Quantum Hall spin liquids
van Lankvelt, F.J.M.

Citation for published version (APA):
Chapter 4

Quantum Hall Liquids of Spin-1 Bosons

In this chapter, we will propose several (non-abelian) quantum Hall spin liquids of spin-1 bosons. These liquids are expected to arise for dilute systems of spin-1 atoms under high rotation. The parameter which tunes the transition between different quantum Hall states and which drives the quantum disordering of the spin-texture lattices is $\nu = N/N_\nu$, the filling factor. In current experiments, $\nu$ is still several orders of magnitude greater than 1, so these experiments are deep within the mean-field regime. In the vigorous experimental pursuit [79, 15] of the quantum liquid regime, this situation is likely to change in the near future. Meanwhile, the theoretical question as to what happens when this extraordinary simple system is subjected to strong quantum fluctuations is interesting in itself.

In the scalar case, the critical $\nu$ was estimated to be $\nu_c \approx 10$ by the Lindemann criterion [20, 85] (that is, the average fluctuation in the position of the vortices equals the separation between them). Explicit calculation of small systems have confirmed this transition and found $\nu_c \approx 6-10$. A similar transition will occur in the spin-1 case, although we have not calculated the appropriate $\nu_c$. We expect this to be of the same order of magnitude as in the scalar case.

In the extreme limit $\omega \rightarrow \omega_0$, we can analyze the quantum liquids analytically in a part of the phase diagram, as we can explicitly find the zero-energy eigenstates of the Hamiltonian. Two of the series we propose, the $SU(4)_k$ and the $SO(5)_k$ series, have a member of this form for $k = 1$. The third series consists of a generalization of a family of fractional quantum Hall states, the hierarchy/composite fermion states, to spin-1 particles. We present some numerical results on small sizes, which unfortunately are probably not conclusive for the nature of the states, due to the restriction to insufficiently large sizes.

We first briefly review the scalar boson quantum Hall regime and discuss the quantum liquids which have been identified.

\footnote{this chapter is based on a collaboration with J.W. Reijnders, K. Schoutens and N. Read [70, 71]}
4.1 SCALAR BOSON LIQUIDS

In one of the first applications of the quantum Hall-CFT connection, by Read and Rezayi, a series of non-abelian quantum Hall states (called RR states) was found. The construction of Moore and Read is generalized by using the affine Lie algebra $SU(2)_k$. This algebra naturally incorporates a $U(1)$ symmetry, as the generators can be written as

\begin{align*}
J^+(z) &= \psi_1 e^{i\sqrt{2/k} \varphi(z)} \\
J^-(z) &= \psi_{k-1} e^{-i\sqrt{2/k} \varphi(z)} \\
J^z(z) &= \sqrt{k/20} \varphi(z)
\end{align*}

where $\varphi$ is a scalar boson field, compactified with radius $\sqrt{k/2}$, and $\psi_l$ ($l = 0, \ldots, k-1$) are $\mathbb{Z}_k$ parafermions. The filling factors for the Read-Rezayi quantum Hall states follow directly from the vertex operator, they are $\nu = k/2$. The parafermions have conformal dimension $(l/\nu - 1)/k$ so that the operators $J^0$ have dimension 1 and are in fact the $SU(2)$ currents. The operator product expansion of the parafermions is

\begin{equation}
\psi_l(z)\psi_{l'}(w) = \left\{ \begin{array}{ll}
(z-w)^{-2l'/k} \psi_{l+l'}(w) + \cdots & l + l' < k \\
(z-w)^{-2(l-l')(k-l')/k} \psi_{l+l'-k}(w) + \cdots & l + l' \geq k
\end{array} \right.
\end{equation}

The parafermion $\psi_0$ is identified with the identity operator.

The OPE shows that if we insert $k$ $J^+$ operators at the same point in a correlation function, the parafermions will fuse to the identity, and the correlator will be non-zero. Approaching with the $k + 1$-th operator, however, kills the correlator. In quantum Hall language, this means that the wavefunction vanishes whenever $k+1$ particles are on the same position. This statement allows us to give a Hamiltonian which exactly reproduces the CFT correlator as a zero-energy eigenstate. It consists of a $k + 1$-body contact interaction:

\begin{equation}
H^k_{SU(2)} = V \sum_{i_1 < \cdots < i_{k+1}} \delta(z_{i_1} - z_{i_2}) \delta(z_{i_2} - z_{i_3}) \cdots \delta(z_{i_k} - z_{i_{k+1}}).
\end{equation}

A particularly elegant representation of the wavefunction was found by Cappelli et al.\cite{17} by using a different coset construction of the $\mathbb{Z}_k$ parafermions:

\begin{equation}
\tilde{\psi}^k_{SU(2)}(\{z_i\}) = S \left[ \prod_{i<j \in \Lambda_1} (z_i - z_j)^2 \prod_{i<j \in \Lambda_2} (z_i - z_j)^2 \cdots \prod_{i<j \in \Lambda_k} (z_i - z_j)^2 \right]
\end{equation}

where $\{\Lambda_i\}$ is a distribution of $N = kn$ particles into $k$ groups of $n$ particles. The symmetrization $S$ sums over all possible distributions. When $k+1$ particles are put in the same position, (at least) two of them are in the same group and so the wavefunction vanishes. It is amusing to note that we can obtain (topological) information, such as torus degeneracy and the Haldane momenta of the degenerate
ground states, by considering a torus with $N_v = 2$. When $k$ particles are present, the Hamiltonian vanishes and every state is a ground state.

These Read-Rezayi (RR) states are incompressible quantum Hall fluids. The special case $k = 1$ is the Laughlin state [48] for bosons at $\nu = 1/2$, while the $k = 2$ state is the Moore-Read "Pfaffian" state [55] (see also chapter 2). The RR quantum Hall states possess a specific "order-$k$" clustering property: there is a so-called composite-boson order parameter [26, 65], an operator that creates $k$ bosons and 2 vortices, and is the minimal order parameter (the one that creates the smallest number of bosons) that has long-range ("off-diagonal") order. Such order implies that the quasiparticles over these liquids are (quantized) vortices in the liquid, carrying fractional vorticity $1/k$, analogous to a fractional magnetic flux $\Phi/\Phi_0$ that occurs in certain quantum Hall states in electronic systems. In quantum Hall liquids, the Hall conductivity implies a fundamental quasiparticle charge $q = \pm k\hbar/e$ in units of the charge of the electron. In the context of neutral bosons in a rotating trap, the same argument implies that a fractional particle number ($q = \pm 1/2$ for the RR states) is present in the quasiparticle, relative to the background density. Furthermore, for $k > 1$ there are non-local degrees of freedom associated with these quasiparticles, that is, the ground states with more than three quasiparticles are degenerate in the limit where all the separations go to infinity. As these degrees of freedom are of a non-local, topological nature, they do not couple to local probes and the degeneracy is protected in the large separation limit. For the case $k = 2$ there is an interpretation in terms of a Majorana fermion in each vortex core [66], more details can be found in chapter 6.

The statistics of quasiparticles in 2D can be defined in terms of adiabatically dragging them along paths, keeping them well-separated, to exchange them. For the RR states, in the case $k = 1$, the statistics are "abelian"; the wavefunction acquires a phase factor when two particles are exchanged, just as for the Laughlin states. For $k > 1$, however, the statistics becomes "non-abelian" [55, 68]. This means that there is a degeneracy, when the positions of the quasiparticles are fixed (in general, this is true only when the quasiparticles are well-separated, though for the $k + 1$-body Hamiltonian and for quasiholes it is exact for any separation). In terms of the trial functions (4.9) and their generalization to quasiholes, this degeneracy is caused by the symmetrization procedure $S$, which destroys the "group" quantum number of the (quasi-) particles [17]. The general framework to obtain these degeneracies from CFT, introduced in chapter 2, has been worked out and agrees with numerical results for the Moore-Read, RR, and other states [55, 67, 28, 5, 6]. We expect the same framework to apply here. When the quasiparticles are exchanged adiabatically, the effect is a matrix operation within these spaces of degenerate states, described by the braiding matrices of the corresponding CFT [55]—hence the term non-abelian statistics.

VORTEX LATTICE MELTING

An extensive study of the quantum melting of the vortex lattice was conducted by Cooper, Wilkin, and Gunn [20], who performed exact diagonalization studies
in the LLL in a toroidal geometry. They predict that the vortex lattice melts into a quantum disordered phase at a critical value $\nu_c \approx 6-10$ of the filling factor $\nu = N/N_v$, with $N_v$ the number of vortices and $N$ the number of bosons. They also found that the quantum state is incompressible at $\nu = k/2 < \nu_c$, with $k = 0, 1, 2, \ldots$, and observed that the ground states at $\nu = k/2$ have substantial overlaps with the Read-Rezayi quantum Hall states.

Figure 4.1 shows the particle-hole excitation gap

$$\Delta(N) = N \left( \frac{E(N-1)}{N-1} + \frac{E(N+1)}{N+1} - 2 \frac{E(N)}{N} \right)^2, \tag{4.7}$$

for a rectangular geometry. The collapse of this gap at $\nu \approx 6$ indicates the transition to the BEC regime. Further support to the vortex lattice interpretation beyond $\nu \approx 6$ is given by the collapse of excitation energies at reciprocal lattice vectors of the vortex lattice [20].

More evidence for the appearance of the Moore-Read state was recently provided by Regnault and Jolicoeur [69], who observed the low-lying two-particle branch in numerical simulations, upon adding one flux quantum in a spherical geometry. They also found evidence for other quantum Hall states not in the RR series.
4.2 \textit{SU}(4)_k\textit{ Series}

We now turn to the rapidly rotating spin-1 bosons, which we analyze using the LLL model, introduced in chapter 3, eq. (3.6). We put $\omega \to \omega_0$ and are thus left with the interaction Hamiltonian eq. (3.2). It is straightforward to construct zero-energy states at $c_2 = 0$. The repulsive contact interaction dictates that the wavefunction should have a node whenever 2 particles are on the same place. Furthermore, two bosons with the same spin should have a double zero in order to maintain a symmetric wavefunction. In terms of the components of the wavefunction for given values $\mu = x, y, z$ of the spin for each particle, we can write such a state down

\begin{equation}
\Psi^{(2,2,2,1,1)}(z_1^x, \ldots, z_{N_x}^x; z_1^y, \ldots, z_{N_y}^y; z_1^z, \ldots, z_{N_z}^z) = \prod_{\mu=x,y,z} \prod_{i<j} (z_i^\mu - z_j^\mu)^2 \prod_{\mu'<\mu''} \prod_{i,j} (z_i^{\mu'} - z_j^{\mu''}),
\end{equation}

where $n_{\alpha}$ denotes the number of particles with spin $\alpha$. The lowest angular momentum $L$ for which this state can be realized is when $N_x = N_y = N_z = N/3$. Notice that $L = N(2N/3 - 1)$, and the filling factor, which can be defined as $\nu = \lim_{N \to \infty} N^2/(2L)$, is $\nu = 3/4$. This state is a straightforward generalization of both the Laughlin $\nu = 1/2$ state for spinless particles, and the (2, 2, 1) Halperin spin-singlet state for spin-1/2 bosons. The full state for $N_x = N_y = N_z = N/3$ is an $SU(3)_{\text{spin}}$ singlet ($(p,q) = (0,0)$), and arguments similar to Laughlin’s plasma mapping show that moreover it has short range spin correlations. Because the state vanishes whenever the particles are at the same point, this state is a zero-energy eigenstate for all values of $c_0$, $c_2$. However, it will only be the ground state (at $L = N(2N/3 - 1)$) when $H_{\text{int}}$ is positive, that is in the region $g_0, g_2 > 0$, within region II. For larger values of $L$, there are many more zero-energy eigenstates, so the ground states are degenerate in the window $g_0, g_2 > 0$. On minimizing $H_\omega$ with respect to $L$, this implies that the lowest possible filling factor as $\omega \to \omega_0$ from below is $\nu = 3/4$ in this regime (within the model in Sec. 3.1).

The state in eq. (4.8) is an exact eigenstate, but in general we are not able to find the exact highly-correlated ground states of $H_{\text{int}}$. Instead, we seek to understand numerical results, and make predictions for the physics at larger sizes by using (among other techniques) trial wavefunctions. These states, which are not generally exact for any known two-body interaction, serve as paradigms for the phases of matter in the thermodynamic limit, as they possess interesting (“universal”) properties such as the quantum numbers and statistics of their excitations that are robust against small changes in the Hamiltonian, until some phase boundary is passed (this philosophy has been discussed e.g. in Ref. [68]). One way to produce trial wavefunctions, that is closely connected to their universal properties, is to use conformal field theory (CFT). We will show how to obtain wavefunctions from a CFT in somewhat more detail in the next section. The CFT describing eq. (4.8) is $SU(4)_1$, so following a strategy in Read-Rezayi [68], and motivated by the analogous results for scalar bosons [20], we can consider a series, $SU(4)_k$, where $k = 1, 2, \ldots$. 

These states have filling factor $\geq 3/4$, and are explicitly $SU(3)_{\text{spin}}$ invariant. Hence we expect them to be relevant near $c_2 = 0$.

The trial states we consider have wavefunctions, in spin components (generalizing those in [17, 76] to spin-1).

$$\tilde{\Psi}_{SU(4)}^k(\{z_i\}) = S_{\text{groups}} \left[ \prod_{\text{groups}} \tilde{\Psi}^{(2,2,1,1,1)} \right] .$$

In this construction the $N = 3pk$ bosons are first partitioned into $k$ groups, each with $p$ particles of each spin component $x, y, z$.

For each group we write a Halperin $\tilde{\Psi}^{(2,2,1,1,1)}$ factor, and these are multiplied together. Finally, the symmetrization operation $S$ over all ways of dividing the particles into groups is applied. The angular momentum is $L = N[2N/(3k) - 1]$, and the filling factor (as $N \to \infty$ at fixed $k$) is therefore $\nu = 3k/4$. It happens that if we put $k = N/3$, the state contains $k$ groups of three bosons each, and the state is exactly the $L = N$ BTC. However, we do not believe this is particularly significant, as the BTC state is the unique $SU(3)_{\text{spin}}$ singlet state at $L = N$.

The states (4.9) are zero-energy eigenstates of a Hamiltonian consisting of a $k + 1$-body interaction:

$$H_{SU(4)}^k = V \sum_{i_1 < \cdots < i_{k+1}} \delta(z_{i_1} - z_{i_2}) \cdots \delta(z_{i_k} - z_{i_{k+1}}) .$$

The interaction is positive for $V > 0$, so all eigenstates have $E \geq 0$. This interaction penalizes $k + 1$ particles at the same point. Therefore, zero-energy eigenstates are those in which the wavefunction vanishes if any $k + 1$ coordinates coincide, regardless of the spins. One can see that the above function has this property, even before the symmetrization over groups, as for any $k + 1$ particles, at least two must be in the same group, forcing the function to vanish. For less than $k + 1$ particles at the same point, it does not necessarily vanish. In fact, for each $k$, (4.9) is the unique zero-energy eigenstate of $H_{SU(4)}^k$ with lowest angular momentum $L$.

For the same Hamiltonian on a torus, there are again zero energy states, at least for $N$ divisible by $3k$. For these cases, the degeneracy of these $SU(4)_k$ ground states is

$$\#_k = \frac{1}{6}(k + 1)(k + 2)(k + 3) .$$

We have verified that this result, which can be inferred from the CFT connection, is reproduced by exact diagonalization of the Hamiltonian eq. (4.10) on the torus.

Like other incompressible quantum Hall states, the phases of matter exemplified by the trial states (4.9) possess point-like quasiparticle excitations which may have fractional particle number (relative to the background density) and/or spin. The particle number associated with the elementary quasiparticles can be found once it is understood that, similar to the RR states [68], the $SU(4)_k$ states are clustered states, in which particles occur in clusters of $3k$ (in an $SU(3)_{\text{spin}}$-singlet). Then a similar argument to that given in section 4.1 shows that they carry charge $q = \pm 1/4$. 
They also have spin 1. The quasiholes, at which there is a deficiency of particle number, can be studied as zero-energy eigenstates of $H_{SU(4)}^{k}$, and fairly explicit trial wavefunctions can be found using the relation with CFT. For $k = 1$, the statistics of the quasi-particles are abelian, like in the case of the Laughlin and Halperin states. With $k > 1$, however, the parafermion fields in the conformal field theory are non-trivial and there are multiple fusion paths for the spin-fields when more than 3 quasi-particles are present. There will be a degeneracy due to this and braiding will induce rotations in the space of ground states. The $SU(4)_k$ states are therefore non-abelian for $k > 1$.

4.3 $SO(5)_k$ Series

Similar to the $SU(4)_k$ case, the $SO(5)_k$ states can be written in the form:

$$\tilde{\Psi}^{k}_{SO(5)}(\{z_i\}) = S_{\text{groups}} \left[ P_{\text{groups}} \tilde{\Psi}^{k-1}_{SO(5)} \right],$$  \hspace{1cm} (4.12)

where now

$$\tilde{\Psi}^{k-1}_{SO(5)}(\{z_i\}) = \text{Pf} \left( \frac{\zeta_i \cdot \zeta_j}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j).$$  \hspace{1cm} (4.13)

Here the spin states for all the particles are included explicitly (the product of spin states $\zeta_i$ being the tensor product) $\text{Pf} M_{ij}$ denotes the Pfaffian of an antisymmetric matrix $M_{ij}$. In the present case the $N = 2kp$ particles are partitioned into $k$ groups, with $2p$ particles in each. The particles in each group form an $SO(3)^{\text{spin}}$ singlet. The product over these groups is then symmetrized. The state as a whole is clearly an $SO(3)^{\text{spin}}$ singlet, and has angular momentum $L = N\lfloor N/(2k) \rfloor - 1$, so the filling factor is $\nu = k$.

The $k = 1$ case, which closely resembles the Moore-Read paired state [55] but for spin-1 particles, is the exact ground state of our two-body Hamiltonian $H_{\text{int}}$ when $g_0 = 0$. That is because, in the state $\tilde{\Psi}_{SO(5)}^{1}$, two particles are found at the same point only if they have total spin 0. Again, the ground state as above is the unique zero-energy eigenstate at the stated angular momentum, but at larger $L$ there are many more zero-energy states. So $\nu = 1$ is the lowest filling factor possible at $g_0 = 0$. This implies that in finite size on the disc a boundary between ground states with the $L$ values of the $SU(4)_1$ and $SO(5)_1$ states must run into $\omega = \omega_0$ at $g_0 = 0$ (this is for $N$ divisible by 6, but there will be similar statements for other values). Such behavior is seen in Fig. 3.2. For $g_0 < 0 (\gamma > 1/2)$, we do not know the exact lowest $\nu$ that occurs as $\omega \rightarrow \omega_0$ from below. The $SO(5)_1$ state can be interpreted in terms of BCS spin-singlet complex-p-wave pairing of composite fermions, in which the Pfaffian represents the pairing in position space [55, 66].

More generally, for each $k$ there is a Hamiltonian for which the $SO(5)_k$ states are exact zero-energy eigenstates, again given by a $k + 1$-body interaction:

$$H_{SO(5)}^{k} = V \sum_{i_1 < \cdots < i_{k+1}} \delta(z_{i_1} - z_{i_2}) \cdots \delta(z_{i_{k}} - z_{i_{k+1}}) P_{k+1}(i_1, \ldots, i_{k+1}).$$  \hspace{1cm} (4.14)
CHAPTER 4. QUANTUM HALL LIQUIDS OF SPIN-1 BOSONS

This interaction includes a projector $P_{k+1}(i_1, \ldots, i_{k+1})$ of the spin state of the $k+1$ particles concerned onto total spin $k+1$.

For general $k$, these states can be considered to be built up out of clusters of $2k$ particles in a spin singlet. From this fact we can obtain the fractional particle number of the elementary quasiparticles, $q = \pm 1/2$. Also, the quasiparticle spin is $1/2$, which is fractionalized compared with the spin 1 of the underlying bosons, and so the number of these quasiparticles must be even. For $k = 1$, there are also excitations with zero particle number that behave as fermions with spin 1. In this case, the universal properties may be understood by a simple extension of the methods of Ref. [66] to this case.

Because it is difficult to see through the symmetrization operation $S$, we will provide some details on the conformal field theory behind these states. Such a CFT description allows us to obtain more insight into the topological properties, such as degeneracies and braiding. For example, to obtain the degeneracy of ground states on the torus, CFT tells us that we only need to know the number of non-trivial representations at level $k$. In the case at hand, this number turns out to be $\frac{1}{2}(k+1)(k+2)$. Again, we verified this number using exact diagonalization of the $(k+1)$-body interaction.

The chiral algebra of the CFT which describes these states is based on the $SO(5)_k$ affine Kac-Moody algebra. $SO(5)$ is a rank 2 Lie algebra, which contains mutually commuting $SO(3)$ and $U(1)$ Lie subalgebras, which we can identify with the symmetries under $SO(3)_{\text{spin}}$ and number conservation. In these subalgebras, the generators are respectively $S^+$, $S^-$, $S_z$ and $c$ respectively. According to the CFT-qH correspondence[55], we can also obtain the quantum Hall state wavefunctions as correlators in the chiral part of a CFT, in which the particles (bosons) should be represented by fields that have abelian braiding properties. In the present case (and the $SU(4)_k$ case is similar), the bosons correspond simply to a different triplet of
current operators of the $SO(5)_k$ affine Kac-Moody algebra. Thus the wavefunction, now in spin components, can be written

$$
\Psi_{SO(5)}^k(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{N/k} e^{-iN\varphi_c/\sqrt{k}(z_\infty)} J_{\alpha_1}(z_1) \cdots J_{\alpha_N}(z_N)
$$

with $J_{\alpha}$ ($\alpha = |, 0, \uparrow$) an $SO(3)_{\text{spin}}$ triplet of currents in the affine Lie algebra, that carry $U(1)$ charge +1. The currents are shown in the $SO(5)$ weight lattice (fig. 4.2), as $\Psi_\uparrow$, $\Psi_0$, $\Psi_\downarrow$. The operator whose position tends to $\infty$ represents a background charge, such that the total $U(1)$ charge of the operators is zero (as it must be in order that the correlator be nonzero). The currents can be expressed as

$$
J_{\alpha}(z) = \psi_{\alpha}(z)e^{i\tilde{\beta}_\alpha \cdot \vec{z}/\sqrt{k}(z)},
$$

where $\psi_{\alpha}$ is a parafermion field, of conformal weight $1 - 1/k$ for the long roots and $1 - 1/2k$ for the short roots. The vertex operator $e^{i\tilde{\beta}_\alpha \cdot \vec{z}/\sqrt{k}(z)}$ contains the two free boson fields $\varphi_c$ (charge) and $\varphi_s$ (spin), with $\tilde{\beta}$ the position in the root lattice. For $J_{\alpha}$ these are $\tilde{\beta}_\uparrow = (1, 1)$, $\tilde{\beta}_0 = (1, 0)$, $\tilde{\beta}_\downarrow = (1, -1)$.

The parafermions simplify when we specialize to the case $k = 1$, where they reduce to the identity operator for the long roots ($\Psi_\uparrow$ and $\Psi_\downarrow$, in particular) and a Majorana fermion for the short roots ($\Psi_0$). The correlator can then be readily written down, as the correlation functions for Majorana fermions is well known:

$$
\langle \psi(z_1) \cdots \psi(z_n) \rangle = \text{Pf} \left( \frac{1}{z_i - z_j} \right).
$$

This reproduces the $SO(5)_1$ wavefunction, in the same way as for the spinless Moore-Read state [55].

We note that the same $SO(5)_k$ algebra will be used in another construction in the next chapter, in a system of spin-$\frac{1}{2}$ particles. The present case differs in that the physical $SO(3)_{\text{spin}}$ symmetry is embedded differently in $SO(5)$, because of the different spin of the underlying particles.

Wavefunctions for zero-energy states containing quasiholes can also be written down as chiral correlators, which now contain vertex operators for primary fields of the $SO(5)_k$ algebra that represent the quasiholes. For the $k = 1$ case, these contain (in the scalar field plus Majorana fermion language) a spin field for the Majorana fermion, and give rise to quasihole wavefunctions analogous to those for the Moore-Read state [55, 67].

### 4.4 Other Spin-1 Quantum Hall Liquids

#### Composite Fermions

Alternative quantum Hall states to the rather exotic series in the previous two sections can be constructed by applying conventional methods to spin-1 bosons.
One such approach, as in the case of scalar bosons, is to map the bosons onto composite fermions, by attaching (say) one vortex to each boson. These fermions see a reduced effective magnetic field, and one can construct an incompressible state when an integer number of Landau levels in the effective magnetic field is filled with all three components. This construction gives states with filling factors \( \nu = \frac{3p}{3p \pm 1} \), which are \( SU(3)_{\text{spin}} \) singlets.

One can interpret the Moore-Read state as the \( p \)-wave pairing of composite fermions \[55\]. In this case, \( p \)-wave \( SO(3) \)-singlet pairing is possible (in contrast to the spin-1/2 case) and indeed, we have seen that the \( SO(5)_1 \) state can be interpreted this way. In the \( SU(3) \) symmetric case at \( \nu = 1 \), no 2-particle \( SU(3) \)-singlet pairing is possible and there are two options for the system. One is to form a Fermi liquid, the other is to spontaneously break the symmetry and form \((p, q) = (2, 0)\) pairs. Note that this last possibility includes the \( SO(3) \) singlet and thus can be continuously connected to the \( SO(5)_1 \) state.

**Vortex Lattices without Polar Order, and Nematic Quantum Hall Liquids**

The earlier discussion of quantum Hall liquid states focussed on singlets under spin rotations, with short range spin correlations. It is interesting to wonder also if quantum Hall liquids with some form of spin ordering could occur. One possibility would be a ferromagnetic quantum Hall liquid. Such states can be easily written down, by using any wavefunction for a quantum Hall state of spinless bosons, with all the boson spins in the \( \sigma = \uparrow \) state (or a global spin rotation of this). One might expect these to occur in the ferromagnetic \( (c_2 < 0) \) part of region II, but in fact we see no sign of them: leaving aside the skyrmion textures in the BEC at low \( L \), at larger \( L \) all ground states are spin singlets. We note that for spin-1/2 electrons, spin-polarized states can occur, e.g. at \( \nu = 1 \), even for spin-independent interaction, due to exchange effects. However, the exchange effects are presumably different for bosons.

A more feasible-looking possibility is quantum Hall states with polar spin order, perhaps in the antiferromagnetic region \( c_2 > 0 \). In the regime at large \( \gamma \) where mean-field theory predicts the Abrikosov vortex lattice, the spin-order is polar. In the polar state, the vector condensate can be written as \( \langle \psi_{\nu, \mu} \rangle = e^{i\varphi} n_{\nu, \mu} \), with \( \varphi \) the phase and \( \hat{n} \) a real vector. In the Abrikosov lattice, the magnitude of the vector \( \hat{n} \) and the phase \( \varphi \) vary to give a triangular lattice of vortices. We can now imagine that quantum fluctuations destroy either part of the order (restoring either the phase or the spin-rotation symmetry) without the other. When the \( U(1) \) and translational symmetry that are broken in the vortex lattice are restored, the ground state is a quantum Hall fluid. For large quantum fluctuations one might expect that the quantum Hall liquid has restored \( SU(2)_{\text{spin}} \) symmetry. However, the two transitions at which these symmetries are restored are independent, and the transitions could in principle occur in either sequence as we go to smaller \( \nu \).

The intermediate phase in which spin symmetry is restored but not the phase would be a vortex lattice in a boson paired state, a BEC of boson pairs. This would
be characterized by having a nonzero expectation value of \( \sum_{\mu} \langle \psi_{\mu}(z) \rangle \), which is
invariant under \( \psi \rightarrow -\psi \). (Such a vortex lattice would also be possible with vortices
containing a half-unit of vorticity each, instead of integers as we assume otherwise,
and this might be reached by restoring symmetry in the \( \pi \)-disclination lattice state.)

The other possible sequence of transitions would be that in which the intermediate
phase is a quantum Hall liquid with restored translational and phase symmetry,
but still has the polar order, which breaks SU(2)_spin. The single-boson expectation
\( \langle \psi_{\mu}(z) \rangle \) would be zero, but if we look at the composite operator \( \psi_{\mu}^\dagger(z) \psi_{\mu}^\r(z) \),
this can have an expectation value. This matrix has a trace equal to the density, which
is uniform by assumption. The traceless Hermitian matrix obtained by subtracting
off the trace contains antisymmetric and symmetric parts. The (imaginary) antisymmetric part corresponds to a spin-1 irreducible tensor that is simply the spin
density, which is assumed to be zero here. The (real) symmetric part corresponds
to a spin-2 irreducible tensor. This is the order parameter of a polar or nematic
state, which represents a vector \( \hat{n} \), but is invariant under \( \hat{n} \rightarrow -\hat{n} \), so it parametrizes
\( S^2/Z_2 = \mathbb{R}P^2 \). Trial wavefunctions for these nematic quantum Hall states can be
written down as those for scalar bosons, times a spin state such as \( \alpha = 0 \) for all
bosons, or as a spin-rotation of this.

It would not be surprising if such nematic quantum Hall liquids occurred in
the phase diagram at large \( \gamma \), now that the corresponding (polar Abrikosov) vortex
lattices are known to be present. In finite size, the ground state would always be
low spin \( S = 0 \) or 1, and there need be no transition separating it from a state
at the same \( L \), \( S \) with short-range correlations, such as the \( SO(5) \), state at \( g_0 = 0 \).
Thus the appearance of such nematic order in a quantum Hall fluid of spin-1 bosons
in the thermodynamic limit cannot yet be ruled out.

4.5 Numerical Results

To examine how well the proposed states describe the true ground states, we have performed exact diagonalization of small systems. In the regime \( L \leq N \), we have
used both the disc and sphere geometries. As we have seen, these results differ
somewhat. But when looking at fast rotation, however, where the filling factor is of
order 1, the system is spread out into a pancake. It makes sense to focus attention
on the interior of the disc and avoid edge effects. This can be done by using an
edgeless geometry such as the sphere or torus. Here we will be interested in the
ground states in which (unlike the work in chapter 3) we find the ground states
without constraining \( L \). Quantum Hall liquid ground states will usually then show
up as \( \tilde{L} = 0 \) states. At finite sizes on the sphere, such ground states that form
a sequence of sizes tending to a particular filling factor \( \nu \) in the thermodynamic
limit lie on a sequence of the form \( N_\nu = N/\nu - S \) [29]. Here \( S \) is known as the
shift, and its appearance is connected with the coupling of the particles to the
curvature of the sphere. The value of \( S \) depends on the liquid state, not only on \( \nu \).
\( N_\nu \) can be obtained from the angular momentum on the disc as \( L = N N_\nu / 2 \) (all
states have \( \tilde{L} = 0 \)). That is, when the states in the \( \tilde{\Psi} \) notation are written for the
sphere, one can take $N_v$ as small as possible, so that $N_v$ equals the highest power of any $z_i$ appearing in the wavefunction (see Sec. 3.1). For quantum Hall ground states, this will usually mean that $\tilde{L} = 0$. For the $SU(4)_k$ ground states, we have $N_v = 4N/(3k) - 2$, while for the $SO(5)_k$ ground states $N_v = N/k - 2$. We will compare the results of numerical solution for the ground states with these series of trial states.

The $SU(4)_k$ states with $k = N/3$ are the exact ground states for $c_2 = 0$ at $N_v = 2$ on the sphere. It is not surprising that they are eigenstates, because they are the only $SU(3)$ singlet states, as in the case of $L = N$ for the disc.

As a further test for the $SU(4)_k$ states, we have looked at sizes $(N, N_v)$ at which such a ground state could lie for $k > 1$. Since $N$ must be divisible by $3k$, such sizes increase rapidly even for $k = 2$. The next case after the trivial BTC for $k = 2$ is $N = 12, N_v = 6$. Here it turns out that the overlap-squared of the exact ground state for $c_2 = 0$ with the trial state is $\langle SU(4)_2 | G.S. \rangle^2 = 0.915226$.

The $SO(5)_1$ state was shown to be an exact ground state for $c_0 > 0, g_0 = 0$. The higher members of this series, however, had a vanishing overlap with the ground states throughout the phase diagram.

As a further test of our proposed wavefunctions, we have performed calculations for torus geometries. On the torus, one simply has $\nu = N/N_v$ for the finite size sequence of ground states that tend to a fluid of filling factor $\nu$ in the thermodynamic limit. We saw in the mean field analysis of the skyrmion lattice, that the lattice can only be observed when the number of flux quanta is a multiple of three. However, at low filling factors, we expect to see quantum Hall states at $\nu = 3k/4$. To be able to observe these, $N_v$ has to be a multiple of 4. Unfortunately, this implies torus sizes which are too large to observe both the quantum liquids and the skyrmion lattice.

To see if the proposed wavefunctions are good candidates, we are therefore forced to look at tori which frustrate the mean field skyrmion lattice. The cases we considered are $N_v = 3, 4, 6$. For $N_v = 4$, we find that the ground states are exactly given by the $SU(4)_k$ series. However, as for the BTC states on the sphere, this is due to the fact that the trial ground states, which are $SU(3)_{\text{spin}}$ singlet states, span the space of all $SU(3)_{\text{spin}}$ singlets on the torus, which has dimension equal to eq. (4.11), the degeneracy of $SU(4)_k$ torus ground states. Clearly this must be independent of the geometry of the torus (described by $\tau$), and we verified this in some cases.

In fig. 4.3 we have plotted the particle-hole excitation gap $\Delta(N)$ (eq. 4.7) for $N_v = 4$. In the thermodynamic limit, this quantity will exhibit upward peaks at filling factors that correspond to incompressible states.

For $N_v = 6$ (figure 4.4), we focussed on the state at $\nu = 3/2, N = 9$, which corresponds to $k = 2$. We have calculated the overlap-squared with the $SU(4)_2$ state to be 0.939804. Another feature in the $N_v = 6$ plot is the state at $\nu = 1$ ($N = 6$). This could be a precursor to a paired composite fermion state; however, the overlap with the $SO(5)_1$ state was small.
4.5. **Numerical results**

Figure 4.3: Particle-hole excitation gap $\Delta(N)$ versus $\nu$, for $N_v = 4$, in a rectangular geometry, $a/b = \sqrt{3}/2$. The peaks can be interpreted as an indication of incompressibility of the corresponding states. For $N = 3, 6, 9$ we verified that the ground states, which are degenerate, are exactly the $SU(4)_k$ quantum Hall trial states with $k = 1, 2, 3$, respectively.

Figure 4.4: Particle-hole excitation gap $\Delta(N)$ versus $\nu$, for $N_v = 6$, in a rectangular geometry, $a/b = \sqrt{3}/3$. 
4.6 Discussion

In this chapter, we have studied several spin-1 spin liquids in the quantum Hall regime. The vortex lattices have melted into translationally-invariant quantum fluids, which we considered in edgeless geometries. The transition regions around the critical filling factor(s) where the quantum melting transition occurs, were not considered.

We have found the conformal field theories underlying the different liquids and ultra-local Hamiltonians that have the proposed states as exact ground states. The exact diagonalization studies are unfortunately not conclusive, since it was only possible to consider small system sizes.

The many-body interactions, considered in this chapter, may seem unphysical. However, as recently shown by Cooper [19], close to a Feshbach resonance it may be possible to create a 3-body (or even 4-body!) contact interaction.