Spin bosons and spin glasses

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Quantum spherical model

In this chapter we are going to study the quantum spherical model. This model is a perfect ground up to understand the concepts of the previous chapter and generally the ideas of critical phenomena, since it is an exactly solvable model that reproduces non-trivial critical phenomena. The fact of being quantum mechanical allows us to study quantum phase transitions analytically. A definite expression is found for the different critical exponents in terms of the dimensionality of the system.

Originally the spherical model, in its classical version, was introduced by Kac. After being introduced in 1947 to Onsager’s rather intricate solution of the 2d-Ising model, he desired to formulate a simpler spin model. As a first step he took the spins to be continuous Gaussian variables, nowadays called the Gaussian model. This had unphysical behavior at low temperatures which led Kac to consider the spherical model. The spherical model has continuous spins that are restricted by the spherical constraint $\sum_{i=1}^{N} S_i^2 = N$, which represents the hypersphere intersecting all vertices of the hypercube sustained by the Ising spins. $S_i = \pm 1$. In the end, the spherical model is formally the same as the Ising model, i.e. the Hamiltonian is the same, with a global constraint instead of a local one: the sum of the square of the length of the spins is constrained rather than each of them. At that time the saddle point method, needed in the solution, was not widely known, and here Berlin came in, leading the celebrated joint publication on the spherical model in 1952 [BK52]. Kac’s personal reminiscence of this history is presented in Ref. [Kac64].

The spherical model for a ferromagnet has been considered in great detail in literature. Actually, the paramagnetic to ferromagnetic transition is similar to an ideal Bose-Einstein condensation. As already pointed out, the critical behavior can be solved exactly. Critical exponents and scaling functions can be derived. In particular, the model with short range interactions exhibits $d_{uc} = 2$ as the lower critical dimension: for $d \leq 2$ no stable ferromagnetic phase occurs. Likewise, $d_{uc} = 4$ is the upper critical dimension: for $d > 4$ critical exponents take their mean-field values. See previous chapter. These analytic results have been used to test approximations and general ideas of phase transitions for a wide range of interactions, short and
long range. For a review on the classical spherical model see Ref. [Joy72].

As said, the spherical model was introduced for its mathematical simplicity. However, Stanley [Sta68] proved that the free energy of a model of arbitrary spin dimension \( \nu \), incorporating thus the Ising model (for spin dimension \( \nu = 1 \)), the \( x-y \) model (spin dimension \( \nu = 2 \)) and Heisenberg model (\( \nu = 3 \)), approaches that of the spherical model in the limit of infinite spin dimensionality \( \nu \to \infty \). Hence, it gives a geometrical interpretation to the spherical model. Since various critical properties where proven to be monotonic functions of the spin dimensionality \( \nu \), the critical properties of the Heisenberg model appeared to be bounded on one side by those of the Ising model and on the other by those of the spherical model.

The spherical model for antiferromagnets was studied by Knops. The spherical constraint imposes \( < S_i^2 > = 1 \) for ferromagnets. However, this does not work for antiferromagnets because of the lack of translational invariance. To recover this, Knops added a second constraint: more generally, one constraint has to be added for each translationally invariant set, which in the case of antiferromagnets means each of the two sublattices. He found that the two constraints reduce to a unique one provided the staggered external field is zero. The fact that the spherical spins are scalars makes it impossible to define an order parameter that can be identified with the spontaneous staggered magnetization. To solve that and get the proper order parameter Knops used a vector version of the spherical model [Kno73a]. He also generalized Stanley’s arguments to non-translational interactions [Kno73b].

The spherical model has also been applied to disordered systems. Though, in view of Knops’ finding, perhaps an infinite number of spherical constraints should be used, typically no analog of the staggered external field is applied, and one may expect that all constraints collapse into a single one. Therefore spherical spin glass models may still give insight in the physics of the problem which would be more difficult to study e.g. with Ising spins. In the case of pair couplings the exact solution exhibits no breaking of replica symmetry and the replica trick need not be used [KTJ76]. The family of \( p \)-spin spin glasses (\( p \)-spin models) [CS92] has been shown to exhibit one step replica symmetry breaking by studying the spherical version. For spin glasses with random pair and quartet interactions \( \{ p = 2 \} + \{ p = 4 \} \), “\( p = 2 + 4 \)”, Nieuwenhuizen showed that an exact solution exists, exposing the full replica symmetry breaking scenario. The simplicity of spherical models thus may give insight in difficult problems for which otherwise no exact solution is available. For an early review on the use of the spherical model in disordered systems, see Ref. [KKPS92].

So far the discussion has been classical. The classicality can be understood in particular because the entropy diverges at low temperature as \( \ln T \) just as for a classical ideal gas. Different quantum versions of the spherical model have been proposed. In this chapter we will discuss the two main approaches and the differences between them. The spins have to become quantum mechanical operators and their adjoint operators also have to be included in the formalism. The spherical constraint can then be kept the same though in operators language, as it was started by Obermair [Obe72], or, conversely, one can constrain both the total spin length as before plus its global kinetic energy, as developed by Nieuwenhuizen [Nie95a, Nie95b]. The
fact of having the dynamics inside the spherical constraint allows one to consider Hamiltonians without an explicit kinetic part. Both approaches, though in different setups, exhibit a quantum phase transition [Voj96, SN04]. Obermair’s model, belongs to a universality class, i.e. $O(n)$ non-linear sigma model for large $n$, while Nieuwenhuizen’s can, depending how the dynamics is considered, belong to the same universality class or to another, i.e. $SU(n)$ Heisenberg ferromagnet in the limit for large $n$.

2.1 Classical spherical model

The spherical constraint was conceived as a relaxation of the Ising constraint. Indeed, Ising spins, $S_i \equiv s_{i,z} = \pm \frac{1}{2} \hbar$, obviously satisfy it. Adjusting the coefficients from the original version it may be written as

$$\frac{1}{2} \sum_{i=1}^{N} S_i^2 = N \sigma.$$  \hspace{1cm} (2.1)

with $\sigma = \hbar^2/8$ having dimension $(J_s)^2$. So the spins are treated as real variables constraint by Eq. (2.1). The Berlin-Kac spherical model is defined by the partition sum

$$Z = \int DS \ e^{-\beta H} \delta\left(\frac{1}{2} \sum_{i=1}^{N} S_i^2 - N \sigma\right) = \int DS \ \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} \ e^{-\beta H - \frac{1}{2} \mu \sum_{i=1}^{N} S_i^2 + \mu N \sigma}$$  \hspace{1cm} (2.2)

where

$$DS = \prod_{i} \int_{-\infty}^{\infty} dS_i.$$  \hspace{1cm} (2.3)

2.1.1 Vector spherical spins

For vector spins the generalization of Eq. (2.1) in the case of $m$ spin dimensions reads

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{a=1}^{m} (S_{i}^a)^2 = Nm \sigma.$$  \hspace{1cm} (2.4)

It is worth mentioning that the spin dimensionality in eq. (2.4) is not related to the approach of Stanley, who started with vector spins and ended up with scalar spherical spins. We only introduce vector spherical spins to avoid the restriction scalar spins have. We benefit from the fact that the vector character allows to study the behavior in a transverse field. A similar step allowed Knops to define a proper order parameter for the antiferromagnetic spherical model [Kno73a].
2.1.2 Quantization

It is natural to consider the $S_i$ analogous to position variables of harmonic oscillators. In quantum mechanics they become hermitian operators $\hat{S}_i$ with the dimension of $h$. Js. The conjugate momentum operator $\hat{\Pi}_i^a$ is dimensionless and postulated to satisfy the commutation relation

$$[\hat{S}_i^a, \hat{\Pi}_j^b] = i\hbar\delta_{i,j}\delta_{a,b}. \quad (2.5)$$

As for harmonic oscillators, this allows to define creation and annihilation operators

$$\hat{\Sigma}_i^{a\dagger} = \frac{1}{\hbar\sqrt{2}}\hat{S}_i^a - \frac{i}{\sqrt{2}}\hat{\Pi}_i^a, \quad \hat{\Sigma}_i^a = \frac{1}{\hbar\sqrt{2}}\hat{S}_i^a + \frac{i}{\sqrt{2}}\hat{\Pi}_i^a \quad (2.6)$$

satisfying the commutation relation

$$[\hat{\Sigma}_i^a, \hat{\Sigma}_j^{b\dagger}] = \delta_{i,j}\delta_{a,b}. \quad (2.7)$$

2.1.3 Spherical constraint on the length of the total spin

There is some freedom to choose the spherical constraint, which amounts to describing different physical situations. The standard quantum constraint considered in literature is just the quantized version of the mean of eq. (2.4).

Constraint 1 : \[ \frac{1}{2} \sum_{i,a} \langle (\hat{S}_i^a)^2 \rangle = Nm\sigma. \quad (2.8) \]

where $\langle ... \rangle$ denotes the quantum expectation value. Obermair took as the quantum Hamiltonian the classical $H(S)$ with spins replaced by operators, and added the kinetic term that one expects for physical rotors.

$$H(\hat{S}, \hat{\Pi}) = \frac{1}{2}g\sum_i \hat{\Pi}_i^2 + H(\hat{S}). \quad (2.9)$$

where $g^{-1}$ is the rotor’s moment of inertia. An effective Hamiltonian which includes the constraint can be derived with a Lagrange multiplier. One ends up with

$$\hat{H}_{\text{tot}} = \frac{1}{2}g\sum_i \hat{\Pi}_i^2 + H(\hat{S}) + \mu(t)\sum_{i,a} (\hat{S}_i^a)^2 - Nm\sigma. \quad (2.10)$$

where $\mu$ is the Lagrange multiplier that enforces the constraint. In equilibrium its value is given by the equation of the spherical constraint $\frac{\partial F}{\partial \mu} = 0$.

2.1.4 Spherical constraint on the number of spin quanta

The constraint can restrict additionally to spin length also the global dynamics of the system. This is possible by using

Constraint 2 : \[ \sum_{i,a} \langle \hat{\Sigma}_i^{a\dagger} \hat{\Sigma}_i^a \rangle = \sum_{i,a} \langle \hat{n}_i^a \rangle = Nm\frac{\sigma}{h^2}. \quad (2.11) \]
A constraint analogous to this one was introduced by Nieuwenhuizen [Nie95a, Nie95b]. The kinetic energy is also constrained here since this constraint includes the momenta as well. This can be seen by writing it in the form

\begin{equation}
\text{Constraint 2 : } \frac{1}{2} \sum_{i,a} (\langle \hat{S}_i^a \rangle^2 + \hbar^2 \langle \hat{\Pi}_i^a \rangle^2) = N m(\sigma + \frac{\hbar^2}{2}). \tag{2.12}
\end{equation}

For a Hamiltonian \( \hat{H}(\hat{\mathbf{S}}, \hat{\mathbf{\Pi}}) \) that may, but need not, depend explicitly on the momenta, the effective spherical Hamiltonian is

\begin{equation}
\hat{H}_\text{tot} = \hat{H}(\hat{\mathbf{S}}, \hat{\mathbf{\Pi}}) + \frac{1}{2} \mu \sum_{i,a} \left[ (\hat{S}_i^a)^2 + \hbar^2 (\hat{\Pi}_i^a)^2 \right] - N m \mu(\sigma + \frac{\hbar^2}{2}). \tag{2.13}
\end{equation}

Now, the situation where the Hamiltonian does not depend explicitly on the momenta (no kinetic term), \( \hat{H}(\hat{\mathbf{S}}, \hat{\mathbf{\Pi}}) \rightarrow \hat{H}(\hat{\mathbf{S}}) \), still leads to sensible dynamics, since the constraint already depends on the momenta. Different constraints describe different physics. However, at high temperatures one expects the differences to become small.

In the remaining of this paper we will simplify the notation by taking units in which \( \hbar = 1 \).

2.1.5 Comparison of the two constraints

The main difference between the two constraints is obviously the presence or absence of momenta. In the second case eq. (2.11), the spherical constraint can carry all the dynamics of the model. On the contrary, using the first constraint eq. (2.8), a kinetic term, with an external parameter \( g \), has to be added to the Hamiltonian [Obe72]. This parameter determines the strength of quantum fluctuations: the classical model can be recovered for \( g = 0 \). This fact makes models with the first constraint describe quantum rotors, as was pointed out in Ref. [Voj96]. The first constraint, eq. (2.8), brings actions which are invariant under orthogonal transformations. Conversely, using the second constraint, eq. (2.11), the choice of Hamiltonian can bring symmetry under unitary transformations or orthogonal ones depending on the question whether the Hamiltonian contains momenta or not. Hamiltonians with unitary transformation symmetry yield free energies analogous to the large \( N \) limit of the generalization of SU(2) Heisenberg spins to SU(\( N \)). Hamiltonians with orthogonal transformation symmetry share the critical phenomena with the large \( N \) limit of O(\( N \)) non-linear sigma model and describe therefore quantum rotors as occurs by using the first constraint, eq. (2.8).

Each of the symmetries belongs to different universality classes in the quantum regime, yet classical critical phenomena are always the same as in the classical model, consistent with the expectation that quantum effects do not lead to qualitative changes at finite temperatures. We will see that the dynamical critical exponent \( z \) is different in both symmetries, causing the difference in critical exponents at the quantum critical point as was pointed out in Ref. [Her76].
2.2 Ferromagnetic Hamiltonians with creation and annihilation operators

We want to study the Hamiltonian

\[
H(\hat{\Sigma}^+, \hat{\Sigma}) = -\sum_{i \neq j} J_{ij} \hat{\Sigma}_i^+ \hat{\Sigma}_j - \sum_i \Gamma_i \left( \hat{\Sigma}_i^+ + \hat{\Sigma}_i \right) / \sqrt{2} \\
= -\frac{1}{2} \sum_{i \neq j} J_{ij} (\hat{S}_i \hat{S}_j + \hat{\Pi}_i \hat{\Pi}_j) - \sum_i \Gamma_i \hat{S}_i, \tag{2.14}
\]

where in the second equality we inserted Eq. (2.6). The $i\hat{S}_i \hat{\Pi}_j$ cancelled since we assumed symmetric couplings, $J_{ij} = J_{ji}$. Obviously, the momentum operators do occur in this expression. The couplings $J_{ij}$ can in principle express any kind of interaction, ferromagnetic, antiferromagnetic, spin glass... The $\Gamma_i$ represent an external field, that can be constant, variable, random... Later on, we will focus on ferromagnetic couplings in the presence of constant magnetic field. This Hamiltonian without the external magnetic field is symmetric under unitary transformations, a fact that will determine the critical behavior.

The first step to get the partition function is to diagonalize the couplings.

\[
\Sigma_i(\tau) = \sum_\lambda \Sigma_\lambda(\tau) e_{\lambda}^i, \\
\Sigma_\lambda(\tau) = \sum_i \Sigma_i(\tau) e_{\lambda}^i. \tag{2.15}
\]

where $e_{\lambda}^i$ is the normalized eigenvector of the coupling matrix $J_{ij}$.

2.2.1 Coherent state representation for spherical spins

In this section we explain, following Ref. [NR98], how to add the spherical constraint to a quantum Hamiltonian using the path integral formalism for models with the second constraint Eq. (2.11). In second quantization the spins are given a bosonic algebra. Then we can use the formalism introduced in section 1.1. We can deal with spherical spins using almost the same approach described there. The operator $\hat{a}_i$ is identified with $\hat{\Sigma}_i^a$, where the index $a$ denotes the spin vector direction, and the corresponding fields $\phi_i$ are denoted as $\Sigma_i^a$.

In order to impose this constraint, Eq. (2.11), in the path integral formalism, the identity definition Eq. (1.16) is modified to adopt to the spherical case, in a way inspired by Ref. [Nie95a, Nie95b, NR98]: one restricts the path integral to states which exactly satisfy the constraint by employing the truncated identity

\[
1 \rightarrow 1_{spherical} \equiv C \int \prod_{ta} d\Sigma_t^a d\Phi(\Sigma_t^a) \frac{e^{-\Sigma^a \cdot \Sigma}}{\pi} \delta(\hat{\Pi} - N m \sigma). \tag{2.16}
\]
where the number operator $\hat{n}$,

$$\hat{n} = \sum_{\lambda} \hat{\Sigma}^\dagger_{\lambda} \hat{\Sigma}_{\lambda}.$$  \hfill (2.17)

counts the total number of spin quanta. We insert

$$\delta(\hat{n} - N m_{\sigma}) = \int_{-\infty}^{\infty} \frac{e^{i\mu_0}}{2\pi} e^{-i\mu(\hat{n} - N m_{\sigma})} = \int_{-i\infty}^{i\infty} \frac{e^{i\mu_0}}{2\pi} e^{-i\mu(\hat{n} - N m_{\sigma})}$$  \hfill (2.18)

where $\mu = i\mu_0$ is imaginary. (Strictly speaking, we should insert a Kronecker-$\delta$ function, rather than the Dirac-$\delta$, but for large $N$ this amounts to the same.) Repeating the same procedure with this new identity we get

$$Z = \int_{\Sigma(\beta) = \Sigma(0)} D\mu D\Sigma D\Sigma^* \exp(-A).$$  \hfill (2.19)

with the action

$$A = \sum_{\tau = 0}^{\beta} d\tau \left[ \Sigma^*(\tau) \cdot \frac{d\Sigma(\tau)}{d\tau} \right.$$  \hfill (2.20)

$$+ \mu(\tau)(\Sigma^*(\tau) \cdot \Sigma(\tau - d\tau) - N m_{\sigma}) + H(\Sigma^*(\tau), \Sigma(\tau - d\tau)) \right]$$

and integration measures defined as

$$\int D\Sigma D\Sigma^* = \prod_{\tau = 0}^{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\Sigma^a(\tau)d\Sigma^a(\tau) \quad \frac{1}{\pi}.$$  \hfill (2.21a)

$$\int D\mu = C\prod_{\tau = -i\infty}^{i\infty} \frac{e^{i\mu(\tau)}}{2\pi i}.$$  \hfill (2.21b)

so $\mu(\tau)$ is the Lagrange multiplier introduced to impose the spherical constraint and the prefactor $C^{(M)}$ is added to ensure, if needed, a proper normalization. Details on this factor were given in Ref. [NR98].

It should be noted that the particle number operator in the definition of the identity eq. (2.16) will be surrounded, as is the case for the Hamiltonian, by spin operators on different timesteps; therefore its creation and annihilation operators will also be projected on different timesteps, $\hat{\Sigma}^\dagger_\lambda \hat{\Sigma}_\lambda \rightarrow \Sigma^\dagger_\lambda(\tau)\Sigma^a_\lambda(\tau - d\tau)$. In Refs. [Nie95a, Nie95b, NR98] the spherical constraint was slightly different from the one presented here but equivalent up to an additive constant.

It is worth remarking that, up to now, we imposed the spherical constraint strictly, no thermal average has been performed. In the following section $\mu$ will be integrated over by the method of steepest descent. This approximation allows the particle number to fluctuate and therefore the satisfiability of the constraint in the end remains only in average.
2.2.2 General solution

We may write the partition function sum as a continuum expression.

\[ Z = \int D\tau D\Sigma D\Sigma^* \exp \left\{ -\int d\tau \sum_{a,\lambda} \left[ \frac{\Sigma^\dagger_a(\tau) d\Sigma^a(\tau)}{d\tau} + \mu(\tau) (\Sigma^a(\tau)\Sigma^a(\tau - d\tau) - Nm\sigma) \right] \right. \]

\[ - J_\lambda \Sigma^a(\tau)\Sigma^a(\tau - d\tau) - \frac{1}{\sqrt{2}} \Gamma_\lambda (\Sigma^a(\tau) + \Sigma^a(\tau - d\tau)) \} \right\}. \]

In discrete notation, the action of Eq. (2.19) reads

\[ A = \sum_j \epsilon \left\{ \frac{1}{\epsilon} \sum_{a,\lambda} \left[ \Sigma^a_{\lambda,j} \Sigma^a_{\lambda,j} - \Sigma^a_{\lambda,j} \Sigma^a_{\lambda,j-1} \right] + \mu(j\epsilon) \left[ \sum_{a,\lambda} \Sigma^a_{\lambda,j} \Sigma^a_{\lambda,j-1} - Nm\sigma \right] \right. \]

\[ - \sum_{a,\lambda} J_\lambda \Sigma^a_{\lambda,j} \Sigma^a_{\lambda,j-1} - \sum_{a,\lambda} \Gamma_\lambda \left( \Sigma^a_{\lambda,j} + \Sigma^a_{\lambda,j-1} \right) \right\}. \]

where \( \epsilon = d\tau \) is the imaginary time step, \( j \) the time index and \( \Gamma_\lambda = \sum_j \Gamma_\lambda \) is the field in the basis of eigenvectors of \( J_{ij} \). Collecting all terms we have

\[ Z = \int D\mu \prod_{\lambda,a} \left\{ \int \Pi_j \left( \frac{d\Sigma_{\lambda,j} d\Sigma^a_{\lambda,j}}{2\pi i} \right) \right. \]

\[ \exp \left[ - \sum_{ij} \Sigma^a_{\lambda,i} B_{ij} \Sigma^a_{\lambda,j} + \sum_j \epsilon \Gamma_\lambda \left( \frac{\Sigma^a_{\lambda,j} + \Sigma^a_{\lambda,j-1}}{\sqrt{2}} \right) \right] e^{\sum_j Nm\sigma(\mu(j\epsilon))}. \]

where \( B_{ij} = \delta_{ij} - (1 + \epsilon J_\lambda - \mu(j\epsilon))\delta_{ij+1} \) here the prime stands for the fact that \( \delta_{1,M+1} \equiv 1 \) due to the trace structure of the partition function. We can now integrate over the spins

\[ Z = \int D\mu \exp \left[ \sum_{\lambda,a} \left\{ -m \ln \det B_{ij} + \frac{\epsilon^2 \Gamma_\lambda^2}{2} \sum_{ij} B_{ij}^{-1} + m\sigma \sum_j \mu(j\epsilon) \right\} \right]. \]

As usual, in thermodynamics, the saddle point value of one-time quantities like \( \mu(\tau) \) can be taken independent of \( \tau \). We will employ this simplification throughout the rest of this chapter. The determinant and the matrix inversion can then be performed [NO98]. Integrating over \( \mu \) by the saddle point method we obtain

\[ \beta F = -m\sigma \beta \mu + \frac{1}{N} \sum_{\lambda,a} \left\{ \ln(1 - a_\lambda) - \frac{Me^2 \Gamma_\lambda^2}{2(1 - a_\lambda)} \right\}. \]
where $a_\lambda = 1 - e^{(\mu - J_\lambda)}$. Sending $M \rightarrow \infty$ we finally get

$$\beta F = -\beta \mu m \sigma + \frac{m}{N} \sum_\lambda \left\{ \ln(1 - e^{-\beta(\mu - J_\lambda)}) - \frac{\beta \Gamma_\lambda^2}{2(\mu - J_\lambda)^2} \right\} =$$

$$- \beta \mu m (\sigma + \frac{1}{2}) + m \int dJ_\lambda \rho(J_\lambda) \left\{ \ln \left[ 2 \sinh \left( \frac{\beta}{2} (\mu - J_\lambda) \right) \right] - \frac{\beta \Gamma_\lambda^2}{2(\mu - J_\lambda)} \right\}.$$  (2.27)

where in the last equality we have assumed that the couplings satisfy $\frac{1}{N} \sum_\lambda J_\lambda = 0$. Otherwise, we would get an additional term $-1/2m\beta\langle J \rangle$. The saddle point equation reads

$$\sigma + 1 = \frac{1}{N} \sum_\lambda \left\{ \frac{1}{1 - e^{-\beta(\mu - J_\lambda)}} + \frac{\Gamma_\lambda^2}{2(\mu - J_\lambda)^2} \right\} = \int dJ_\lambda \rho(J_\lambda) \left\{ \frac{1}{1 - e^{-\beta(\mu - J_\lambda)}} + \frac{\Gamma_\lambda^2}{2(\mu - J_\lambda)^2} \right\}.$$  (2.28)

The sums over the different eigenvalues of the coupling matrix have been changed into integrals. Each $J_\lambda$ has a weight in this integral given by $\rho(J_\lambda)$. The actual form for this weight function will depend on the type of couplings. A set of weight functions for ferromagnets in different cubic lattices can be found in Ref. [Joy72], and for spin glasses with long range interactions in Refs. [NR98, KTJ76].

At large temperatures these equations reduce to

$$\beta F = -\beta \mu m \sigma + \int dJ_\lambda \rho(J_\lambda) \left\{ \ln \beta(\mu - J_\lambda) - \frac{\beta \Gamma_\lambda^2}{2(\mu - J_\lambda)} \right\}.$$  (2.29)

$$\sigma + 1 = \int dJ_\lambda \rho(J_\lambda) \left\{ \frac{T}{\mu - J_\lambda} + \frac{\Gamma_\lambda^2}{2(\mu - J_\lambda)^2} \right\}.$$  (2.30)

Apart from a factor two, these are exactly the equations of the classical spherical model, see e.g. [Joy72]. This factor two arises because the momenta double the degrees of freedom, see e.g. [Nie95a, Nie95b]. Near the phase transition they are already approximate, but the transition stays within the classical universality class.

### 2.2.3 Ferromagnetic couplings with transversal field in d dimensions

In this section we will use the results given in the previous one for the concrete case of ferromagnetic couplings with uniform transversal field. The Hamiltonian in this case differs from the one before Eq. (2.14) in the fact that the couplings only act in the $z$-direction while the external field only acts in the $x$-direction (we restrict ourselves therefore to $m = 2$). The free energy reads
\[ 3F = -3\mu(2\sigma + 1) + \int \frac{d^d k}{(2\pi)^d} \ln \left[ 2 \sinh \left( \frac{3}{2}(\mu - J(k)) \right) \right] + \ln \left[ 2 \sinh \left( \frac{3\mu}{2} \right) \right] - \frac{3\Gamma^2}{2\mu} \]

and the saddle point equation

\[ 2(\sigma + 1) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-\beta(\mu - J(k))}} + \frac{1}{1 - e^{-\beta\mu}} + \frac{\Gamma^2}{2\mu^2}. \]  

where we have applied the changes \( J_\lambda \to J(k) \) and

\[ \int dJ_\lambda \rho(J_\lambda) = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d}. \]  

We choose \( J(k) \approx J_0 - J'|k|^x \) for \( |k| \to 0 \). In the case of short range couplings, for instance, one has \( x = 2 \) since \( J(k) = \sum J \cos k_i \approx J(0) - \frac{1}{2} J |k|^2 \). A long range coupling that decays as \( J(r) \sim 1/r^x \) at large \( r \) gives \( x = \alpha - d \).

As in the theory of Bose-Einstein condensation, the saddle point equation fixes the dependence of \( \mu \) on temperature. There should be a solution at any \( T \). In order to have a real free energy, \( \mu \) cannot be smaller than the maximum value for \( J(k) \). Therefore, we should investigate the convergence of the integral in the limit \( \mu \to J_0 \). If the integral diverges, \( \beta \) must go to infinity before \( \mu \) reaches \( J_0 \) in order to satisfy the saddle point equation, so there exists a \( \mu \) for all temperatures and no phase transition occurs. If the integral converges, however, there will be a range of temperatures in which the saddle point as it stands cannot hold. This indicates that we have overlooked a macroscopic occupation of the ground state, as occurs in Bose-Einstein condensation. The relevant integral behaves as

\[ \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-\beta(\mu - J_0 + J(k))}} \approx \frac{\Omega_d}{(2\pi)^d} \int_0^1 dk k^{d-1} \int_0^1 \frac{1}{1 - e^{-\beta J_0 k^x}} \int_0^1 dk k^{d-1-x}. \]  

where \( \Omega_d \) is the hypersurface of a sphere in \( d \) dimensions. At \( k = 0 \), this integral converges for \( d > x \), hence there will be a phase transition for dimensions larger than \( x \).

At low temperatures, \( \mu \) may get stuck at \( J_0 \) and the saddle point equation as it is in Eq. (2.32) is no longer valid. This is because, as in Bose-Einstein condensation calculations, the ground state is not properly included in the integral. It should be taken out of the sum before this one is converted to an integral. This causes a change in the free energy by a factor \( (\mu - J_0)q \), where \( q = \frac{1}{N} \sum_{k=0}^{\tilde{z}} \tilde{\Sigma}_{k=0}^{\tilde{z}} \) is the ground state occupation, and the saddle point equation becomes

\[ 2(\sigma + 1) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-\beta(\mu - J(k))}} + \frac{1}{1 - e^{-\beta\mu}} + \frac{\Gamma^2}{2\mu^2} + q. \]  

$q$ can be evaluated from the saddle point equation $(\mu - J_0) \sqrt{q} = 0$. Thus when $\mu = J_0$ the occupation of the ground state can take non-zero values that can be determined using Eq. (2.35). Hence the ground state occupation is macroscopic in the ordered phase.

A transversal field will lower the transition temperature. Above a certain value $\Gamma_c$, the transition does not exist anymore, thus $T = 0, \Gamma = \Gamma_c$ is a quantum critical point, see chapter 1 or for a complete study over quantum phase transitions, see e.g. [Sac99]. We will now first study the classical critical point, at $\Gamma = 0$.

**Finite temperature phase transition**

For the dimensions where the phase transition exists, the critical temperature is found by solving the equation

$$2(\sigma + 1) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-\beta(J_0 - J(k))}} + \frac{1}{1 - e^{-\beta J_0}}. \quad (2.36)$$

The dependence of the chemical potential (the Lagrange multiplier introduced in Eq. (2.18) is equivalent to the chemical potential and both terminologies will be used) on temperature near the transition is the first thing needed. To get it, we expand the saddle point equation around the critical point $T = T_c + \tau, \mu = J_0 + \delta \mu$. The integral gives, up to first order in $\delta \mu$ and $\tau$

$$\int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-J_0 - J(k)}} \approx \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{1 - e^{-\beta(J_0 - J(k))}} \right] \quad (2.37)$$

The coefficient of $\delta \mu$ is an integral that diverges for $d \leq 2x$. This means that for these dimensions the leading term in the $\delta \mu$ expansion of Eq. (2.37) has a power smaller than one. For dimensions $d > 2x$ we will have $\delta \mu \propto \tau$ which will lead to the mean field exponents, thus $d_{uc} = 2x$ is therefore the upper critical dimension. To study the system near the critical point we subtract Eq. (2.36) from the saddle point equation, a procedure that will cancel the zeroth order term in the expansion in $\tau$ and $\delta \mu$, giving finally

$$\tau \approx a_{d<2x} \delta \mu \frac{d-x}{\tau}, \quad a_{d<2x} = \frac{4\Omega_d T_c^3}{(2\pi)^d J^4} \alpha \frac{x \sin \left( \frac{(d-x)\pi}{x} \right)}{x} \quad \text{for } x < d < 2x$$

$$\tau \approx a_{d=2x} \delta \mu \ln \delta \mu, \quad a_{d=2x} = \frac{4\Omega_d T_c^3}{(2\pi)^d J^2} \alpha \frac{1}{x} \quad \text{for } d = 2x \quad (2.38)$$

$$\tau \approx a_{d>2x} \delta \mu \quad \text{for } d > 2x.$$
a_{d>2x} = \alpha T_c \left[ \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sinh^2 \left( \frac{J_0 - J(k)}{2T_c} \right)} + \frac{1}{\sinh^2 \left( \frac{J_0}{2T_c} \right)} \right]

where

\alpha = \left[ \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{J_0 - J(k)}{\sinh^2 \left( \frac{J_0 - J(k)}{2T_c} \right)} + \frac{J_0}{\sinh^2 \left( \frac{J_0}{2T_c} \right)} \right]^{-1}

(2.39)

is a finite, positive number.

The internal energy of the system reads

\[ U = -\mu(2\sigma + 1) + \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{\mu - J(k)}{2} \coth \left( \frac{\beta \mu}{2} \right) \]

\[ + \frac{\mu}{2} \coth \left( \frac{\beta \mu}{2} \right) - \frac{\mu^2}{2} \]

(2.40)

The specific heat close to the transition from the paramagnetic side can be written as

\[ C \approx \begin{cases} C_0 + \frac{\mu}{\sigma} & \text{for } x = 2x \lessgtr \frac{d}{d-x} \end{cases} \]

\[ C_1 \]

(2.41)

where

\[ C_0 = \frac{1}{4T^2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left[ \frac{\mu - J(k)}{\sinh^2 \left( \frac{\mu - J(k)}{2T} \right)} + \frac{\mu^2}{4T^2 \sinh^2 \left( \frac{\mu}{2T} \right)} \right]. \]

(2.42)

\[ C_1 = -2\sigma - 1 + \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{1}{2} \coth \left( \frac{\mu - J(k)}{2T} \right) - \frac{\mu - J(k)}{4T \sinh^2 \left( \frac{\mu - J(k)}{2T} \right)} \right\} \]

\[ + \frac{1}{2} \coth \left( \frac{\mu}{2T} \right) - \frac{\mu}{4T \sinh^2 \left( \frac{\mu}{2T} \right)} + \frac{\mu^2}{2 \mu^2}. \]

(2.43)

where \( a_d \) are the prefactors in Eq. (2.38) for the corresponding dimension. In the ordered phase \( \mu \) is stuck in its minimum value \( (\mu = J_0) \) for any temperature. Hence, \( C = C_0(\mu = J_0) \) in the ordered phase. The critical exponent \( \alpha \), see section 1 for definition of the critical exponents, is the expected one: \( \alpha = \frac{d - 2x}{d - x} \) for \( x < d < 2x \), and the mean field value \( \alpha = 0 \) holds for \( d > 2x \), which describes a jump in the specific heat.

Adding a small longitudinal field \( h \), the free energy reads

\[ \beta F = -\beta \mu(2\sigma + 1) + \int \frac{d^d k}{(2\pi)^d} \ln \left[ 2 \sinh \left( \frac{\beta \mu}{2} \right) \right] \]

\[ + \ln \left[ 2 \sinh \left( \frac{\beta \mu}{2} \right) \right] - \frac{\beta \mu^2}{2} - \frac{\beta h^2}{2(\mu - J_0)} \]

(2.44)
and the saddle point equation becomes

$$2(\sigma + 1) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{1 - e^{-\beta(\mu - J(k))}} + \frac{1}{1 - e^{-\beta\mu}} + \frac{\Gamma^2}{2\mu^2} + \frac{h^2}{2(\mu - J_0)^2}$$

(2.45)

By differentiating the free energy with respect to $h$ can be seen that the magnetization is

$$M_z = \frac{h}{\mu - J_0}.$$  

(2.46)

In the limit $h \to 0$, it is proportional to the square root of the occupation of the ground state, since by comparing Eq. (2.45) with Eq. (2.35) one finds $q = \frac{1}{N}\langle \sum_{k=0}^{x} \sum_{k=0}^{y} \rangle = \frac{M^2}{2^x}$. The factor $\frac{1}{2}$ appears because it is actually the real part of the spin field that is macroscopically occupied. This factor comes in from the transformation Eq. (2.6). From Eq. (2.45), we can approach the transition by sending the longitudinal field to zero at the critical temperature. The saddle point equation now accounts for the dependence of the chemical potential on the field. The calculation is similar, yielding finally

$$h \approx \left( \frac{2\Omega d T_c \pi}{(2\pi)^d J d/x} \sin \left( \frac{\pi(d-x)}{x} \right) \right)^{\frac{1}{2}} \delta \mu \frac{d-1}{2}$$  

for $x < d < 2x$,

$$h \approx \left[ \frac{2\Omega d T_c}{(2\pi)^d J 2x^2} \ln \delta \mu \right]^{\frac{1}{2}} \delta \mu \frac{3}{2}$$  

for $d = 2x$,  

$$h \approx \left( \int \frac{d^d k}{(2\pi)^d} \frac{1}{2T_c \sinh^2 \left( \frac{J_0 - J(k)}{2T_c} \right)} \right)^{\frac{1}{2}} \delta \mu^{3/2}$$  

for $d > 2x$.

Inserting these expressions in Eq. (2.46), the critical exponent $\delta$ is found to be $\delta = \frac{d-x}{2-x}$ for dimensions $x < d < 2x$ and the mean field value $\delta = 3$ is recovered for $d > 2x$. From the magnetization, the susceptibility follows as $\chi \approx \frac{1}{\delta \mu}$. Therefore we find $\gamma = \frac{x}{d-x}$ for $x < d < 2x$ and $\gamma = 1$ for $d > 2x$. In the ordered phase, the expansion of the saddle point equation, Eq. (2.45), for $T$ near the transition yields

$$M_z^2 \approx \frac{\tau}{2T_c} \left[ \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{J_0 - J(k)}{\sinh^2 \left( \frac{J_0 - J(k)}{2T_c} \right)} + \frac{J_0}{\sinh^2 \left( \frac{J_0}{2T_c} \right)} \right].$$

(2.48)

Since $M_z^2 \propto \tau$ for all dimensions where the phase transition exists, one has $\beta = \frac{1}{2}$.

For other critical exponents the correlation function is needed. It can be computed adding the right source term to the Hamiltonian, $\sum_{\lambda}(g_{\lambda}(\tau_q)\hat{\Sigma}_{\lambda} + g_{\lambda}^*(\tau_r)\hat{\Sigma}_{\lambda})$ and differentiating
where $T$ stands for the time ordered product. $Z(g^*, g)$ is the partition function of the Hamiltonian including the source terms and $Z_0$ is the partition function without them. This procedure is carefully explained in Ref. [NO98] giving the result

$$G(k, \tau) = G(k, \tau | k, 0) = e^{\tau \mu - J(k)} \left\{ \theta(\tau - \eta)(1 + n_k) + \theta(-\tau + \eta)n_k \right\}$$

where $\theta$ is a Heaviside step function and $\eta$ is a positive infinitesimal that indicates that the second term is the relevant one at $\tau = 0$. Furthermore,

$$n_k = \frac{1}{e^{\beta \Omega} - 1}$$

is the boson occupation probability, with $\Omega = (\mu - J(k))$. Fourier transforming this last result to Matsubara frequencies we get

$$G(k, \omega_n) = \frac{1}{J(k) - \mu - i\omega_n} \frac{k=0}{J' |k|^x + \delta \mu + i\omega_n} -1.$$  \hspace{1cm} (2.52)

So when we approach the critical point, we can see from this equation that $\xi^{-x} \propto \delta \mu$, then using Eq. (2.38) we find that $\nu = \frac{1}{d - x}$ for dimensions $x < d < 2x$ and $\nu = \frac{1}{4}$ for $d > 2x$. $\eta = 2 - x$ due to the fact that the couplings depend on $k^x$, and $z = x$ because in the denominator $\omega_n$ appears as a linear term. Both $\eta$ and $z$ are valid for any dimension.

This finally gives all the critical exponents of this finite temperature phase transition, which are exactly the same as in the classical model. This is expected from renormalization group arguments [Her76]. The critical behavior is controlled by a classical fixed point, therefore quantum dynamics does not play a qualitatively new role. Hence, the results are the same as in the classical spherical model [Joy72] or other models with different quantum dynamics considered at finite temperatures [Voj96].

**T=0 quantum phase transition**

In this section we analyze the behavior of the system at $T = 0$. As it can be seen from Eq. (2.32), when the transversal field increases, the temperature of the transition decreases till it reaches zero. This defines a quantum critical point $T_c = 0$ at $\Gamma = \Gamma_c$. In order to study it, an analogous procedure as before should be followed. At $T = 0$ everything happens to be rather simple. The free energy reduces to

$$F = -2\sigma \mu - \frac{\Gamma^2}{2\mu}.$$  \hspace{1cm} (2.53)
The saddle point equation turns out to be

\[ 2\sigma = \frac{\Gamma^2}{2\mu^2} \quad \text{in the paramagnetic phase.} \]

\[ 2\sigma = \frac{\Gamma^2}{2J_0^2} + q \quad \text{in the ferromagnetic phase.} \tag{2.54} \]

where \( q = \frac{1}{2}M_c^2 \) is the occupation of the ground state for small transversal fields. Since the temperature vanishes, quantum fluctuations, controlled by \( \Gamma \), give rise to the phase transition. Therefore, the parameter that should be used to control the transition is the transversal field and not the temperature. Then the proper analog of the specific heat will be proportional to the second derivative of the free energy with respect to the source of fluctuations, the transversal field:

\[ C_T \equiv \frac{\partial^2 F}{\partial \sigma^2} = -\frac{1}{\mu} + \frac{\Gamma}{\mu^2} \frac{\partial \mu}{\partial \Gamma}. \tag{2.55} \]

As before, we must know the dependence of \( \delta \mu (\mu = J_0 + \delta \mu) \) on the distance to the critical point \((\delta \Gamma)\) in the paramagnetic phase. The upper critical dimension will be \( d_{uc} = x \). Between those two dimensions, \( 0 < d < x \), the analysis of the saddle point equation, Eq. (2.32), yield that the product \( \beta \delta \mu \) goes to a finite, strictly positive value for \( T \to 0 \). This leads to a scaling form \( \delta \mu \propto \delta \Gamma^{x/d} \). On the \( T = 0 \) line, the analog of the specific heat goes continuously from the paramagnetic value \( C_T \approx -1/J_0 + \Gamma_c/J_0^2 \delta \Gamma^{(x-d)/d} \) to the simple ferromagnetic value \( C_T = -1/J_0 \). This implies that \( \alpha = (d - x)/d \). Adding a longitudinal field we find the dependence \( \delta \mu \propto \delta \Gamma^{2x/(d+2x)} \) bringing \( \delta = (d + 2x)/d \) and \( \gamma = x/d \). Subtracting the saddle point equation near the transition in the ferromagnetic phase from the one at the transition, we get \( q = (\Gamma^2 - \Gamma_c^2)/2J_0^2 \), which in the lowest order gives \( M_c^2 \propto \delta \Gamma \) and therefore, as always, \( \beta = \frac{1}{2} \). Equation (2.52) can be used here once it is transformed to real frequencies, \( \omega_n = \omega + in \eta \). Then we find \( \nu = 1/d, \eta = 2 - x \) and \( z = x \).

For dimensions \( d > x \), from eqs. (2.55,2.54), it can be seen that the analog of the specific heat has a jump discontinuity, implying \( \alpha = 0 \).
Summary of the critical exponents

Finally all the critical exponents together read

<table>
<thead>
<tr>
<th>Exponent</th>
<th>Classical $(x = d_{\text{lc}} &lt; d &lt; d_{\text{uc}} = 2x)$</th>
<th>Quantum $(d &lt; d_{\text{uc}} = x)$</th>
<th>Classical $(d &gt; 2x)$</th>
<th>Quantum $(d &gt; x)$</th>
</tr>
</thead>
<tbody>
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<td>$\alpha$</td>
<td>$d-2x$</td>
<td>$d-2x$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\frac{d}{2}$</td>
<td>$\frac{d}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\gamma$</td>
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<td>$\frac{d}{2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\delta$</td>
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<td>$\frac{d}{2}$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\nu$</td>
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<td>$\frac{d}{2}$</td>
<td>$\frac{1}{x}$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$2 - x$</td>
<td>$2 - x$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$z$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Table 2.1: Critical exponents for the Hamiltonian with momenta

We can see how the third column corresponds to the table 1.1 for the normal nearest neighbors interaction where $x = 2$.

### 2.3 Hamiltonians involving spins but not their momenta

In this section we extend the analysis of section 2.2 to a Hamiltonian which only depends on the spin operators $\hat{S}$ and not on the momenta $\Pi$. When going from the classical to the quantum model, we have to keep in mind that the Hamiltonian must be Hermitian. To be precise, the Hamiltonian we will deal with is

$$ H = -\frac{1}{2} \sum_{ij} J_{ij} \hat{S}_i \hat{S}_j - \sum_i \Gamma_i \hat{S}_i, \quad (2.57) $$

with real valued $J_{ij}$ and $\Gamma_i$ and where $\hat{S} = (\hat{\Sigma}^\dagger + \hat{\Sigma})/\sqrt{2}$ is the real part of the former spin field. Hence, the Hamiltonian does not involve momenta, but the spherical constraint does. see Eq. (2.11). This changes the symmetry of the problem from invariance under unitary transformations to orthogonal ones. In terms of boson creation and annihilation operators the coupling term for symmetric interactions, $J_{ij} = J_{ji}$, is proportional to $J_{ij}(2\hat{\Sigma}_i^\dagger \hat{\Sigma}_j + \hat{\Sigma}_i \hat{\Sigma}_j^\dagger + \hat{\Sigma}_i^\dagger \hat{\Sigma}_j^\dagger)$, where we can notice the symmetry of the problem. We will see that this model reproduces the $O(N)$ quantum rotor model.

We can get the partition function in many ways. A similar procedure using discrete imaginary time path integrals can be done as before. This gives us many problems due to the fact that creation and annihilation operators are projected on different time steps which is a lengthy and tedious procedure. However, the form of the Hamiltonian makes it suitable to apply a Bogoliubov transformation (for details see e.g. [Aue94]). Due to that, we get the same Hamiltonian as before but
1.2.3 Hamiltonians involving spins but not their momenta

with different coefficients. In order to do so, we must add the spherical constraint directly to the Hamiltonian. The procedure is as follows: first the couplings matrix is diagonalized as done before in Eq. (2.15) and then the $\hat{S}$'s are shifted to absorb the field term, i.e. $\hat{S}_\lambda \rightarrow \hat{S}_\lambda - \frac{\Gamma_{\lambda}}{\mu - J_{\lambda}}$. This finally gives

$$H = \sum_\lambda \left\{ (\mu - \frac{J_{\lambda}}{2}) \hat{\Sigma}_\lambda^\dagger \hat{\Sigma}_\lambda - \frac{J_{\lambda}}{4} (\hat{\Sigma}_\lambda^\dagger \hat{\Sigma}_\lambda^\dagger - \hat{\Sigma}_\lambda \hat{\Sigma}_\lambda) - \frac{\Gamma_{\lambda}^2}{2(\mu - J_{\lambda})} \right\} - Nm\mu \sigma. \quad (2.58)$$

Performing the Bogoliubov transformation it turns into

$$H = \sum_\lambda \left\{ \sqrt{\mu(\mu - J_{\lambda})} \hat{\alpha}_\lambda^\dagger \hat{\alpha}_\lambda - \frac{\Gamma_{\lambda}^2}{2(\mu - J_{\lambda})} \right\} - Nm\mu \sigma, \quad (2.59)$$

where the canonical transformation consists of an imaginary rotation

$$\hat{\Sigma}_\lambda = \cosh \theta_{\lambda} \hat{\alpha}_\lambda + \sinh \theta_{\lambda} \hat{\alpha}_\lambda^\dagger ; \quad \hat{\Sigma}_\lambda^\dagger = \cosh \theta_{\lambda} \hat{\alpha}_\lambda^\dagger + \sinh \theta_{\lambda} \hat{\alpha}_\lambda \quad (2.60)$$

where $\theta_{\lambda}$ are real and even in $\lambda \rightarrow -\lambda$. The condition to get 2.59 from 2.58 is that

$$\tanh 2\theta_{\lambda} = \frac{J_{\lambda}}{2(\mu - \frac{J_{\lambda}}{2})}. \quad (2.61)$$

Finally Eq. (2.59) is a Hamiltonian analogous to Eq. (2.14). So it can be diagonalized as explained, giving finally

$$\beta F = -\beta \mu m (\sigma + \frac{1}{2})$$
$$+ m \int dJ_\lambda \rho(J_\lambda) \left\{ \ln \left[ 2 \sinh \left( \frac{\beta}{2} \sqrt{\mu(\mu - J_\lambda)} \right) \right] - \frac{\beta \Gamma_{\lambda}^2}{2(\mu - J_\lambda)} \right\} \quad (2.62)$$

where we have put back the factor $m$ standing for the number of components of the vector spin. The saddle point equation is obtained as

$$\sigma + \frac{1}{2} = \int dJ_\lambda \rho(J_\lambda) \left\{ \frac{2\mu - J_\lambda}{4\sqrt{\mu(\mu - J_\lambda)}} \coth \frac{\beta}{2} \sqrt{\mu(\mu - J_\lambda)} + \frac{\beta \Gamma_{\lambda}^2}{2(\mu - J_\lambda)^2} \right\}. \quad (2.63)$$

At large temperatures these equations reduce to

$$\beta F = -\beta \mu m (\sigma + \frac{1}{2}) + m \int dJ_\lambda \rho(J_\lambda) \left\{ \ln \beta \sqrt{\mu(\mu - J_\lambda)} - \frac{\beta \Gamma_{\lambda}^2}{2(\mu - J_\lambda)} \right\} ; \quad (2.64)$$
$$\sigma + \frac{1}{2} = \int dJ_\lambda \rho(J_\lambda) \left\{ \frac{T}{2\mu} + \frac{T}{2(\mu - J_\lambda)} + \frac{\beta \Gamma_{\lambda}^2}{2(\mu - J_\lambda)^2} \right\}. \quad (2.65)$$

These equations are very similar to the standard ones of the classical spherical model (up to a factor 2), see Eq. (2.29), but they are only identical where they should be, namely at large $T$, where also $\mu \sim T$ is very large, see also Ref. [Joy72].
2.3.1 Ferromagnetic couplings in the presence of a transversal field

Analyzing the phase transition of the Hamiltonian that does not contain momenta is analogous to the previous case. We begin again by choosing the coupling term in the $z$ direction and the external field in the $x$ and we assume ferromagnetic couplings. The saddle point equation gives a phase transition via a macroscopic occupation of the ground state, which in the present case is a bit more complicated. The critical exponents are different, due to the fact that the symmetries of the system have changed. The free energy reads

$$\beta F = -\beta \mu (2\sigma + 1) + \int \frac{d^d k}{(2\pi)^d} \ln \left[ 2 \sinh \left( \frac{\beta}{2} \sqrt{\mu(\mu - J(k))} \right) \right]$$

$$+ \ln \left[ 2 \sinh \left( \frac{\beta \mu}{2} \right) \right] - \frac{\beta \Gamma^2}{2\mu}.$$ (2.66)

and the saddle point equation

$$4\sigma + 2 = \int \frac{d^d k}{(2\pi)^d} \frac{2\mu - J(k)}{2 \sqrt{\mu(\mu - J(k))}} \coth \left[ \frac{\beta}{2} \sqrt{\mu(\mu - J(k))} \right] + \coth \left( \frac{\beta \mu}{2} \right) + \frac{\Gamma^2}{\mu^2}.$$ (2.67)

We now analyze this model in detail.

**Finite temperature phase transition**

Following the same procedure as before we can find that the transition exists for $d > x$ and that the upper critical dimension is $d = 2x$. The critical temperature is the solution of

$$4\sigma + 2 = \int \frac{d^d k}{(2\pi)^d} \frac{2J_0 - J(k)}{2 \sqrt{J_0(J_0 - J(k))}} \coth \left[ \frac{\beta}{2} \sqrt{J_0(J_0 - J(k))} \right] + \coth \left( \frac{\beta J_0}{2} \right).$$ (2.68)

The dependence of the chemical potential in the temperature near the classical critical point reads
\[ \tau \approx a_{d<2x}\delta \mu \frac{d-x}{x}. \]

\[ a_{d<2x} = \alpha \frac{2\Omega_d T_c^3 \pi}{(2\pi)^d J^2 x \sin \left( \frac{(d-x)\pi}{x} \right)} \quad \text{for } x < d < 2x \]

\[ \tau \approx a_{d=2x}\delta \mu \ln \delta \mu. \]

\[ a_{d=2x} = \alpha \frac{2\Omega_d T_c^3}{(2\pi)^d J^2 x} \quad \text{for } d = 2x \quad (2.69) \]

\[ \tau \approx a_{d>2x}\delta \mu. \]

\[ a_{d>2x} = \alpha \left( \frac{\partial}{\partial \mu} \left\{ \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{2\mu - J(k)}{2\sqrt{\mu(\mu - J(k))}} \coth \left( \frac{\beta}{2} \sqrt{\mu(\mu - J(k))} \right) + \coth \left( \frac{3\mu}{2} \right) \right\} \right)_{\mu=J_0} \]

where

\[ \alpha = \left[ \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{2J_0 - J(k)}{2 \sinh^2 \left( \frac{\sqrt{J_0(J_0-J(k))}}{2T_c} \right)} + \frac{J_0}{\sinh^2 \left( \frac{J_0}{2T_c} \right)} \right]^{-1}. \quad (2.70) \]

The internal energy of the system reads

\[ U = -\mu(2\sigma + 1) + \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{\sqrt{\mu(\mu - J(k))}}{2} \coth \left( \frac{\beta}{2} \sqrt{\mu(\mu - J(k))} \right) + \frac{\mu}{2} \coth \left( \frac{\beta \mu}{2} \right) - \frac{\Gamma^2}{2\mu}. \quad (2.71) \]

The specific heat has the same expression as in Eq. (2.41) where \( a_d \) now correspond to the prefactors of Eq. (2.69) and with coefficients

\[ C_0 = \frac{1}{4T^2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{\mu(\mu - J(k))}{\sinh^2 \left( \frac{\sqrt{\mu(\mu - J(k))}}{2T_c} \right)} + \frac{\mu^2}{4T^2 \sinh^2 \left( \frac{\mu}{2T_c} \right)}. \quad (2.72) \]

\[ C_1 = -(2\sigma + 1) + \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \frac{2\mu - J(k)}{4\sqrt{\mu(\mu - J(k))}} \coth \left( \frac{\beta}{2} \sqrt{\mu(\mu - J(k))} \right) \right. \]

\[ \left. - \frac{2\mu - J(k)}{8T \sinh^2 \left( \frac{\sqrt{\mu(\mu - J(k))}}{2T_c} \right)} \right\} + \frac{1}{2} \coth \left( \frac{\mu}{2T} \right) - \frac{\mu}{4T \sinh^2 \left( \frac{\mu}{2T_c} \right)}. \quad (2.73) \]

This is analogous to the previous model and gives the same exponent, \( \alpha = \frac{d-2x}{d-x} \) for \( x < d < 2x \) and \( \alpha = 0 \) for \( d > 2x \). Adding a small magnetic field longitudinal to the couplings, the free energy becomes
\[ \beta F = -\beta \mu (2\sigma + 1) + \int \frac{d^d k}{(2\pi)^d} \ln \left[ 2 \sinh \left( \frac{\beta}{2} \sqrt{\mu J(k)} \right) \right] + \ln \left[ 2 \sinh \left( \frac{\beta \mu}{2} \right) \right] - \frac{\beta \Gamma^2}{2\mu} - \frac{\beta \hbar^2}{2(\mu - J_0)}. \]  

(2.74)

Therefore the magnetization is \( M_z = \frac{\hbar}{\mu - J_0} \), which is as before the square root of the occupation of the ground state \( q = \frac{1}{N} < S_z (|k| = 0)^2 >= M_z^2 \). The saddle point equation is now

\[ 2(2\sigma + 1) = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{2\mu - J(k)}{2\sqrt{\mu J(k)}} \coth \left\{ \frac{\beta}{2} \sqrt{\mu (\mu - J(k))} \right\} \]

(2.75)

\[ + \coth \left( \frac{3\mu}{2} \right) + \frac{\Gamma^2}{\mu^2} + \frac{\hbar^2}{(\mu - J_0)^2} \cdot \]

With all these and following the algebra of the previous section one finds the same critical exponents for the magnetization for the same dimensions since we are in the classical critical point.

The time ordered correlation function \( \langle \hat{S}_{\tau}^x (\tau) \hat{S}_{-\tau}^x (0) \rangle \) differs from the previous one, Eq. (2.50) since in this case the \( \hat{S} \) are not the variables that diagonalize the Hamiltonian in Eq. (2.59). We must write \( \hat{S} \) in terms of \( \hat{\alpha} \) and then compute the correlations. This brings

\[ G(k, \tau | k, 0) = \frac{J(k)}{4 \sqrt{\mu (\mu - J(k))}} \left\{ n_k \cosh \left[ \tau \sqrt{\mu (\mu - J(k))} \right] \right. \]

\[ + e^{-\tau/\sqrt{\mu (\mu - J(k))}} \left. \right\}. \]

(2.76)

where

\[ n_k = \frac{1}{e^{\mu \Omega} - 1} \]

(2.77)

is just like before in Eq. 2.51 but with \( \Omega = \sqrt{\mu (\mu - J(k))} \). The correlation function in frequency space reads

\[ G(k, i\omega_n) = \frac{-J(k)}{2[\omega_n^2 - \mu (\mu - J(k))]} . \]

(2.78)

When approaching the critical point we find that \( \xi^{-\nu} \propto \delta \mu \) as before and we get the same critical exponent \( \nu = \frac{1}{d-x} \) for dimensions \( x < d < 2x \) and \( \nu = \frac{1}{x} \) for \( d > 2x \). Since couplings appear in the same way as before we also get the same critical exponent. \( \eta = 2-x \) for all dimensions. The difference appears in the
dynamical critical exponent. Here \( \omega_n \) appears squared, therefore \( z = x/2 \). Here we see how the model reproduces the critical exponents of the rotor model as in Ref. [Voj96] bringing thus a different behavior at the \( T = 0 \) quantum critical point from the model of section 2.2.3.

**\( T=0 \) quantum phase transition**

In this case the \( T = 0 \) phase transition the dynamical critical exponent \( z = \frac{x}{2} \) is smaller than \( z = x \) of the previous section. The free energy reads

\[
F = -\mu (2\sigma + 1) + \int \frac{d^d k}{(2\pi)^d} \frac{\sqrt{\mu (\mu - J(k))}}{2} + \frac{\mu}{2} - \frac{\Gamma^2}{2\mu}
\]

and the saddle point is set by

\[
4\sigma + 1 = \int \frac{d^d k}{(2\pi)^d} \frac{2\mu - J(k)}{2\sqrt{\mu (\mu - J(k))}} + \frac{\Gamma^2}{\mu^2}.
\]

We find that the transition exists for dimensions larger than \( d > \frac{x}{2} \) and \( d = \frac{3x}{2} \) is the upper critical dimension. The chemical potential depends on the source of fluctuations \( \delta \Gamma = \Gamma - \Gamma_c \) as

\[
\delta \Gamma \approx a_{d<3x/2} \delta \mu \frac{2^{d-x} \mu}{2x}, \quad a_{d<3x/2} = \frac{\Omega_d}{2(2\pi)^d J_0^{d/2}} \frac{\Gamma (\frac{3}{2} - \frac{d}{x}) \Gamma (\frac{d}{x})}{(2d - x) x \sqrt{\pi}} \quad \text{for } \frac{x}{2} < d < \frac{3x}{2}
\]

\[
\delta \Gamma \approx a_{d=3x/2} \delta \mu \ln \delta \mu, \quad a_{d=3x/2} = \frac{\Omega_d}{(2\pi)^d 8x J_0^{d/2}} \quad \text{for } d = \frac{3x}{2}
\]

\[
\delta \Gamma \approx a_{d>3x/2} \delta \mu, \quad a_{d=3x/2} = \frac{J_0^2}{2} \left[ \frac{2\Gamma_c^2}{J_0^3} - \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{J_0 (J_0 - J(k))}} - \frac{2J_0 - J(k)}{4 (J_0 (J_0 - J(k)))^{3/2}} \right]
\]

where the \( \Gamma_c \)'s on the right hand side of the first equality are Euler's Gamma functions. The specific heat (see eq. (2.55) ) coming from the disordered region will behave as

\[
C_{\Gamma} \approx \left\{ \begin{array}{ll}
\frac{-1}{J_0} - \frac{4\Gamma_c^2}{J_0^2 (2d-x)} \delta \Gamma^{-2d+3x/2} & \text{for } \frac{x}{2} < d < \frac{3x}{2} \\
\frac{-1}{J_0} + \frac{2\Gamma_c^2}{J_0^2} \left( \frac{2r^2}{J_0} - a_{d>3x/2} \right)^{-1} & \text{for } d > \frac{3x}{2}
\end{array} \right.
\]

where \( a_{d} \) is the prefactor in Eq. (2.81) for the proper dimension. Coming from the ordered region, conversely, \( C_{\Gamma} \approx \frac{-1}{J_0} \). Therefore \( \alpha = \frac{2d-3x}{2d-x} \) for \( \frac{x}{2} < d < \frac{3x}{2} \) and \( \alpha = 0 \) for \( d > \frac{3x}{2} \).

The dependence of a small longitudinal field on the chemical potential, in case the transversal field is at its critical value, reads...
Quantum spherical model

\[
h \approx a_{d<\frac{3x}{2}} \delta \mu \frac{2d+3x}{2d-x} \quad \text{for } \frac{x}{2} < d < \frac{3x}{2}.
\]

\[
h \approx \left[ a_{d=\frac{3x}{2}} \ln \delta \mu \right]^{\frac{1}{2}} \delta \mu^{\frac{1}{2}} \quad \text{for } d = \frac{3x}{2}.
\]

\[
h \approx \left( -\frac{2\Gamma^c}{J^3} - a_{d>\frac{3x}{2}} \right)^{\frac{1}{2}} \delta \mu^{\frac{1}{2}} \quad \text{for } d > \frac{3x}{2}.
\]

From these equations and the ones for the magnetization and the susceptibility, we can find that \( \delta = \frac{2d+3x}{2d-x} \) and \( \gamma = \frac{2x}{2d-x} \) for \( \frac{x}{2} < d < \frac{3x}{2} \), while \( \delta = 3 \) and \( \gamma = 1 \) for \( d > \frac{3x}{2} \). As before, \( \beta = \frac{1}{2} \) for every dimension. For the correlation function the calculation is the same as in the finite temperature case, projected into real time, the exponents are \( \nu = \frac{2}{2d-x} \) for dimensions \( \frac{x}{2} < d < \frac{3x}{2} \) and \( \nu = \frac{1}{x} \) above the critical dimension, \( \eta = 2 - x \) and \( z = \frac{x}{2} \) for all dimensions.

Summary of the critical exponents

Finally, the critical exponents read

<table>
<thead>
<tr>
<th>Exponent</th>
<th>Classical ((x = d_{lc} &lt; d &lt; d_{uc} = 2x))</th>
<th>Quantum ((\frac{x}{2} &lt; d &lt; \frac{3x}{2}))</th>
<th>Classical ((d &gt; 2x)) Quantum ((d &gt; \frac{3x}{2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>(\frac{d-2x}{d-x}) (\frac{2d-3x}{2d-x})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(\beta)</td>
<td>(\frac{1}{2}) (\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(\gamma)</td>
<td>(\frac{d-x}{d-x}) (\frac{2d-x}{2d-x})</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(\delta)</td>
<td>(\frac{d+x}{d-x}) (\frac{2d+x}{2d-x})</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(\nu)</td>
<td>(\frac{d-x}{d-x}) (\frac{2d-x}{2d-x})</td>
<td>(\frac{1}{x})</td>
<td></td>
</tr>
<tr>
<td>(\eta)</td>
<td>(2-x) (2-x)</td>
<td>(2-x)</td>
<td></td>
</tr>
<tr>
<td>(z)</td>
<td>(\frac{d}{2}) (\frac{x}{2})</td>
<td>(\frac{x}{2})</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Critical exponents for the Hamiltonian without momenta

2.4 Generalization and mapping from Heisenberg spins

In this section we generalize the two preceding Hamiltonians and we map the Heisenberg model onto the spherical model. In a more compact way, we can write the former Hamiltonians in absence of external field as

\[
H = -\sum_{ij} \left( A_{ij} \hat{S}_i^+ \hat{S}_j + \frac{B_{ij}}{2} \left[ \hat{S}_i^+ \hat{S}_j^+ \right] \right).
\]

If the matrices \( A_{ij} \) and \( B_{ij} \) can be diagonalized simultaneously, the techniques from previous sections can be used. The free energy reads
\[ \beta F = - \beta \mu m(\sigma + \frac{1}{2}) + \frac{m}{N} \sum_{\lambda} \left[ \frac{\beta A_{\lambda}}{2} + \ln \left( 2 \sinh \left( \frac{\beta}{2} \sqrt{(\mu - A_{\lambda})^2 - B_{\lambda}^2} \right) \right) \right] \]  

(2.85)

where \( \mu \) satisfies the saddle point equation

\[ \sigma + \frac{1}{2} = \frac{1}{N} \sum_{\lambda} \frac{\mu - A_{\lambda}}{2 \sqrt{(\mu - A_{\lambda})^2 - B_{\lambda}^2}} \coth \left( \frac{\beta}{2} \sqrt{(\mu - A_{\lambda})^2 - B_{\lambda}^2} \right). \]  

(2.86)

The coefficient \( B_{ij} \) in Eq. (2.84) is the responsible for a change in the symmetries of the problem. If \( B_{ij} \) is zero, the action is symmetric under unitary transformations while if it is non-zero the symmetry is reduced to orthogonal.

The mapping from Heisenberg spins comes as follows. The Hamiltonian can be written in terms of Schwinger bosons [Aue94]. The Schwinger boson transformation for SU(2) spins reads

\[ S^+ = a^+_1 a^+_2, \quad S^- = a^+_1 a^+_2, \quad S^z = \frac{1}{2} (a^+_1 a^+_2 - a^+_2 a^+_1). \]  

(2.87)

This can be generalized to SU(\( N \)) spins and expand around the large-\( N \) limit [AA88]. In a path integral formalism for ferromagnetic interactions

\[ H = -\frac{1}{2} \sum_{ij} J_{ij} S_i \cdot S_j \rightarrow -\frac{1}{2N} \sum_{ij,m} J_{ij} a^*_{jm} a_{im} a^*_{in} a_{jn}. \]  

(2.88)

where \( i,j \) represent lattice sites and \( m,n \) represent the boson flavor. The Hilbert space spanned by Schwinger bosons is much larger than the one given by Heisenberg spins. The constraint needed to restrict it to the physical Hilbert space is that the number of Schwinger bosons at each site has to be kept fixed \( \sum_{m} n_{m} = NS \). This is inserted into the formalism in the same way as we have done it with the spherical constraint, a Lagrange multiplier \( \mu_i(\tau) \) appears. The biquadratic terms can be decoupled by a Hubbard-Stratonovich transformation (see e.g. [NO98]). In the case of a ferromagnet, the transformation at each time step and for each flavor in the path integral reads

\[ \exp \left\{ -\frac{\epsilon}{2N} \sum_{ij} J_{ij} a^*_{j} a_{i} a^*_{i} a_{j} \right\} \]

\[ \times \int \prod_{i,j} dQ_{ij} \exp \left\{ \frac{\epsilon N}{2} \sum_{i,j} Q_{ij} J_{ij} Q_{ji} - \frac{\epsilon}{2} \sum_{i,j} Q_{ij} J_{ij} a^*_{i} a_{i} \right\} \]  

(2.89)

where a field \( Q_{ij}(\tau) \) has been generated. In the mean field approximation, one puts \( Q_{ij}(\tau) = Q \) and \( \mu_i(\tau) = \mu \). Hence, one gets up to a non interesting constant
\[ H_{MF}^{N-B}(\mathcal{N}) = \sum_{i,m} \mu a_{im}^+ a_{im} - Q \sum_{ij,m} J_{ij} a_{jm}^+ a_{im} + \frac{NQ^2}{2} \sum_{ij} J_{ij} - \mathcal{N} S \mu \] (2.90)

where we have already added the Schwinger boson constraint \( \sum_{i,m} a_{im}^+ a_{im} = \mathcal{N} S \).

The free energy per particle reads

\[ \beta F = \frac{\mathcal{N}}{N} \sum_{\mathbf{k}} \ln(1 - e^{-\beta (\mu - QJ(\mathbf{k})}) + \frac{\beta NQ^2}{2} J(\mathbf{k} = 0) - \beta S \mu \mathcal{N} \] (2.91)

and the saddle point equations are

\[ \frac{1}{\mathcal{N}} \sum_{\mathbf{k}} n_{\mathbf{k}} = S. \] (2.92)

\[ \frac{1}{\mathcal{N}} \sum_{\mathbf{k}} J(\mathbf{k}) n_{\mathbf{k}} = QJ(\mathbf{k} = 0). \] (2.93)

where \( n_{\mathbf{k}} \) is the boson occupation number Eq. (2.51) with \( \omega = \mu - QJ(\mathbf{k}) \). Subtracting the two saddle point equations we can see that for large \( S \) and small \( T \) we can approximate \( Q \approx S \) recovering then the spherical model Eq. (2.27,2.28) for zero external field or Eq. (2.85,2.86) for \( B_{ij} = 0 \). From this approach we thus see that the free energy of a SU(\( \mathcal{N} \)) Heisenberg ferromagnet for large \( \mathcal{N} \) is formally the same as the quantum spherical model proposed in Eq. (2.14) in the thermodynamic limit, so when the radius of the hypersphere that defines the model (\( \mathcal{N} \) in eq. (2.11)) is also very large. Thus the large \( \mathcal{N} \) limit is analogous to Stanley’s large spin dimensionality limit.

In the case of an SU(\( \mathcal{N} \)) antiferromagnet the procedure is more or less the same but the symmetries are different. The lattice is divided in two sublattices \( A,B \). In one of the sublattices a spin rotation is performed that allows us to write the Hamiltonian in the form [AA88]

\[ H = \frac{1}{2} \sum_{ij} J_{ij} S_i \cdot S_j - \frac{1}{2N} \sum_{ij,mn} J_{ij} a_{im}^* a_{jm}^* a_{jn} a_{jn}. \] (2.94)

Performing a Hubbard-Stratonovich transformation as before, the Hamiltonian with the Schwinger boson constraint in the mean field approximation finally reads

\[ H_{MF}^{AF-M-B}(\mathcal{N}) = \sum_{i,m} \mu a_{im}^+ a_{im} - \frac{Q}{2} \sum_{ij,m} J_{ij} (a_{im}^+ a_{jm}^+ + a_{im} a_{jm}) \]

\[ + \frac{NQ^2}{2} \sum_{ij} J_{ij} - \mathcal{N} S \mu. \] (2.95)
It is important to stress that here the SU(\(N\)) symmetry has been reduced to a residual O(\(N\)). The free energy per particle reads

\[
\beta F = \frac{N}{N} \sum_{k} \ln \left( 2 \sinh \left( \frac{\beta}{2} \sqrt{\mu^2 - Q^2 J^2(k)} \right) \right) - \beta N \left( S + \frac{1}{2} \right) \mu + \frac{\beta N Q^2}{2} J(k = 0)
\]

and the saddle point equations read

\[
\frac{1}{N} \sum_{k} \frac{\mu}{\sqrt{\mu^2 - Q^2 J^2(k)}} \left( n_k + \frac{1}{2} \right) = S + \frac{1}{2}, \tag{2.97}
\]

\[
\frac{1}{N} \sum_{k} \frac{J^2(k)Q}{\sqrt{\mu^2 - Q^2 J^2(k)}} \left( n_k + \frac{1}{2} \right) = QJ(k = 0), \tag{2.98}
\]

where \(n_k\) is Eq. (2.51) for \(\omega = \sqrt{\mu^2 - Q^2 J^2(k)}\). Subtracting the first equation times \(\mu\) from the second times \(Q\) we get

\[
\frac{1}{N} \sum_{k} \sqrt{\mu^2 - Q^2 J^2(k)} \left( n_k - \frac{1}{2} \right) = \mu \left( S + \frac{1}{2} \right) - Q^2 J(k = 0). \tag{2.99}
\]

The first term is proportional to \(T\), so for very small temperatures and very large \(S\), near the transition where \(\mu \approx QJ(k = 0)\), we can approximate \(Q \approx S + \frac{1}{2}\). Then eqs. (2.96,2.97) are analogous to eqs. (2.85,2.86) for \(A_{ij} = 0\). This will have the same critical behavior as the model in section 2.3 due to the fact that it comes from the term \(\coth[\sqrt{\mu - J(k)}]/\sqrt{\mu - J(k)}\) which also appears here due to the equality \(2n_k + 1 = \coth[\sqrt{\mu + QJ(k)}(\mu - QJ(k))]\).

2.5 Summary and discussion

In this chapter we have discussed a way of working with quantum spherical spin models using path integrals and coherent states. Some examples of the use of this formalism are given, eqs. (2.14,2.57), and their critical phenomena are studied. We propose a comparison with SU(\(N\)) Heisenberg models that gives a geometrical interpretation to the quantum spherical spins. The spherical constraint we use, fixes the number of spin quanta \(\hat{S}\), Eq. (2.11); in other words, it fixes both the average length square of the spin operator, \(\hat{S}^2\), and the one of its conjugate momentum, \(\hat{P}^2\). The usual version of the quantum spherical model, on the contrary, involves only the spin part \(\hat{S}\). The presence of momenta in the spherical constraint allows to consider Hamiltonians that have no kinetic term, since it can be induced by the constraint, a fact that can change the symmetries of the problem, and due to that, the critical behavior.
The Hamiltonian in Eq. (2.14) yields an action invariant under unitary transformations. It brings formally the same free energy as a SU($\mathcal{N}$) Heisenberg ferromagnet in the limit of large $\mathcal{N}$. The other Hamiltonian studied, Eq. (2.57), brings an action invariant under orthogonal transformations: it gives the same critical behavior as an SU($\mathcal{N}$) Heisenberg antiferromagnet in the limit of large $\mathcal{N}$, which is, in its turn, analogous to an O($\mathcal{N}$) nonlinear $\sigma$-model or quantum rotor model [Voj96, Sac99].

The main difference between these models lies in the dynamical critical exponent $\nu$ which brings a different behavior at the quantum critical point. Classical critical phenomena are, as expected, the same in both models and equal to those of the classical spherical model.

In the formulation of the model, the strict spherical constraint has been used where fluctuation on the particle number are not allowed. The constraint is added to the action via a Lagrange multiplier. The strict approach has to be abandoned when we integrate this Lagrange multiplier since we cannot perform the integration exactly and we use the saddle point approximation. In this step, we automatically allow fluctuations on the particle number and therefore the constraint ends being satisfied only in average. These effects are immaterial in the considered thermodynamic limit, but do enter finite size corrections.

When performing the analogy between the spherical model and the Heisenberg one, we did not include the external field. Mapping such term was not possible since in the spherical model the external field comes linearly, as a source term. In spite of that, the phase diagram follows the expected behavior for a spin model with an external transversal field. The critical exponents for the classical and the quantum model are the ones expected by renormalization group arguments [Her76]. The quantum critical point behaves as the classical one for dimensions $D_{\text{quant}} = d_{\text{class}} + \nu$ where $\nu$ is the dynamical critical exponent.

Many works in literature are closely related to the one described here. For instance, starting from the SU($\mathcal{N}$) Heisenberg model and to do the already stated large $\mathcal{N}$ limit to get to a solvable model. In order to have a transversal field that competes with the ordering of the interacting spins one could introduce anisotropy in the model. A study of this type has been done for 2 dimensions by Timm et al. [TJ00] in terms of Schwinger bosons and in terms of Holstein-Primakoff bosons by Kaganov et al. [KC87] for any dimension. The anisotropy term brings a residual spin symmetry describing Ising or XY spins. The phase transition depends on the type of this residual symmetry; an additional transversal field decreases the transition temperature towards zero giving a quantum critical point, result qualitatively reproduced by our model.

Additionally, Sachdev and Bhatt [SB90] represented pairs of spins in a square lattice with a bond representation; they form either a singlet or a triplet. These elements can be written down in terms of the canonical “Schwinger boson” representation of the generators of $SU(2) \otimes SU(2) = SO(4)$. Since a couple of spins either form a singlet or a triplet, a constraint must be added $s^\dagger s + \sum_\alpha t_\alpha^\dagger t_\alpha = 1$, where $s$ represents the singlet annihilation operator, and $t_\alpha$ represents a triplet annihilation operator in the $\alpha$ direction. Sachdev and Bhatt study using this formalism systems with interactions up to third nearest neighbors. They make the further assumption
that the singlet part condenses and replace the $s$ operator by its mean field value $\langle s \rangle = \bar{s}$, and solve the rest for the triplets. The final Hamiltonian is very close to our Eq. (2.57), or, better, the generalization of our model Eq. (2.84) with the proper couplings. A minor difference is the role played by the non-constant mean value of the singlet part.