2 \neq 0$, one expects and finds that, typically, from a generic minimum there are 5 flat directions leading to adjacent minima with equal energy. These flat directions correspond to the 3 generators of the $SO(3)$ spin symmetry, an overall phase, and an orbital $O(2)$ rotation. For spin-independent interactions the symmetry orbits are generically 10-dimensional.

One convenient quantity to plot is the expectation value $\langle S \rangle$ of the spin, whose length is conserved under global spin rotations. In special cases, this expectation value is axi-symmetric; in the more general case it is non-axisymmetric and the mean field configuration breaks the orbital $O(2)$ symmetry. Another useful quantity is the three-component condensate wavefunction (analogous to the familiar spinor for spin-1/2), which is the expectation value of the field operator, $\langle \psi_\alpha(z) \rangle = \sum_m b_{m\alpha} \phi_m(z)$ (see Sec. 3.1). It is a vector in the $\alpha = \uparrow, 0, \downarrow$ basis. From this we can plot the density in each spin component in position space. This could be accessed experimentally if after switching off the trap to allow the particle cloud to expand, a Zeeman term is switched on, which causes the three $\alpha$ components to separate as they expand.

As an example, we plot in figure 3.6 the 2D density profile in each spin component of two different mean field ground state configurations at $c_2 = 0, L = N$. The top frame shows the densities for the condensate proportional to $(\phi_0, \phi_1, \phi_2)$; the lower
frame shows a configuration that is related to this by an SU(3) rotation. The total density in each of the \( m = 0, 1, 2 \) orbitals is an SU(3) invariant, and it is the same for both configurations shown in figure 3.6. The mean field energy of these configurations is \( E_{MF} = \frac{11}{48} N^2 \), in agreement with order \( N^2 \) term in the energy of the exact quantum (BTC) ground state, eq. (3.12) with \( N_v = 2 \), \( p = q = 0 \) and \( n = N/3 \).

First we consider the disc geometry with \( \epsilon_2 = 0 \). Carrying out the mean field minimizations, we find in terms of \( \ell = L/N \) that for \( 0 \leq \ell \leq 2/3 \) the number densities \( \langle n_m \rangle = \sum_\alpha b^*_{m\alpha} b_{m\alpha} \) in the orbitals of the mean field ground states behave like (here and in the remainder of this section, these numbers are normalized so that they sum to 1) \( \langle n_0 \rangle = 1 - \ell, \langle n_1 \rangle = \ell \) and \( \langle n_2 \rangle = 0 \). For \( 2/3 \leq \ell \leq 1 \) we find \( \langle n_0 \rangle = \frac{1}{3}, \langle n_1 \rangle = \frac{1}{3} - \ell \) and \( \langle n_2 \rangle = \ell - \frac{2}{3} \). All this is in agreement with the results derived from the exact quantum ground states in section 3.3.

For very small interaction ratios \( |\gamma| \ll 1 \), the total densities in the orbitals remain the same as for \( \gamma = 0 \), but there is non-trivial structure in the spin dependence, leading to spin transitions at critical values of \( \ell = L/N \), as we will describe shortly.

In figure 3.7 we have plotted region II of the phase diagram, this time with \( \ell \) radially. The shaded regions show where only the first two orbitals \( (m = 0, 1) \) are present in the condensate. One region is a tiny strip near \( \ell = 1 \) for \( \gamma \geq (7 + 4\sqrt{2})/17 \approx 0.75 \), where the \( (m, \alpha) = (1, 0) \) state is occupied by all the bosons. This state can be seen as a "polar" vortex, since it has the same spin state as the polar BEC. The other region, centered (roughly) around the \( \epsilon_2 = 0 \) axis, contains states in which both the \( m = 3 \) orbital is used.

In the anti-ferromagnetic regime for \( \ell < 1 \) there is a large area where the \( m = 3 \) orbital requires a non-zero density; in this area, mean field theory in which only the first three orbitals are used is not valid. However, around and at the SU(3)-axis and around the polar vortex as well as in the ferromagnetic regime, the density in the \( m = 3 \) orbital is very small for \( \ell \leq 1 \) and can safely be ignored. Besides, if the energy \( H_{\omega} \) in a rotating frame (see eq. (3.6)) is minimized, only the states which use the first three orbitals \( m = 0, 1, 2 \) are of interest for \( \ell \leq 1 \). (This is with the exception of the vicinities of the boundaries of region II (see figure 3.1) at \( \gamma \to \infty \) and at \( \gamma = -1 \)).

In the following subsections we present results for the LLL mean field ground state for \( |\gamma| \ll 1 \), in both the ferromagnetic and anti-ferromagnetic regimes, and we discuss the ground states at \( \ell = 1 \) for general values of \( \gamma \).

Our mean field results pertain to the LLL, relevant for the regime of weak interactions, and they thus differ from the mean field solutions of the GP equations [52, 50]. Nevertheless, there is agreement on some of the important features, such as the smooth dependence of \( L \) on \( \omega \) in the ferro regime, and the role of the state with a single \( \pi \)-disclination near \( \ell = 0.5 \) in the antiferromagnetic regime [50].

**Anti-ferromagnetic interactions**

We now specify the mean field ground states, given in the form of a three-component condensate wave function, for small, positive \( \gamma = +\epsilon \), and for \( \ell \leq 1 \). As before, the
Figure 3.7: Regions in the $\gamma$, $\ell$ plane in which only the $m = 0$ and $m = 1$ orbitals are present in the mean field ground state on the disc are shaded. The angular coordinate is $\phi = \arctan \gamma$ and $\ell = L/N$ is plotted radially. In the shaded strip near $\ell = 1$, a "polar vortex" forms the ground state. The dotted lines mark the $\gamma = \pm 1$ and $\gamma = 0$ directions.
condensate wave function is a vector in the $\alpha = |\uparrow, 0, \downarrow\rangle$ basis. In the table below we specify the mean occupation numbers of the four states that we found.

<table>
<thead>
<tr>
<th>$n_0\uparrow$</th>
<th>$n_{0\downarrow}$</th>
<th>$n_{10}$</th>
<th>$n_{1\downarrow}$</th>
<th>$n_{20}$</th>
<th>$n_{2\downarrow}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \ell \leq \ell_a$</td>
<td>$\frac{1}{2}(1 - \ell)$</td>
<td>$\frac{1}{2}(1 - \ell)$</td>
<td>$\ell$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\ell_a \leq \ell \leq \frac{2}{3}$</td>
<td>0</td>
<td>$1 - \ell$</td>
<td>0</td>
<td>$\ell$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{2}{3} \leq \ell \leq \ell_b$</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>$\frac{4}{3} - \ell$</td>
<td>$\ell - \frac{2}{3}$</td>
</tr>
<tr>
<td>$\ell_b \leq \ell \leq 1$</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{4}{3} - \ell$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.1: Occupation numbers of the LLL mean field ground state with small anti-ferromagnetic interaction $\gamma = \varepsilon$.

Note that the condensates given in this table are specific representatives of families of condensates that are related by the $SO(3)_{\text{spin}}$ symmetry. There are two critical values, $\ell_a = \frac{4}{7} - \frac{\sqrt{2}}{7} \approx 0.37$ and $\ell_b = 10 - 4\sqrt{5} - \frac{4}{3}\sqrt{85 - 38\sqrt{5}} \approx 0.83$, where we see a discontinuous rearrangement of the condensate configuration and of $(S)$. For nonzero $\gamma$, these changes in the condensate are continuous; they become singular (discontinuous) only as $\gamma \to 0^+$. For $\ell < \ell_a$, the condensate can be represented by $(\frac{1}{\sqrt{2}}\phi_0(z), \eta\phi_1(z), \frac{1}{\sqrt{2}}\phi_0(z))$ with $\lambda = \sqrt{N-L}$, $\eta = \sqrt{L}$. Applying $SO(3)_{\text{spin}}$ rotations, one finds alternative representations such as $(\frac{1}{\sqrt{2}}[\lambda\phi_0(z) - \eta\phi_1(z)], 0, \frac{1}{\sqrt{2}}[\lambda\phi_0(z) + \eta\phi_1(z)])$. The $SO(3)_{\text{spin}}$-invariant quantity $|\langle S\rangle|^2$ is found to be

$$|\langle S\rangle|^2 = \frac{N^2}{\pi^2} \ell(1-\ell)(z+z\varepsilon)^2 e^{-2|z|^2}.$$  \hspace{1cm} (3.17)

The state that emerges at $\ell > \ell_a$ corresponds to $(\eta\phi_1(z), 0, \lambda\phi_0(z))$, leading to

$$|\langle S\rangle|^2 = \frac{N^2}{\pi^2} (|1-\ell| - |z|^2)^2 e^{-2|z|^2}.$$  \hspace{1cm} (3.18)

For $\ell \neq 1/2$, the integrated value of $\langle S\rangle$ for this state is non-zero and there is a spontaneous magnetization. In figure 3.8 a two-dimensional plot of $\langle S\rangle$ at both sides of the spin transition at $\ell = \ell_a$ is shown. The state at $0 < \ell < \ell_a$ can be viewed as a configuration of two $\pi$-disclinations off the center of the trap, while the state in the regime $\ell_a < \ell < 2/3$ (or possibly even as far as $\ell_b$) can be understood as a single $\pi$-disclination in the polar state.

The angular momentum for which the $m = 2$ orbital is first occupied in the mean field ground state, $\ell = \frac{2}{3}$, is robust against small anti-ferromagnetic interactions. For $\gamma = +\varepsilon$, $2/3 < \ell < \ell_b$, the condensate can be represented as $(\tau\phi_1(z), \sigma\phi_2(z), \xi\phi_0(z))$, with $\xi = \sqrt{\frac{N}{3}}$, $\sigma = \sqrt{L - \frac{2N}{3}}$, $\tau = \sqrt{\frac{4N}{3} - L}$, while for $\ell_b < \ell \leq 1$ we have $(-\sigma\phi_2(z), \tau\phi_1(z), \xi\phi_0(z))$. 
3.4. LLL MEAN FIELD THEORY

Figure 3.8: Two-dimensional plot of \((S)\) at both sides of the spin transition at \(\ell = \ell_a^c\). The intensity codes the length \(|(S)|\), while the color indicates the direction on the spin sphere as in figure 3.11 below. The left and right pictures correspond to eqs. (3.17) and (3.18), respectively.

In figure 3.9, we have depicted the ground state angular momentum per particle, \(\ell\), as a function of the rotation frequency \(\omega\) for some positive values of \(\gamma\). It is seen that upon increasing \(\gamma\) a semi-plateau (a distinguished part of the curve on which the angular momentum increases gradually) develops. Upon increasing \(\gamma\) further, the semi-plateau becomes flatter and the width decreases, until for \(\gamma\) larger than some critical value \(\gamma_c \approx 1.19\), \(\ell(\omega)\) jumps from \(\ell = 0\) to an \(\ell = 1\) plateau at a critical frequency \(\omega_c\) given by \(\omega_0 - \omega_c \approx 0.15c_0N\). This is a transition from the non-rotating state to the polar vortex, analogous to what occurs in the scalar boson case.

Ferromagnetic interactions

With small negative \(\gamma = -\epsilon\) the mean field ground states for slow rotation are characterized (up to \(SO(3)_{\text{spin}}\) rotations) by the occupation numbers given in table 3.2. Again, we find two spin transitions, the first at \(\ell_a^{-\epsilon} = 2 - \sqrt{2} \approx 0.59\) and the second at \(\ell_b^{-\epsilon} \approx 0.69\).

<table>
<thead>
<tr>
<th>(\ell)</th>
<th>(\langle n_{00} \rangle)</th>
<th>(\langle n_{01} \rangle)</th>
<th>(\langle n_{10} \rangle)</th>
<th>(\langle n_{11} \rangle)</th>
<th>(\langle n_{21} \rangle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 \leq \ell \leq \ell_a^{-\epsilon})</td>
<td>0</td>
<td>(1 - \ell)</td>
<td>(\ell)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\ell_a^{-\epsilon} \leq \ell \leq \frac{2}{3})</td>
<td>(1 - \ell)</td>
<td>0</td>
<td>0</td>
<td>(\ell)</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{2}{3} \leq \ell \leq \ell_b^{-\epsilon})</td>
<td>(\frac{1}{3})</td>
<td>0</td>
<td>0</td>
<td>(\frac{4}{3} - \ell)</td>
<td>(\ell - \frac{2}{3})</td>
</tr>
<tr>
<td>(\ell_b^{-\epsilon} \leq \ell \leq 1)</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{4}{3} - \ell)</td>
<td>0</td>
<td>(\ell - \frac{2}{3})</td>
</tr>
</tbody>
</table>

Table 3.2: Mean occupation numbers of the LLL condensate for small ferromagnetic interaction \(\gamma = -\epsilon\).
Figure 3.9: The ground state angular momentum per particle \( \ell \) on the disc as a function of \( \tilde{\omega} = (\omega - \omega_0)/c_0 N \) for various interaction strengths and slow rotation. Upper figures: anti-ferromagnetic regime, \( \phi = 0.1, 0.5, 0.75 \). Lower figures: ferromagnetic regime, \( \phi = -0.1, -0.3, -0.5 \).

For \( \ell < \ell_\alpha^- \) the condensate can be represented by \( (0, \eta \phi_1(z), \lambda \phi_0(z)) \) with \( \lambda \) and \( \eta \) as given above. In this state, the expectation values of the components of the spin vector take the following form

\[
\langle S_x \rangle = \frac{N}{\pi \sqrt{2}} \sqrt{\ell(1-\ell)} (z + \bar{z}) e^{-|z|^2}
\]
\[
\langle S_y \rangle = \frac{N}{\pi \sqrt{2}} \sqrt{\ell(1-\ell)} (-i)(z - \bar{z}) e^{-|z|^2}
\]
\[
\langle S_z \rangle = \frac{N}{\pi} (1 - \ell) e^{-|z|^2}.
\]  

(3.19)

The state at \( \ell_\alpha^- < \ell < 2/3 \) corresponds to \( (0, \lambda \phi_0(z), \eta \phi_1(z)) \), leading to a spin vector that vanishes at the center of the disc. The spin-textures for \( \gamma = -\epsilon, \ell < 2/3 \), can be interpreted as half-skyrmions (or merons).

For \( \gamma = -\epsilon \), the condensate can be represented as \( (\sigma \phi_2(z), \xi \phi_0(z), \tau \phi_1(z)) \) for \( 2/3 < \ell < \ell_\alpha^- \), with \( \xi, \sigma \) and \( \tau \) as above, while for \( \ell_\alpha^- < \ell \leq 1 \) we have \( (\sigma \phi_2(z), \tau \phi_1(z), \xi \phi_0(z)) \).

In the ferromagnetic regime the \( \omega \) dependence of the ground state angular momentum becomes a smooth curve; see Figure 3.9.
3.4. LLL MEAN FIELD THEORY

MEAN FIELD CONFIGURATION AT $L = N$

Assuming that only the first three $m = 0, 1, 2$ orbitals participate in the ground state, the mean field ground states at $\ell = 1$ take the form $(\pm \phi_2(z), \sigma \phi_1(z), \xi \phi_0(z))$, with $\xi = \sigma = \sqrt{k_\pm}$ and $\tau = \sqrt{1 - 2k_\pm}$, and with the $(\pm)$—sign corresponding to (anti-)ferromagnetic interactions. The parameters $k_\pm$ depend on $\gamma$ according to

$$k_\pm(\gamma) = \frac{\mp (19 + 28\sqrt{2}) \gamma^2 + (42 \pm 4\sqrt{2}) \gamma - 3}{71 \gamma^2 + 126 \gamma - 9}. \quad (3.20)$$

The orbital occupation numbers, given as

$$\langle n_{10} \rangle = 1 - 2k_+, \quad \langle n_{01} \rangle = \langle n_{21} \rangle = k_+, \quad (3.21)$$

are continuous for $\gamma$ going through 0, but the spin-texture, which is sensitive to the phases in the condensate wave function, is not. We find that for $\gamma = \pm \epsilon$, up to an overall constant,

$$\langle S_x \rangle = (1 + \frac{1}{\sqrt{2}} \bar{z} z)(z + \bar{z}) e^{-|z|^2},$$

$$\langle S_y \rangle = (1 + \frac{1}{\sqrt{2}} \bar{z} z)(-i)(z - \bar{z}) e^{-|z|^2},$$

$$\langle S_z \rangle = (1 - \frac{1}{2} (\bar{z} z)^2) e^{-|z|^2}. \quad (3.22)$$

Note that in the anti-ferromagnetic case, the expectation value of the spin vector is vanishing on the circle $\bar{z} z = \sqrt{2}$, while in the ferromagnetic case we see a single skyrmion texture with $\langle S \rangle$ non-vanishing everywhere. Figure 3.10 shows the spin texture at $\ell = 1$ for $\gamma = \pm \epsilon$.

From (3.20) it is possible to derive the critical anti-ferromagnetic interaction ratio for which the polar vortex appears, by simply solving $k_+(\gamma) = 0$. The critical value found then is $\gamma^* = (7 + 4\sqrt{2})/17 \approx 0.75$ (see figure 3.7). If $\gamma$ increases towards $\gamma^*$, the density in the $m = 3$ orbital acquires a small value. So, strictly speaking, the states discussed here are not the true mean field ground state in the whole intermediate region. Around $\gamma = 0$ and $\gamma = \gamma^*$ however, $\langle n_{30} \rangle$ is zero and the value of $\gamma^*$ is in agreement with numerical results.

In the ferromagnetic regime, upon lowering $\gamma$ the parameter $k_-$ gradually decreases from $k_- = 1/3$ at $\gamma = 0$ to $k_- = 1 - 1/\sqrt{2}$ at $\gamma = -1$, with the corresponding occupation numbers given in eq. (3.21).

THE SPHERE WITH $N_v = 1, 2$

It is instructive to perform LLL mean field theory on a system of spin-1 bosons in a spherical geometry, with $N_v = 1$ or $N_v = 2$, meaning that 2 or 3 orbitals are available to the particles. To compare with the disc as before, we write these results in terms of $\ell = (1/2 N_v - L)/N$. Notice, however, that by flattening out the sphere by
stereographic projection, the results are qualitatively similar to those for the disc when only the first two or three orbitals are occupied. This is especially true for the states at \( N_v = 2, \ell = 1 \). Even though the topological classification of textures (see Appendix A) does not strictly apply to the plane, the form of the spin textures on the sphere is a useful guide to those in the disc for \( \ell \leq 1 \).

For the case \( N_v = 1 \) (two orbitals on the sphere), we mention the following results. With \( 0 < \gamma \leq \pi/4 \) the ground state configuration is the same as the one we found on the disc for \( \ell < \ell_i^{\pi} \). This configuration can be interpreted as two \( \pi \)-disclinations at opposite poles of the sphere. For \( \pi/4 \leq \gamma < \pi/2 \) all bosons occupy the \( \alpha = 0 \) spin component, forming a polar state with a single vortex. In the ferromagnetic regime, with very small \( \gamma \) we find the same spin transition as the one on the disc at \( \ell = \ell_i^{\pi} \). With \( N_v = 1 \) this transition lies at \( \ell = 1/2 \).

These configurations can be interpreted as a half-skyrmion (or meron) in the spin texture, with the spin density vanishing at one point on the sphere, around which the spin density winds around the equator in S-space, passing over one pole at the opposite end of the sphere. If the interaction is deformed by increasing \( N_v \) towards \( N_v \rightarrow \infty \), the location of the spin transition is gradually shifted towards \( \ell = \ell_i^{\pi} \). With finite ferromagnetic interaction (\( N_v = 1 \) again), there is a finite region where the core traces a path over the sphere (from the south pole to the north pole, as \( \ell \) increases) and connects the two sides of the transition. The interaction energy is clearly independent of \( \ell \). This region is bounded by \( \gamma(\ell) = -\left| \arctan(2\ell - 1) \right| \) for \( 0 \leq \ell \leq 1 \).

For \( N_v = 2 \) and \( \gamma = 0 \), the occupation numbers, summed over spin, in the three available orbitals are given by \( \langle n_0 \rangle = 1 - \ell, \langle n_1 \rangle = \ell \) and \( \langle n_2 \rangle = 0 \) for \( 0 \leq \ell \leq 1/3 \), followed by \( \langle n_0 \rangle = \frac{5}{6} - \frac{1}{3} \ell, \langle n_1 \rangle = \frac{1}{3}, \langle n_2 \rangle = \frac{1}{3} \ell - \frac{1}{6} \) for \( 1/3 \leq \ell \leq 1 \). These mean field results agree with the exact quantum ground state results obtained in section 3.3.

For \( N_v = 2 \) and small ferromagnetic interactions, \( \gamma = -\epsilon \), the mean occupation numbers in the condensate are given in table 3.3 (up to \( SO(3)_{\text{spin}} \) and \( SO(3)_{\text{orb}} \).
3.4. LLL MEAN FIELD THEORY

In the trajectory from $\ell = 0$ to $\ell = 1$ there are no spin transitions. The $\ell = 1$ state, which has $\langle n_{01} \rangle = \langle n_{10} \rangle = \langle n_{21} \rangle = \frac{1}{3}$, is the mean field ground state for arbitrary ferromagnetic spin interactions, $0 > \gamma > -1$. It is a single skyrmion texture with both uniform number density and magnitude of the spin density, and is discussed further in the Appendix.

| $0 \leq \ell \leq \frac{1}{3}$ | $\langle n_{01} \rangle$ | $\ell$ | $0$
|-----------------------------|-----------------|-----|-----
| $\frac{1}{3} \leq \ell \leq 1$ | $\frac{5}{6} - \frac{1}{2}\ell$ | $\frac{1}{3}$ | $\frac{1}{2}\ell - \frac{1}{6}$

Table 3.3: Mean occupation numbers of the LLL mean field ground state in spherical geometry, $N_v = 2$, with small ferromagnetic interaction $\gamma = -\epsilon$.

In the case $N_v = 2$, and small anti-ferromagnetic interactions, $\gamma = +\epsilon$, for $0 \leq \ell \leq \frac{1}{3}$ the mean occupation numbers per orbital of the condensate are the same as in the ferromagnetic case, but the spin structure is different. For $\ell \geq \frac{1}{3}$ the spin expectation values in the $m = 0$ and 2 orbitals become non-zero (without a discontinuity) and are not linear functions of $\ell$. Since $|\langle S_{m=1} \rangle|^2 = 0$ and the number density is constant in the $m = 1$ orbital, the spin state describing the bosons in this orbital can be arranged by an $SO(3)$ rotation to be $(0, 1, 0)/\sqrt{3}$. The vectors representing the bosons in the $m = 0, 2$ orbitals then are simply constructed. Together with the previously-mentioned vector they form a mutually orthogonal set which minimizes $H_n$. Provided that the spin-vectors are properly normalized, the energy can be expressed in terms of one parameter $\alpha(\ell)$, which is connected to the spin densities by $\cos(2\alpha(\ell)) = |\langle S_0 \rangle|/\langle n_0 \rangle = |\langle S_2 \rangle|/\langle n_2 \rangle$. Minimizing the energy with respect to $\alpha(\ell)$ gives

$$\alpha(\ell) = \arccos \left( \frac{1}{2} + \frac{1}{4}\sqrt{\frac{\lambda (\frac{2}{3} - \lambda)}{1 - 4\lambda + 6\lambda^2}} \right),$$

with $\lambda = \frac{1}{2}(\ell - \frac{1}{3})$. The maximum of the anti-ferromagnetic energy is not dependent on the angular momentum and lies at $\alpha = \pi/2$. In the ferromagnetic case this point minimizes the energy, corresponding exactly to the occupation numbers in Table 3.3.

At $\ell = 1$, there are solutions with uniform density, with an unbroken $SO(3)$ subgroup of the $SO(3)_{\text{orb}} \times SO(3)_{\text{spin}}$ symmetry, as the limiting case of the previous $\ell < 1$ states. This case is also discussed in the Appendix. There are also solutions in which the orbital distribution in the mean field configuration of the ground state is not unique. For instance, among the degenerate states at large $\gamma$ we find the polar vortex with $\langle n_{10} \rangle = 1$ and a configuration with $\langle n_{20} \rangle = \langle n_{00} \rangle = \frac{1}{2}$.
It is of interest to try to match the mean field states with ground states found in diagonalization studies, such as those shown in Figure 3.2 for the disc. Since these have definite values of the quantum numbers, they can be compared with the mean field states only by projecting the latter to components with definite quantum numbers [2]. For each \( N \) and \( L \), the value of the spin picked out should reflect the form of the interaction, and should presumably be the maximal value in the ferromagnetic regime, and the minimal value in the antiferromagnetic. For the spin-independent case \( e_2 = 0 \), the lowest \( SU(3)/_{\text{spin}} \) quantum numbers are favored as ground states.

We will not attempt to identify all the states in Fig. 3.2 in this way, but only some of the more prominent. We have already mentioned that the mean field state at \( \ell = 1 \) and \( e_2 = 0 \) corresponds to the BTC singlet state. Since the mean field state has equal mean occupation of \((m, n) = (0, 1), (1, 0), \) and \((2, 1)\), it does contain a unique singlet component which is exactly the BTC state. When \( e_2 \) is turned on, the quantum numbers remain at \((L, S) = (6, 0)\), but the state will be slightly altered in its details. The corresponding skyrmion spin textures on the sphere are also discussed in the Appendix.

The BDC multiplet at \( L = N/2 \) for \( e_2 = 0 \) that uses only \( m = 0, 1 \) also deserves comment. This corresponds in mean field theory to the \( N_v = 1 \) case discussed above and in the Appendix. When \( e_2 < 0 \), it becomes a half-skyrmion or meron, which survives for all \(-1 \leq \gamma \leq 0\). This meron has no projection to spin 0, and anyway for this regime maximal spin is expected in the ground state. Indeed, for \( N = 6 \) the corresponding \((L = 3)\) state has \( S = 3 \). The whole regime \( L/N \leq 1 \) for ferromagnetic interactions resembles what one expects for skyrmions, that is \( L \) (corresponding to \( N - L \) on the sphere) decreasing as \( S \) increases, as \( S = N - L \) [88]. For \( e_2 > 0 \) and \( L = N/2 \), the lowest-spin part of the BDC state becomes the ground state.

At larger positive \( \gamma \), there is a prominent region of \((L, S) = (6, 0)\) in the \( N = 6 \) data. At the largest \( \gamma \), we expect that this can be identified (in the same sense as the preceding discussion, or as in Ref. [35]) with the polar vortex state of this section. (In a finite size study, one would not expect to see a transition from the BTC state at \( e_2 = 0 \) to this polar vortex with the same quantum numbers at large \( \gamma \).) The jump from \( L = 0 \) to \( L = 6 \) expected from the mean field is seen in Fig. 3.2. At smaller \( \gamma \), a \((3, 0)\) region is seen. We speculate that this state corresponds to a single \( \pi \)-disclination at the center of the trap (with a second one at infinity, or the opposite pole on the sphere), and that the region corresponds to \( \ell'_a < \ell < 2/3 \) in the mean field results. Notice also the prominent semi-plateaus near \( \ell = 0.5 \) in the plots in Fig. 3.9 at larger \( \gamma \).
3.5 VORTEX AND SKYRMION LATTICES

Upon driving the system faster, multiple skyrmions are induced. These are expected to form a lattice and can be well treated in a mean field analysis. Such an analysis was performed by Kita et al. [45], who found a range of different lattices for $c_2 > 0$, depending on the relative strength to $c_0$ and the rotation. By including higher Landau levels, they were able to show that some of these lattices are qualitatively identical at high and low rotation. Near $c_2 = 0$, the (scalar) vortex breaks up into three vortices, one for each spin component, forming a triangular lattice. For $c_2 \geq 0.069c_0$, the vortex splits into two $\pi$-disclinations, which make up a square (anti-ferromagnetic) lattice.

We have carried out a program, similar to Mueller and Ho[57], appropriate for a mean field LLL description of a multi-component condensate. The LLL approximation (in the limit $\omega \to \omega_0$) fixes the vortex lattice spacing to be equal to the harmonic oscillator length. Note that this is different from the Thomas-Fermi regime, where the distance is fixed by the number of vortices, as the density of bosons is the same as in a non-rotating trap.

TORUS WAVEFUNCTIONS

The single-particle wavefunctions in a toroidal geometry are easily obtained, starting from the cylinder in the Landau gauge $A_y = x$:

$$\psi^\text{cylinder}_k(r) = \frac{1}{\sqrt{\pi L}} e^{-iky} e^{-(x-k)^2/2}, \quad k \in \frac{2\pi}{L} \mathbb{Z} \quad (3.24)$$

$$\psi^\text{torus}_i(r) = \sum_k c_{ik} \psi^\text{cylinder}_k(r) \quad (3.25)$$

where $L$ is the width in the $y$ direction, $L_1 = (0, L)$. Under a translation of $L_2 = (l_x, l_y)$ we demand invariance up to a gauge transformation $A = -yl_x$ (corresponds to $A'(r + L_2) = A(r)$):

$$\psi^\text{torus}_i(r + L_2) = e^{-il_x} \psi^\text{torus}_i(r) \quad (3.26)$$

$$\implies c_{ik} = e^{-ikl_y} c_{i(k-l_x)}, \quad l_x \in \frac{2\pi}{L} \mathbb{Z}. \quad (3.27)$$

This last expression is the flux quantization condition. Setting $l_x = 2\pi N_v/L$, we obtain for the total flux $\Phi = L_1 \times L_2 = 2\pi N_v$. There are $N_v$ independent solutions, $i = 0 \ldots N_v - 1$:

$$c_{ik} \propto e^{-il_yl_x k^2/2 - ikl_y}/2 \quad (3.28)$$

and $k = \frac{2\pi}{L}(i + nN_v)$, $n \in \mathbb{Z}$.

Note that in the above, we have assumed periodic boundary conditions. These can be generalized and solved as

$$\psi^\text{torus}_i(r + L_1) = e^{i\varphi} \psi^\text{torus}_i(r) \quad (3.29)$$
\[ \psi_i^{\text{torus}}(\mathbf{r} + \mathbf{L}_2) = e^{i\varphi_2} e^{-i\varphi_1} \psi_i^{\text{torus}}(\mathbf{r}) \]
\[ \Rightarrow \psi_i^{\text{torus}}(\mathbf{r}) = \psi_i^{\text{torus}}(\mathbf{r} + \varphi_2 \mathbf{L}_1 / 2\pi N_v - \varphi_1 \mathbf{L}_2 / 2\pi N_v) . \] (3.31)

**Multi-component condensates**

Under the assumption that the vortices in each spin component form a Bravais lattice, we can choose the one-particle wavefunctions to be the torus wavefunctions with \( N_v \) flux quanta (typically, \( N_v = 1 \) or \( N_v = 2 \)). For a scalar condensate, the lattice is completely specified by the geometry \( \tau \) of the torus.

In the case of multi-component condensates, however, more general boundary conditions are possible. We only need to demand

\[ \psi'(\mathbf{r} + \mathbf{L}_i) = e^{i\Lambda_i(\mathbf{r})} U_i \psi(\mathbf{r}), \] (3.32)

where \( \mathbf{L}_i \) \((i = 1, 2)\) define the geometry, and \( \Lambda_i \) is the gauge transformation mentioned above. The matrices \( U_1 \) and \( U_2 \) should commute, \( U_1 U_2 U_1^{-1} U_2^{-1} = 1 \), to obtain single-valued wavefunctions.

We require that the translations commute with the Hamiltonian, so that the energy of a unit cell is well-defined. For \( \gamma \neq 0 \), this implies \( U_i \in SO(3)_{\text{spin}} \). The common eigenvectors of \( U_1, U_2 \) then have eigenvalues \((1, e^{i\varphi_1}, e^{-i\varphi_1})\) and \((1, e^{i\varphi_2}, e^{-i\varphi_2})\).

With an overall \( SO(3) \) rotation, we can fix the direction of the vector with eigenvalue 1 to be parallel to \( \hat{z} \) in spin space. With this, the unit cell of the magnetic order (seen in the spin density which is gauge invariant, for example) is larger than that of the density, but always contains an integer number of the latter.

Using this approach, we can confirm a large part of the phase diagram of Kita et al [45], but we also find additional phases in the ground states at large \( \gamma \). These are polar phases, for which we use a unit cell with a single flux quantum. We will use \( \phi = \arctan \gamma \) as the parameter. The minimization procedure uses a simplex downhill algorithm in the geometry \( \tau \) and the phases \( \varphi_1, \varphi_2 \). The wavefunction is obtained from the polynomial free energy by using a conjugate gradient algorithm, starting from a random point. The wavefunction in general is unique up to a phase and a \( SO(3) \) rotation along the \( \hat{z} \)-axis.

The phases we obtain, as illustrated in figure 3.11, are as follows:

**Ferro lattice.** A major part of the ferromagnetic phase diagram \((-\pi/4 \leq \phi \leq -0.08)\) is covered by a lattice with \( N_v = 2 \) flux quanta in the unit cell. This is the same lattice as one obtains for the spin-\( \frac{1}{2} \) bosons with full \( SU(2) \) symmetry or, equivalently, the quantum Hall ferromagnet with the Landé factor \( g = 0 \) [84, 57]. If we consider the spin-1 to be composed of two spin-1/2 particles, then \( N_v = 2 \) for the spin-1 bosons corresponds to \( N_v = 1 \) for the spin-\( \frac{1}{2} \) particles. This structure is related to the \( N_v = 2 \) skyrmions discussed in the Appendix.

**Skyrmion-vortex lattice.** At \( \phi \approx -0.08 \), it becomes beneficial to include vortices ("merons"). The unit cell now has \( N_v = 3 \), with both a skyrmion and a
3.5. VORTEX AND SKYRMION LATTICES

Figure 3.11: The different lattices found in rotating spin-1 boson condensates. The first picture is the Hammer-Aitoff projection of the colors on the spin-sphere. Top and bottom correspond respectively to the north and south pole. The intensity codes $|\langle S \rangle|$, the size of the spin-vector. Other pictures are the spin expectations at different ratios $c_2/c_0$: $\phi = -0.1, -0.05, 0.01, 0.016, 0.04, 0.1, 0.54, 0.7$ and $0.9$. The last picture shows the density, as the spin vanishes.

- vortex. Based on direct computations in disc geometry (see below), we expect that this phase does not extend to $\phi = 0$, but that there are other phases in the weakly ferro regime $-0.02 < \phi < 0$.

- triangular vortex lattice. Exactly at $\phi = 0$, the nodes in the three components are arranged in a triangular lattice. This lattice can be realized with $N_v = 3$ and $\varphi_1 = \varphi_2 = 0$. The mean-field components $b_{m\mu}$ ($m = 0, 1, 2$), form a unitary ($U(3)$) matrix. This lattice is not shown in figure 3.11, as the $SO(3)$-spin is not well defined.

- square ladder. The triangular vortex lattice of the $c_2 \neq 0$ case is essentially unchanged up to $\phi = 0.0143$, being squeezed only. However, the $SU(3)$ symmetry is broken. This spin shows a ladder structure, where adjacent ladders are shifted by $3/2$ rung-spacings.

- canted ladder, $0.0143 \leq \phi \leq 0.0193$. The ladder structure stays intact, however the rungs are now canted.

- triangular ladder, $0.0193 \leq \phi \leq 0.069$.

- square $\pi$-disclination, at $\phi \approx 0.069$, there is a first order phase transition to the square $\pi$-disclination lattice. Only the $\uparrow$ and $\downarrow$ components are present in this lattice.

- squeezed $\pi$-discl., $0.428 \leq \phi \leq 0.62$. The lattice is squeezed in one direction and expanded in the other.
triangular \( \pi \)-discl., \( 0.62 \leq \phi \leq 0.786 \). At \( \phi \approx 0.62 \), there is a first order phase transition to a triangular \( \pi \)-disclination lattice.

polar Abrikosov, beyond \( \phi \approx 0.786 \), the \( \pi \)-disclinations are unstable and the systems prefers to have only one component, such that \( \langle \mathbf{S} \rangle = 0 \) everywhere. The vortices of this component form an Abrikosov lattice, with vanishing density at the cores.

The phases at \( \phi > 0.428 \), and at \( \phi < 0 \) have not been observed before. Figure 3.11 shows the spin texture in the various lattices, with colors coding the direction of the spin vector and the intensity marking its length, so that black regions indicate places where all components of the spin vector vanish. [For the lattice at \( \phi = 0.9 \), which is the polar Abrikosov lattice, the spin density vanishes and we plotted the particle density instead.] The particle density is finite in all lattices except the polar Abrikosov one.

To check whether the Ansatz is sufficiently general in the complete phase diagram, we have supplemented the above analysis by direct numerical computations of LLL mean field ground states in a disc geometry, with \( \omega < \omega_0 \). Since no periodic structure is imposed, the lattices form spontaneously. These computations show that the torus correctly reproduces the dominant phases such as the square lattice of \( \pi \)-disclinations and the skyrmion and skyrmion-vortex lattices. In the region \(-0.02 < \phi < 0 \) the two geometries showed different lattice structures, possibly due to finite size effects. We leave conclusive results in this region for future work.

At special values of \( \varphi_1, \varphi_2 \), when they are both of the form \( p \pi / q \) (\( p, q \) integer), it is possible to realize the lattice by using a larger unit cell and identical phases for all three spin components. An example of this is the triangular lattice at \( c_2 = 0 \), where \( \varphi_1 = -\varphi_2 = 2\pi/3 \). In this case, we can realize the same lattice by using a torus with 3 flux quanta and \( \varphi_1 = \varphi_2 = 0 \). The other example is the square \( \pi \)-disclination lattice, which can be described by using 2 flux quanta. This can be compared to the spin-\( \frac{1}{2} \) situation\[57\], where the lattice at \( g_{12} = g_1 = g_2 \) (unbroken \( SU(2) \) symmetry) can equivalently be described by using a torus with 2 flux quanta\[45, 83\].
3.6 DISCUSSION

In this chapter we have studied the phase diagram of spin-1 bosons in a rotating trap, within the LLL approximation, using a variety of techniques (numerical diagonalization, mean field theory, and analytical constructions). We concentrated on certain regimes. These were

- low rotation, such that the angular momentum $L$ is less than or equal to the particle number $N$, where the system is beginning to contain some vorticity.
- higher rotation, where the bulk of the fluid accommodates vorticity and is occupied by a lattice of (possibly coreless) vortices, which we considered as infinite periodic structures.

There is a rich variety of phases as the interaction parameters, especially the ratio of the coefficients of spin-dependent and spin-independent interaction terms are varied. The results obtained here, especially those at lower rotations which should be more easily accessible, should motivate further experiments to rotate spin-1 bosons with unbroken spin-rotation symmetry.

Non-rotating spin-full $^{87}$Rb condensates are currently controlled with great precision. This has allowed a detailed study of the $(F = 1$ and $F = 2)$ spinor dynamics [78, 18].

In a recent preprint [58], Mueller discusses the situation with an angular momentum per particle up to $\approx 8$, with $\gamma = \pm 0.05$. In a detailed analysis of the symmetries of the configurations with a few textures present, the formation of the lattice is illustrated beautifully.

The regime where quantum Hall conditions are reached will be considered in the next chapter. Loosely speaking, this happens when the average number of bosons in an orbital becomes of order 1, where fluctuations will become so strong that the condensate is quantum disordered.
CHAPTER 3. SLOWLY ROTATING SPIN-1 CONDENSATES