Quantum Hall spin liquids
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In this chapter, we derive an effective theory for the $SO(5)_1$ spin-charge separated state. We have seen that we can find the (first-quantized) states which describe elementary excitations by the use of conformal field theory. However, in such a description, we assume that these excitations behave like free (or at least, weakly interacting) particles. So it is natural to ask what field theory could describe their topological properties and how such a field theory can be derived. In principle, this theory could be used to make detailed predictions. An example of this was given by Halperin, Lee and Read [32], who calculated response functions of the “Composite Fermion” Fermi liquid at $\nu = 1/2$.

The discussion will be phenomenological to a large extent. There is limited hope to derive the effective parameters analytically, since they depend on the highly non-trivial ground states. A straightforward approach to calculate such parameters uses e.g. Monte Carlo techniques to integrate over all different particle configurations. We will not pursue this approach. We know that the Coulomb interaction does not favor spin-charge separation at $\nu = 2/3$ (see the previous chapter). So the effective theory would be strongly interacting and quite without physical meaning.

In general, we will need Chern-Simons terms to describe the more general braiding that can exist in $2+1$ dimensions. It is possible to find operators in second quantization that reproduce the states given by the qH-CFT connection. In a path-integral representation, however, we can only use (complex) scalar or Grassmannian fields.

We will formulate a theory that captures the topological information, retrieved from the qH-CFT connection, in this $2+1$ dimensional description. It is in a large extent phenomenological, due to a mean-field treatment of a strongly (infinitely strong) coupled gauge theory. The effective theory turns out to be a $\mathbb{Z}_2 \otimes U(1)_c \otimes$
SU(2) gauge theory. It is related to the Z2-gauge theory of fractionalization of Senthil and Fisher[80, 82]. The derivation that follows is basically dual, in that the vortices in that theory appear as particles in ours and vice versa.

6.1 SEPARATING SPIN AND CHARGE

The state we will try to understand, looks in first quantization like:

\[ \Psi^{(m)}_{\text{paired}}\left(\left\{ z_i^\uparrow; z_j^\downarrow\right\} \right) = \text{Pf}\left(\frac{1}{x_i - x_j}\right) \Psi^{(m+1,m+1,m)}_{\text{Halperin}}(z_i^\uparrow; z_j^\downarrow) \]  

where \( x_i = z_i^\uparrow, z_j^\downarrow, m \) is an odd integer (for fermionic electrons), \( \Psi^{(m+1,m+1,m)}_{\text{Halperin}} \) is the Halperin wave function and the filling factor is \( \nu = \frac{m}{2m+1} \). We will consider the case \( m = 1 \), which corresponds to physical electrons.

The field theory which describes the spin-charge separated state derives from the same observation Balatsky and Fradkin made for the singlet quantum Hall effect. They noted that the Halperin wavefunction naturally factorizes into a charge related part and a spin part:

\[ \Psi^{(m+1,m+1,m)}_{\text{Halperin}} = \Psi^{m+1/2}_{\text{seions}}(\{x_i\}) \Psi_{\text{CSL}}(\{z_i^\uparrow; z_j^\downarrow\}) \]  

where

\[ \Psi^{m+1/2}_{\text{seions}}(\{x_i\}) = \prod_{i<j}(x_i - x_j)^{m+1/2} \]  

and

\[ \Psi_{\text{CSL}}(\{z_i^\uparrow; z_j^\downarrow\}) = \prod_{i<j}(z_i^\uparrow - z_j^\downarrow)^{1/2} \prod_{i<j}(z_i^\downarrow - z_j^\uparrow)^{1/2} \prod_{i,j}(z_i^\uparrow - z_j^\downarrow)^{-1/2} \]

The charge part describes a collection of semions forming a Laughlin state. The spin part is the chiral spin liquid, in continuum form. We will follow their notation and introduce a slave-semicolon decomposition of the electron operator.

An important distinction occurs in the charge sector. While Balatsky and Fradkin considered filling factor \( \nu = 2/(2m+1) \) with \( m \) even, we are now in an equivalent position with \( m \) odd. We will follow their discussion and emphasize the differences with the current situation.

We start out by splitting the electron field \( c_\sigma \):

\[ c_\sigma = \psi \phi_\sigma \]

with \( \psi \) and \( \phi_\sigma \) both semionic fields. We will first consider the properties of both sectors separately and later glue them together. The spin and charge sectors, although in this case defined for semions, both define well-known states.
6.2 THE CHARGE SECTOR

The two equivalent approaches to a field theoretical description of the fractional quantum Hall effect are based on the application of statistical transformation to either bosons[102] or fermions[49]. Both constructions use Chern-Simons theory to transform the statistics of bare electrons. On a mean-field level, the magnetic field is (partially) cancelled by the Chern-Simons gauge field. Fluctuations around this mean-field solution will be strong but both theories give well-defined values for the quantum numbers of quasi-particles. Numerical methods which solve the full theory for small numbers of particles support the belief that these are in fact the correct values and that we can use the mean-field theory to start a perturbative expansion.

In the composite boson construction, the electro-magnetic field is cancelled on a mean-field level by the Chern-Simons field. The bosons are effectively in zero magnetic field and will form a Bose-Einstein condensate. In the composite fermion approach, only part of the external magnetic field is cancelled and in the mean-field picture, the fermions will exhibit the integer quantum Hall effect.

We will pursue the second approach, and transform the charged semion into a fermion. In general, this can be achieved by choosing the Chern-Simons coupling constant \( \theta \) to be \( \theta = 1/2\pi(p + 1/2) \), with \( p \) an odd integer. The most convenient choice here is \( p = m \) and we obtain \( \theta = \nu/2\pi \). A natural action to describe the (non-interacting) fermions is

\[
\mathcal{L}_{\text{charge}} = \psi^*(iD_0^\mu + \mu)\psi + \frac{1}{2m}\psi^* (D^\mu)^2 \psi + \nu \mathcal{L}_{\text{CS}}(a)
\]

where we have introduced the gauge field \( a \) to describe the transformation. \( A \) is the electro-magnetic gauge field. With some abuse of notation we use \( \psi \) as a Grassmann field to describe the fermions.

On a mean-field level, we find the field equation for the Chern-Simons gauge field:

\[
\frac{\nu}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho = \langle J^\mu \rangle ,
\]

where \( J^\mu \) is the current of fermions, \( J^0 = \bar{\psi}, J^i = 0 \). Solving this equation, we see that the Chern-Simons magnetic field \( b = e^{ij} \partial_i a_j \) is constant. Moreover, it exactly cancels the external (electro-) magnetic field!

The situation that arises is identical to the "composite fermion Fermi liquid" at half-integer filling factor. Different behaviour is seen for these liquids in the quantum Hall setting. In the lowest Landau level, the liquid seems to survive to low temperatures and is well described by a (modified) Fermi-liquid theory [32]. The second LL supports the non-abelian Moore-Read state [55] which can alternatively be interpreted as a BCS superconductor [27]. In higher LL's a charge-density wave, well described by Hartree-Fock theory, develops.

We will assume that the interaction between the fermions is such that we end up with an incompressible state. In particular, there is a gap to all excitations. The
most natural candidate is the $p$-wave BCS superconductor. In appendix 6.A, this superconductor is discussed in more detail.

We assume that the interaction for the "charge" field $\psi$ can be transformed into the following form by a Hubbard-Stratonovich transformation of the interaction:

$$\mathcal{L}_{\text{charge}} + \mathcal{L}_{\text{int}} = \psi^*(iD_0^c + \mu)\psi + \frac{1}{2m}\psi^* (D^c)^2 \psi + \Delta^* \psi (\partial_x + i\partial_y)\psi + c.c. + \frac{1}{u} |\Delta|^2, \quad (6.9)$$

with $u$ the interaction strength. We furthermore assume that in mean-field approximation, there is a BCS condensate, $\langle \Delta \rangle \neq 0$.

As is clear from the above, the $\psi$ field forms $p$-wave Cooper pairs, and in the spectrum a gap opens. The excitations are the neutral BCS quasi-particles and vortices in the condensate.

Vortices in the condensate have the general form:

$$\varphi = \theta/2 \quad \text{(6.10)}$$

$$a_i = \bar{a}_i + \frac{1}{2} e^{i\theta} x_j / r^2, \quad \text{for } r \to \infty \quad \text{(6.11)}$$

where $(r, \theta)$ are polar coordinates and $\varphi$ is the phase of the condensate, $\Delta = e^{i\varphi} |\Delta|$. This solution of the gap-equation introduces a branch-cut in $\psi$. If a quasi-particle is adiabatically transported around the vortex (braided), the resulting wavefunction is identical to the original wavefunction but with an additional minus sign.

If the Bogoliubov-deGennes equations (which determine the spectrum of quasi-particles) are solved in the presence of such a vortex, a zero-energy mode is found [93]. (The limiting case $\mu \to 0$, which leads to the same topological properties, was discussed by Read and Green [66].) Explicitly, this zero-mode has the following form in Nambu space [93]:

$$\chi(r, \theta) \propto e^{ip_F r} \exp \left( - \int_0^r ds |\Delta(s)|/v_F \right) \left( \frac{1}{i} \right), \quad (6.12)$$

with $p_F = \sqrt{p/\pi}$ the Fermi momentum and $v_F = p_F / m$ the velocity. With $2n$ vortices present, there are $2n$ real fermionic operators $c_i$ satisfying

$$\{c_i, c_j\} = 2\delta_{ij} \quad \text{(6.13)}$$

When the vortices $(i, i+1)$ are braided clockwise, one of the fermions $(i)$ will see the branchcut due to the presence of the other $(i+1)$. The operator $T_i$ which performs this action as $c_i \to T_i c_i T_i^{-1}$, was found by Ivanov to equal $T_i = \exp(\frac{\pi}{4} c_{i+1} c_i)$ [38]. The $2n$ real fermions constitute $n$ fermionic zero-energy states. Upon braiding, these states will be filled (emptied) by the breakup (formation) of a Cooper pair. The parity of the number of filled levels is conserved, so that the number of states is $2^{n-1}$ in each sector.
6.3. Spin sector

As is known from the Moore-Read state [38], the composite of a half flux vortex and a Majorana fermion generate the same representation of the braid group as a doublet, coupled to a $SU(2)_2$ Chern-Simons term. This representation is non-abelian, in the sense that with more than 4 vortices the ground state has a finite degeneracy and braiding of vortices in real space induces rotations among these ground states [55, 59].

At this point, it is possible to integrate $\psi$ out and we obtain the following effective theory:

$$\exp \left( i \int L'_{\text{charge}} \right) = \int D\psi D\psi^* \exp \left( i \int (L_{\text{charge}} + L_{\text{int}}) \right), \quad (6.14)$$

$$L'_{\text{charge}} = \frac{1}{2m^*} \Delta^* (\partial_i - 2i\bar{a}_i)^2 \Delta + \frac{1}{2} |\Delta|^2 + \lambda |\Delta|^4$$
$$+ \frac{1}{8\pi^2} D(\frac{1}{2} \partial_0 \varphi - \bar{a}_0)^2 + L_{\text{top}} , \quad (6.15)$$

$$L_{\text{top}} = \frac{1}{4\pi} \varepsilon^{\mu
u\rho} (\frac{1}{2} \partial_\mu \varphi - \bar{a}_\mu) \partial_\nu (\frac{1}{2} \partial_\rho \varphi - \bar{a}_\rho) . \quad (6.16)$$

Here, $m^*$ is the effective mass of the Cooper pair and $\lambda$ is an effective parameter. The density of states at the Fermi energy ($D = 4\pi m$) provides the screening of electric current. The "topological" term $L_{\text{top}}$ has no effect, since the combination $\frac{1}{2} \partial_\mu \varphi - \bar{a}_\mu$ is gauge-invariant. Furthermore, the finite energy constraint ensures that it vanishes faster than $1/r^2$ as $r \to \infty$.

6.3. Spin sector

The chiral spin liquid has been analyzed by various authors in the context of e.g. the triangular Heisenberg model[42] or the singlet quantum Hall effect[10]. We can choose a description with a bosonic field coupled to a (non-abelian) $SU(2)$ Chern-Simons term:

$$L_{\text{spin}} = \frac{1}{2} |D_0 \phi|^2 + \frac{1}{2} |D^* \phi|^2 + \lambda (\bar{\phi} \cdot \phi - \rho)^2 + L_{CS}(A) , \quad (6.17)$$

where $\phi$ is a bosonic field in the fundamental representation of $SU(2)$. The gauge field $A$ transforms this boson into a semion. The interaction fixes the expectation value $\langle|\phi|^2\rangle$ to be equal to $\rho$ on microscopic length scales. The Chern-Simons term disorders the field on length scales $l \sim 1/\sqrt{\pi \rho}$. Excitations in this liquid correspond to non-compensated spins or, equivalently, flux in the gauge field $A$. The assumption of a spin liquid ground state implies that such excitations are gapped. The low-energy theory for the quasi-particles will therefore be Lorentz-invariant:

$$L_s' = \frac{1}{2} |D_0 b|^2 + \frac{1}{2} v_s^2 |D^* b|^2 + \frac{1}{2} v_s^2 m^2 |b|^2 + L_{CS}(A) , \quad (6.18)$$

where $v_s$ is the "speed of light" and $m$ is the effective mass of the quasi-particles. Since this mass is essentially the screening length in the liquid, it will be approx-
approximately given by \( m = \sqrt{\pi \rho} \). The velocity \( v_s \) is determined by short distance spin interactions and we will treat it as an effective parameter.

The flux tube which arises from the presence of a spinon creates a branch cut in the wavefunction for the original particles \( \phi \). One can see this easily in the explicit wavefunction for a spinon (spin up at position \( \eta \)) and \( N \) particles:

\[
\overline{\Psi}_{\text{spinon}}(\{z_i\}; \eta) = \prod_i (z_i - \eta)^{\sigma_i \sigma' / 2} \overline{\Psi}_{\text{CSL}}(\{z_i\}) .
\] (6.19)

The spin of the particles is \( \sigma_i = \pm 1 \) and \( \sigma' \), the spin of the quasi-particle is \( \sigma' = 1 \). To have a well-defined wavefunction for the \( N \) particles, there should be an additional gauge field. If the particles carry charge under this gauge field, while the spinons carry (half) a flux, then the branch cut in the wavefunction of the particles is compensated by the minus sign one obtains when braiding a charge around a half quantum of flux.

### 6.4 Gluing Charge and Spin

The original decomposition of the electron operator into two semion operators implies the presence of a strong-coupling \( U(1) \) ("RVB") gauge field. The invariance of the electron operator under the gauge transformation \( \phi \rightarrow e^i \sigma \phi, \psi \rightarrow e^{-i \sigma} \psi \) imposes the same invariance on all observable quantities. We can implement the RVB gauge field by coupling the charge and spin:

\[
\mathcal{L} = \mathcal{L}_{\text{charge}} + \mathcal{L}_{\text{spin}} + c \mu (J^\mu - J_s^\mu) .
\] (6.20)

The field \( c \) acts as a Lagrange multiplier and by the explicit form is minimally coupled to both spin and charge fields.

Since we want to obtain a (weakly-coupled) theory expressed in terms of the physical excitations, we should look for the excitations which are invariant under the gauge symmetry.

**BCS quasi-particle:** This particle, present in the charge sector of the theory, only carries charge under the \( \mathbb{Z}_2 \) gauge field which transforms \( \psi \rightarrow -\psi \) and leaves \( \Delta \) invariant. So we immediately find that this (EM) neutral fermionic excitation is also a valid excitation of the coupled theory.

**holon:** The finite winding number of the phase of the BCS order parameter is cancelled as \( r \rightarrow \infty \) by a vortex in \( a_\mu \). The Chern-Simons term induces a finite charge \( (\nu / 2) \), associated to this quasi-particle. This extra density can be accommodated for in the spin liquid, without introducing quasi-particles. The neutral fermion sees a branchcut and induces a Majorana fermion in the core of the vortex.

For the other particles in the theory, we can use the presence of a charged order parameter \( \Delta \). As in usual BCS theory, the superconducting order parameter \( \Delta \)
acts as a charge 2 scalar Higgs field. In the low-energy limit, only the unbroken subgroup $Z_2$ of $U(1)$ will remain.

We can now identify the gauge field which the spinon needed to give a single-valued wavefunction. The spin-semion carries a charge under the remaining $Z_2$ gauge symmetry. A natural choice for the additional flux of the spinon is now a $Z_2$ flux. This leads to the

**spinon:** The $Z_2$ flux of the spinon also couples to the BCS quasi-particle. By the same reasoning as for the holon, a Majorana fermion is induced in the core.

6.5 **DISCUSSION**

The presence of the zero-energy modes in the core of both the holon and the spinon leads to non-abelian braiding. Upon braiding, the zero modes are occupied (or emptied), exactly as in the scalar Moore-Read case. From the conformal field theory connection, we know that the spinon and the holon together transform as the 4-dimensional representation of $SO(5)$. It should be possible to transform the theory into a form in which these quasi-particles are represented by a field transforming in this representation, coupled to a $SO(5)$ Chern-Simons term.

It should be noted that upon fusing an even number of spinons and holons, the resulting particle will have abelian braiding relations. When two quasi-particles are within a distance of $l \sim v_F/|\Delta|$, the zero-mode wavefunctions will overlap. The isolated vortex approximation breaks down and the zero mode shifts to finite energy. One such composite is the spin-charge composite, with charge $\nu/2$ and spin-$\frac{1}{2}$. This particle is nothing but the familiar quasi-particle over spin-singlet Halperin states.
6.A (scalar) $p$-wave BCS theory

For a number of scalar paired quantum Hall states, Read and Green[66] performed an extensive analysis of their topological properties in terms of (more or less) conventional BCS theory. An important role is played by zero modes. Such modes are at the heart of such effects as statistical transformation and instanton interference.

The $p$-wave (spin polarized) BCS superconductor has been an important special case. In the presence of a vortex, the Dirac equation predicts a mode in which the two components have support in very different regions. One component lives near the core of the vortex, while the other is supported by the edge of the sample. We can therefore treat them as independent modes, each described by a Majorana fermion. By doing so, we can quickly recover the state counting for the Moore-Read state. In the presence of $2n$ (well separated) vortices, there are $n$ Dirac zero energy solutions. This gives a degeneracy of $2^n$ of the ground state, even if all positions are fixed. Since the breakup of a Cooper pair will always produce two fermions, these ground states fall into two categories: one with an even number and one with odd number of fermions. In each of these categories the degeneracy will therefore exactly be equal to $2^{n-1}$, the same number we found in the analysis of quasi-particles over the Moore-Read state.

Note that the fact that the electrons are spin polarized is particularly important. In the unpolarized $\langle S \rangle = 0$ spin-full $p$-wave paired case, a situation analogous to the quantum Hall (331) state [33], no such degeneracies exist. It is anticipated [66] that the phase transition between these two paired states could be observed in quantum Hall experiments with two layers. The nature of the transition is very closely related to the Helium A to $A_1$ transition.