Spin bosons and spin glasses
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Solvable model

The most general version of the spin boson model is intractable analytically. There are many different approximations to solve the dynamics defined in Eqs. (3.1-3.4), see e.g. [L87, HG98]. Our approach consists in making all approximations in the definition of the model. Then we end up having a model with restricted validity, which, however, has an exact solution under the constraints imposed. This way we have a better control on the approximations we perform. That is very important for our purposes as it will be described later 4.4.1.

4.1 The model

In this section we will describe the precise version we take from the generic spin-boson model and its motivation. In the following sections, we will show how to get through the analytic details to find the dynamics. As before, and in general for open systems we start from Eq. (3.1). The two level system (spin $\frac{1}{2}$) is described by the Hamiltonian:

$$H_S = \frac{\varepsilon}{2} \hat{\sigma}_z, \quad \varepsilon \equiv \hbar \Omega. \quad (4.1)$$

so a spin system with energy levels $\pm \frac{\varepsilon}{2}$. This basically represents a spin under the action of a constant magnetic field where we take the direction of the magnetic field as the $z$ direction.

The spin interacts with a thermal bath which is a set of harmonic oscillators as before in Eq. (3.3). Later on, the thermodynamic limit for the bath will be taken.

The next important point is to specify the interaction between the spin and the bath. In general, we should use Eq. (3.4) since the bath perturbs/damps the spin in all directions. In this respect, for two-level systems, one distinguishes three types of relaxation processes and the corresponding time scales:

i) $T_2$-time scale, related to the relaxation of the average transversal components $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_y \rangle$ of the spin (decoherence). Note that the very notion of the
transversal components is defined by the form (4.1) of the spin Hamiltonian.

\( T_1 \)-time scale, related to the relaxation of \( \langle \sigma_z \rangle \).

It is customary to have situations, where

\[ T_2 \ll T_1. \tag{4.2} \]

the main physical reason being that the transversal components are not directly related to the energy of the spin and their decay is not directly connected with energy exchange between the spin and the bath.

Our basic assumption on the relaxation times is (4.2). Moreover, to facilitate the solution of the model we will disregard \( T_2 \) time as being very large, thereby restricting the times of our interest to those much shorter than \( T_1 \). The interaction Hamiltonian is thus chosen such that it induces only transversal relaxation:

\[ H_1 = \frac{\hbar}{2} \hat{\hat{X}} \dot{\dot{\sigma}}_z, \quad \hat{\dot{X}} = \sum_k g_k (a_k^\dagger + a_k). \tag{4.3} \]

where \( g_k \) are the coupling constants to be specified later, and where \( \hat{\dot{X}} \) is the collective coordinate operator of the bath.

A third relaxation time \( T_2^* \) has a completely different origin. It only appears when dealing with an ensemble of non-interacting spins each having Hamiltonian (4.1) with a randomly distributed energy \( \varepsilon \) (dephasing). This randomness can be caused by inhomogeneous fields contributing into energy \( \varepsilon \), or by action of environment, e.g., chemical shifts for nuclear spins or effective g-factors for electronic spins in a quantum dot. Due to this dispersion in \( \varepsilon \), different spins will precess at different speed causing a relaxation in the transversal components of the collective magnetization, i.e., the average of \( \langle \sigma \rangle \) over the whole ensemble. If the distribution of \( \varepsilon \) is sufficiently wide, the relaxation \( T_2^* \) is directly related to its dispersion.

As a starting point we will consider situations where \( T_2^* \) is infinity, i.e., we consider only one spin or identical copies of one spin. In sections 5.4 and 6.2 we will consider work extraction and spin cooling respectively in the case of arbitrary short \( T_2^* \). In order to cancel the effect of this relaxation we will use the spin-echo technique [Hah50], bringing qualitatively the same results as with only one spin. The effect, however, is reduced quantitatively.

The model up to now described has been considered in literature [Luc90, Unr95, PSE96, VL98, SL04] to study decoherence. This model describes pure decoherence, no damping is considered, since \( \langle \sigma_{\pm} \rangle \) decay but \( \langle \sigma_z \rangle \) does not. In order to study work extraction and spin cooling we need to use external fields to perturb the system. These external fields will be in the form of pulses in order to keep the model analytically solvable. The approximations needed and the formalism used is described in section 4.2.
4.1.1 Heisenberg equations and their exact solution

The Heisenberg equations for the operators $\hat{\sigma}_z(t)$ and $\hat{a}_k(t)$ read from Eqs. (3.1, 3.3, 4.1, 4.3):

$$\dot{\hat{\sigma}}_z = 0, \quad \hat{\sigma}_z(t) = \hat{\sigma}_z(0).$$

$$\dot{\hat{a}}_k = \frac{i}{\hbar} [H, \hat{a}_k] = -i\omega_k \hat{a}_k - \frac{i}{2} g_k \hat{\sigma}_z.$$  

Eqs. (4.4, 4.5) are solved as

$$\hat{a}_k(t) = e^{-i\omega_k t} \hat{a}_k(0) + \frac{g_k \hat{\sigma}_z}{2\omega_k} (e^{-i\omega_k t} - 1),$$

and then

$$\hat{X}(t) = \hat{\eta}(t) - \hat{\sigma}_z G(t).$$

where

$$G(t) \equiv \sum_k \frac{g_k^2}{\omega_k} (1 - \cos \omega_k t).$$

quantifies the reaction of the spin on the collective operator of the bath, and where we denoted

$$\hat{\eta}(t) = \sum_k g_k [\hat{a}_k(0) e^{i\omega_k t} + \hat{a}_k(0) e^{-i\omega_k t}].$$

for the quantum noise operator $^1$. Recalling the standard relations

$$\hat{\sigma}_\pm = \hat{\sigma}_x \pm i \hat{\sigma}_y, \quad [\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \quad \hat{\sigma}_z \hat{\sigma}_\pm = \pm \hat{\sigma}_\pm,$$

and using (4.7) and $[\hat{X}(t), \hat{\sigma}_\pm(t)] = 0$ since they belong to different Hilbert spaces.— one derives

$$\dot{\hat{\sigma}}_\pm = \frac{i}{\hbar} [H, \hat{\sigma}_\pm] = \pm i \left( \Omega + \hat{X} \right) \hat{\sigma}_\pm = i \left( \pm \Omega \pm \hat{\eta}(t) - G(t) \right) \hat{\sigma}_\pm. \quad (4.11)$$

These equations are solved as:

$$\hat{\sigma}_\pm(t) = \exp \left[ \pm i\Omega t - iG_1(t) \right] \hat{\Pi}_\pm(0, t) \hat{\sigma}_\pm(0). \quad (4.12)$$

$$\hat{\Pi}_\pm(t_0, t_1) \equiv T \exp \left[ \pm i \int_{t_0}^{t_1} ds \hat{\eta}(s) \right]. \quad (4.13)$$

$$F(t) \equiv \int_0^t ds \ G(s) = \sum_k \frac{g_k^2}{\omega_k} \left( t - \frac{\sin \omega_k t}{\omega_k} \right). \quad (4.14)$$

$^1$Note that the commutator of the quantum noise is a c-number: $[\hat{\eta}(t), \hat{\eta}(s)] = -2i \sum_k g_k^2 \sin \omega_k (t - s) = -2i\hbar \hat{\eta}(t - s)$.
II.4. Solvable model

where $T$ stands for the time-ordering operator.

It is also useful for later to consider the Heisenberg time evolution of the $\hat{\Pi}_\pm$ operator:

$$e^{i\hat{\Pi}_\pm \Delta t/\hbar} e^{-i\hat{\Pi}_\pm \Delta t/\hbar} = \hat{\Pi}_\pm(t_1 + t, t_2 + t) \exp[\pm i\hat{\sigma}_z \chi(t_1, t_2, t)]$$

where

$$\chi(t_1, t_2, t) \equiv F(t_2) - F(t_1) + F(t_1 + t) - F(t_2 + t).$$

with $F(t)$ defined in Eq. (4.14).

4.1.2 Factorized initial conditions

We assume that initially, at the moment $t = 0$, the bath and the spin were in the following factorized state:

$$\rho(0) = \rho_S(0) \otimes \rho_B(0) = \rho_S(0) \otimes \frac{e^{-\beta H_B}}{\operatorname{tr} e^{-\beta H_B}}$$

where $\rho_S(0)$ is the initial density matrix of the spin, and where the bath is initially at equilibrium with temperature $T = 1/\beta$.

Factorized initial conditions are adequate when the spin was prepared independently from the equilibrium bath and then is brought in contact to the bath at the initial time. For example, injection of an electronic spin into a quantum dot, or creation of an exciton by external radiation. Yet another situation where factorized initial conditions are adequate is selective measurement of $\hat{\sigma}_z$ by an external apparatus. In this case $\rho_S(0)$ is an eigenstate of $\hat{\sigma}_z$ upon which the selection was done. Non-factorized initial states are commented upon below, in section 4.1.3.

The equilibrium relation

$$\langle \hat{a}_k^\dagger(0)\hat{a}_k(0) + \hat{a}_k(0)\hat{a}_k^\dagger(0) \rangle = \coth \left( \frac{\beta \omega_k}{2} \right).$$

implies that the quantum noise is stationary Gaussian operator with

$$\langle \hat{n}(t) \rangle = 0.$$ 

and with the following time-ordered correlation function:

$$K_T(t-t') = \langle T [\hat{n}(t)\hat{n}(t')] \rangle =$$

$$\sum_k g_k^2 \left[ \coth \left( \frac{\beta \omega_k}{2} \right) \cos \omega_k(t-t') - i \operatorname{sgn}(t-t') \sin \omega_k(t-t') \right].$$

\footnote{It is useful to note that this process of bringing spin in contact to the bath need by itself not be connected with any fundamental energy cost. Imagine, fo example, a sudden switching of the interaction Hamiltonian $H_I = \frac{1}{2} \hat{X} \hat{\sigma}_z$. Since in the equilibrium state of the bath $\langle \hat{X} \rangle = 0$, the work done for the realization of this switching is zero.}
where the average $\langle \ldots \rangle$ is taken over the initial state (4.17). It can be written as

$$K_T(t) = K(t) - i \dot{G}(t). \quad (4.21)$$

where

$$K(t - t') = \mathcal{R} K_T(t - t') = \frac{1}{2} \langle \dot{\eta}(t) \dot{\eta}(t') + \dot{\eta}(t') \dot{\eta}(t) \rangle$$

$$= \sum_k g_k^2 \coth \left( \frac{\beta \hbar \omega_k}{2} \right) \cos \omega_k(t - t'). \quad (4.22)$$

is the symmetrized correlation function.

Since $\dot{\eta}(t)$ is a gaussian random operator, one can use Wick's theorem for decomposing higher-order products: Any correlation of an odd number of $\dot{\eta}$'s vanishes. A correlation of an even number of $\dot{\eta}$'s is equal to the sum of products of pair correlations, the sum being taken over all pairings. For example:

$$\langle T \dot{\eta}(t_1) \dot{\eta}(t_2) \dot{\eta}(t_3) \dot{\eta}(t_4) \rangle = \langle T \dot{\eta}(t_1) \dot{\eta}(t_2) \rangle \langle T \dot{\eta}(t_3) \dot{\eta}(t_4) \rangle$$

$$+ \langle T \dot{\eta}(t_1) \dot{\eta}(t_3) \rangle \langle T \dot{\eta}(t_2) \dot{\eta}(t_4) \rangle + \langle T \dot{\eta}(t_1) \dot{\eta}(t_4) \rangle \langle T \dot{\eta}(t_2) \dot{\eta}(t_3) \rangle.$$  

Note that the similar Wick-decomposition of $\langle T \dot{\eta}(t_1) ... \dot{\eta}(t_{2k}) \rangle$ will be a sum of $(2k-1)!! = (2k-1)(2k-3)...3$ terms.

Due to the factorized structure (4.17) of the initial state, the common averages of $\dot{\eta}$ and various spin operators can be taken independently. For example, averaging Eq. (4.12) and using Wick's theorem together with $k! 2^k (2k-1)!! = (2k)!$ one gets:

$$\langle \dot{\sigma}_\pm(t) \rangle = e^{\pm i \Omega t - i G_1(t)} \left\langle \dot{\Pi}_\pm(0,t) \right\rangle \langle \dot{\sigma}_\pm(0) \rangle = e^{\pm i \Omega t - \xi(t)} \langle \dot{\sigma}_\pm(0) \rangle. \quad (4.23)$$

where for $t_2 \geq t_1$:

$$\left\langle \dot{\Pi}_\pm(t_1,t_2) \right\rangle = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_2} ds_1 \cdots ds_{2k} \langle T [\dot{\eta}(s_1) ... \dot{\eta}(s_{2k})] \rangle$$

$$= \exp \left[ -\frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} ds_1 ds_2 K_T(s_1 - s_2) \right]$$

$$= \exp [-\xi(t_2 - t_1) + i G_1(t_2 - t_1)].$$

and where

$$\xi(t) = \frac{1}{2} \int_0^t \int_0^t ds_1 ds_2 K(s_1 - s_2) = \int_0^t ds_1 \int_0^{s_1} ds_2 K(s_2). \quad (4.24)$$

As seen from (4.23), $\xi(t)$ characterizes the decay of $\dot{\sigma}_\pm$ due to the interaction with the bath.
4.1.3 Correlated initial conditions

For some applications it is more sensible to use the correlated initial conditions for the spin and the bath:

$$\rho(0) = \frac{1}{Z} \exp \left[ -J_S H_S - \beta(\hat{H}_1 + \hat{H}_B) \right], \quad Z = \text{tr} e^{-J_S H_S - \beta(\hat{H}_1 + \hat{H}_B)}. \quad (4.25)$$

where $J_S$ is the inverse temperature of the spin and $\beta$ is that of the bath. The temperature of the spin is defined by Eq. (4.27) below or analogously by

$$\langle \sigma_z \rangle = -\tanh \left[ \frac{J_S \varepsilon}{2} \right]. \quad (4.26)$$

This difference between the temperature of the bath and that of the spin will be used as an initial out of equilibrium situation for the work extraction in chapter 5. For cooling purposes, chapter 6, this is not needed and both temperatures can initially be the same.

This initial condition with $J_S \neq \beta$ can be generated from the overall equilibrium equal-temperature state of the spin and the bath via cooling or heating the bath by means of some superbath. During this process $\sigma_z$ is conserved, and the bath relaxes to its new temperature under an "external field" $\pm \frac{i}{2} \hat{X}$ generated by the interaction Hamiltonian $\hat{H}_1$ with $\sigma_z = \pm 1$. More details of this procedure are given in Appendix 4.A.

In the thermodynamical limit for the bath the correlated initial condition (4.25) is equivalent to the factorized condition (4.17) with

$$\rho_S(0) = \frac{1}{\text{tr} e^{-J_S H_S}} e^{-J_S H_S}. \quad (4.27)$$

that is, when starting from the factorized initial condition (4.17, 4.27), the dynamics of the system builds up a correlated state which at times $t$ much longer than the response time of the bath, $t \gg 1/\Gamma$, is equivalent to (4.25). By saying equivalent we mean that the initial conditions (4.17, 4.27) and (4.25) produce the same values for spin observables and for collective observables of the bath (i.e., the ones involving summation over all bath oscillators).

A formal reason for this can be seen as follows. One can write the full Hamiltonian $\hat{H}$ defined in (3.1) as

$$\hat{H} = \sum_k \hbar \omega_k \left( \hat{a}_k^\dagger + \frac{g_k \hat{\sigma}_z}{2\omega_k} \right) \left( \hat{a}_k + \frac{g_k \hat{\sigma}_z}{2\omega_k} \right) + \frac{\varepsilon}{2} \hat{\sigma}_z - \sum_k \frac{\hbar g_k^2}{4\omega_k}. \quad (4.28)$$

and diagonalize it via a unitary operator:

$$\hat{U} = \exp \left[ \sum_k \frac{g_k \hat{\sigma}_z}{2\omega_k} (\hat{a}_k^\dagger - \hat{a}_k) \right], \quad \hat{U} \hat{a}_k \hat{U}^\dagger = \hat{a}_k - \frac{g_k \hat{\sigma}_z}{2\omega_k}, \quad \hat{U} \hat{\sigma}_z \hat{U}^\dagger = \hat{\sigma}_z. \quad (4.29)$$
Thus the operators
\[ \hat{b}_k = \hat{a}_k + \frac{g_k \hat{\sigma}_z}{2\omega_k}, \quad [\hat{b}_k, \hat{b}_l^\dagger] = \delta_{kl} \] (4.30)
are distributed —over the initial state (4.25)— independently of the operators of spin. Moreover, as follows from (4.28, 4.25), the operators \( \hat{b}_k \) have on the state (4.25) exactly the same statistics (i.e., the same correlators) as the corresponding operators \( \hat{a}_k \) on the factorized state (4.17).

Now note that for the initial condition (4.25), \( \hat{\sigma}_z(0) \) and the quantum noise operator \( \hat{\eta}(t) \) are in general not independent variables, in contrast to the case of the factorized initial condition (4.17, 4.27). However, for \( t \gg 1/\Gamma \) they do become independent:
\[ \hat{\eta}(t) = \hat{\eta}_b(t) + \hat{\sigma}_z(G(t) - G), \quad \hat{\eta}_b(t) \equiv \sum_k g_k [\hat{\sigma}_z(0)e^{i\omega_k t} + \hat{b}_k(0)e^{-i\omega_k t}], \] (4.31)
where
\[ G \equiv \sum_k \frac{g_k^2}{\omega_k} \] (4.32)
is the limit of \( G(t) \) for \( t \gg 1/\Gamma \). Taking the latter limit in (4.31), one sees that \( \hat{\eta}(t) \) becomes equal to \( \hat{\eta}_b(t) \) and thus independent of \( \hat{\sigma}_z \). Recalling that \( \hat{\eta}_b(t) \) has in the state (4.25) the same statistics as \( \hat{\eta}(t) \) in the factorized state (4.17), finishes the argument.

Note that the thermodynamical limit for the bath is essential for this conclusion. Otherwise, \( G(t) \) will be a finite sum of cosines, and will not converge to \( G \).

In the following we will use the factorized initial condition Eq. (4.17) since it is technically simpler. The time limit \( t \to \infty \) will be taken before any perturbation acts on the system to ensure the equivalence with correlated initial conditions Eq. (4.25). This is the so-called ergodic limit. In a more general model, the one considering \( T_1 \), this limit is in fact \( t \gg 1/\Gamma \) and \( t \ll T_1 \).

### 4.2 Pulsed Dynamics

The external fields acting on the spin are described by a time-dependent Hamiltonian
\[ \hat{H}_F(t) = \frac{1}{2} \sum_{k=x,y,z} h_k(t) \hat{\sigma}_k, \] (4.33)
with magnitudes \( h_k(t) \), which is to be added to \( \hat{H} \) defined in (3.1) such that the overall Hamiltonian is time-dependent:
\[ \hat{H}(t) = \hat{H} + \hat{H}_F(t). \] (4.34)
Eq. (4.33) represents the most general external field acting on the spin. We shall concentrate on \textit{pulsed} regime of external fields which is well known in NMR and ESR physics [Sie71, Sli90, EBW87, AG82, Abr61, SS66, Hah50, Wau92]. For example, it was used to describe spin-echo phenomena [Hah50, Wau92] or processes of switching off undesired interactions, such as those causing decoherence [Sli90, EBW87, VL98, SL04].

A pulse of duration $\delta$ is defined by sudden switching on the external fields at some time $t > 0$, and then suddenly switching them off at time $t + \delta$. It is well-known that during a sudden switching the density matrix does not change [LL87], while the Hamiltonian gets a finite change. Let us for the moment keep arbitrary the concrete form of external fields in the interval $(t, t + \delta)$. The Schrödinger evolution operator of the spin+bath from time zero till some time $t + \tau$, $\tau > \delta$, reads:

$$
\mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{0}^{t+\tau} ds \, \hat{H}(s) \right] = e^{-it(t+\tau-t-\delta)\hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{t}^{t+\delta} ds \, \hat{H}(s) \right] e^{-it\hat{H}/\hbar} \quad (4.35)
$$

$$
= e^{-i\tau\hat{H}/\hbar} \hat{U}_P(t) e^{-it\hat{H}/\hbar}. \quad (4.36)
$$

The LHS of Eq. (4.35) contains the full time-dependent Schrödinger-representation Hamiltonian $\hat{H}(s)$, while in the RHS of this equation we took into account that the actual time-dependence is present only between $t$ and $t + \delta$. The terms $e^{-it\hat{H}/\hbar}$ and $e^{-i(t+\tau-t-\delta)\hat{H}/\hbar}$ stand for the free (unpulsed) evolution in time-intervals $(0, t)$ and $(t + \delta, t + \tau)$, respectively. In Eq. (4.36) we denoted

$$
\hat{U}_P(t) = e^{i\delta\hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{t}^{t+\delta} ds \, \hat{H}(s) \right] \quad (4.37)
$$

$$
= \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{t}^{t+\delta} ds \, e^{i(s-t)\hat{H}/\hbar} \hat{H}_F(s) e^{i(t-s)\hat{H}/\hbar} \right] \quad (4.38)
$$

$$
= \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{0}^{\delta} ds \, e^{i\hat{H}/\hbar} \hat{H}_F(s + t) e^{-is\hat{H}/\hbar} \right]. \quad (4.39)
$$

for the pulse evolution operator. The transition from (4.37) to (4.38) can be made by recalling $\hat{H}(t) = \hat{H} + \hat{H}_F(t)$ and then by noting that the expressions in these equations satisfy the same first-order differential equation in $\delta$ with the same boundary
condition at $\delta = 0$, implying that they coincide:

$$
\frac{\partial}{\partial \delta} e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] = \frac{i}{\hbar} \left( \hat{H} - e^{i\delta \hat{H}/\hbar} \hat{H}(t+\delta) e^{-i\delta \hat{H}/\hbar} \right) e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \tag{4.40}
$$

$$
= -\frac{i}{\hbar} e^{i\delta \hat{H}/\hbar} \hat{H}_F(t+\delta) e^{-i\delta \hat{H}/\hbar} e^{i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right].
$$

$$
\frac{\partial}{\partial \delta} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} \right] =
$$

$$
-\frac{i}{\hbar} e^{i\delta \hat{H}/\hbar} \hat{H}_F(t+\delta) e^{-i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} \right]. \tag{4.41}
$$

We focus on pulses so short that the influence of the spin Hamiltonian $\hbar \Omega \sigma_z/2$ and the interaction Hamiltonian $\hat{H}_I$ can be neglected during the interval $\delta$, that is, in a Taylor-expansion we may keep only the leading term.

$$
e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} = \hat{H}_F(s+t) + \frac{is}{\hbar} \left[ \hat{H}, \hat{H}_F(s+t) \right] + ...
$$

$$
= \hat{H}_F(s+t) + \frac{is}{\hbar} \left[ \frac{\hbar \Omega}{2} \sigma_z + \hat{H}_I, \hat{H}_F(s+t) \right] + ...
$$

$$
\approx \hat{H}_F(s+t), \tag{4.42}
$$

for $0 < s < \delta$. Thus, for the pulse evolution operator one gets

$$
\hat{U}_F(t) = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds \hat{H}_F(s+t) \right]. \tag{4.43}
$$

The generalization of the evolution operator (4.36) to an arbitrary number of short pulses is straightforward.

Note that in obtaining (4.43) we do not require that the bath Hamiltonian $\hat{H}_B$ during the pulse is neglected. Since the external fields are acting on the spin only, the influence of the bath Hamiltonian disappears automatically from $e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar}$ once the interaction Hamiltonian $\hat{H}_I$ has been neglected.

Recalling the orders of magnitude $\hbar \Omega$ and $\gamma \Gamma$ of the spin energy and the interaction energy, respectively, in particular, recall Eq. (4.7), $\hbar G = \hbar \int_0^\infty d\omega J(\omega)/\omega$ and Eq. (4.55).— one gets the following qualitative criteria for the validity of the short pulsing regime

$$
\delta \ll \min \left( \Omega^{-1}, \left[ \gamma \Gamma \right]^{-1} \right). \tag{4.44}
$$

As should be, for very small $\gamma$ and a fixed $\Gamma$, the second restriction on $\delta$ is weaker than the first one. More quantitative conditions for the validity of the pulsed regime
were studied recently in the context of decoherence suppression by external pulses [SL04].

To deal with the pulsed dynamics in the Heisenberg representation, one introduces the following superoperators:

\[ \mathcal{E}_t \hat{A} = e^{iHt/\hbar} \hat{A} e^{-iHt/\hbar}. \]
\[ \mathcal{P}_t \hat{A} = U_p^\dagger(t) \hat{A} U_p(t). \]  

Then the Heisenberg evolution of an operator \( \hat{A} \) corresponding to Eqs. (4.36, 4.43) reads

\[ \hat{A}(t + \tau) = \mathcal{E}_t \mathcal{P}_t \mathcal{E}_t \hat{A}. \]  

### 4.2.1 Parametrization of pulses

As seen from (4.33, 4.46), any pulse corresponds to the most general unitary operation in the Hilbert space of the spin, which would correspond to a rotation in the classical language since \( \text{SU}(2) = \text{O}(3) \). It turns out to be convenient to parametrize pulses as:

\[ \mathcal{P} \hat{\sigma}_a \equiv e^{i\delta H_F/\hbar} \hat{\sigma}_a e^{-i\delta H_F/\hbar} = \sum_{b=\pm, z} c_{a,b} \hat{\sigma}_b, \quad a = \pm, z. \]

For more detailed applications we will need the explicit form of \( e^{i\delta H_F/\hbar} \) as a \( 2 \times 2 \) unitary matrix whose determinant can be taken equal to unity without loss of generality:

\[ e^{i\delta H_F/\hbar} = \begin{pmatrix} e^{-i\phi} \cos \psi & -e^{-i\phi} \sin \psi \\ e^{i\psi} \sin \phi & e^{i\phi} \cos \phi \end{pmatrix}. \]

where

\[ 0 \leq \phi, \psi \leq 2\pi, \quad 0 \leq \vartheta \leq \frac{\pi}{2}. \]

### 4.3 Setup of pulsing

- The spin and the bath are prepared in the state (4.17), with the initial state of the spin having the definite temperature \( T_S \) as given by (4.27). Thus, for the initial average population difference one has:

\[ \langle \hat{\sigma}_z \rangle = -\tanh \left[ \frac{\beta_S \varepsilon}{2} \right] < 0. \]

- One waits for a time \( t \) which for the present model with the Hamiltonian (3.1) can be arbitrarily large, but in reality should be much smaller than the \( T_1 \) time-scale. This waiting is done for ensuring the robustness of the setup.
II.4.4 Realizations of the model

For our purposes it will suffice to take $t \gg 1/\Gamma$. This imposes $T_1 \gg 1/\Gamma$, a restriction much weaker than the ones already discussed— to the applicability of the model and then the setup does not depend on details of the initial preparation, since the initial conditions (4.17) and (4.25) are equivalent. see section 4.1.3.

- We shall restrict ourselves with two-pulse or three-pulse setup. In the first case the pulses

$$P_1 = P_2, \quad P_{t+\tau} = P_2,$$

are applied at times $t$ and $t + \tau$, respectively.

In the second case the pulses

$$P_1 = P_1, \quad P_{t+\tau_1} = P_2, \quad P_{t+\tau_1+\tau_2} = P_3,$$

are applied at times $t$, $t + \tau_1$ and $t + \tau_1 + \tau_2$, respectively.

4.4 Realizations of the model

Once the model with all its ingredients has been defined, we discuss some of its realizations and provide some numbers. A two-level system coupled to a thermal bath is a standard model for practically all fields where quantum systems are studied: NMR, ESR, quantum optics, spintronics, Josephson junctions, etc. Two particular conditions are however necessary to apply this model: the condition $T_1 \gg T_2$ on the characteristic relaxation times and availability of sufficiently strong pulses. On the other hand, we can admit rather short times $T_2^*$, since this timescale can be overcome with the spin-echo technique.

Here are experimentally realized examples of two-level systems, which have sufficiently long $T_2$ times. satisfy to $T_1 \gg T_2$. E.g. $T_1$ exceeds $T_2$ by several orders of magnitude, and admit strong pulses of external fields. For atoms in optical traps, where $T_2 \sim 1s$, $1/\Gamma \sim 10^{-8}s$, there are efficient methods for creating non-equilibrium initial states and for manipulating atoms by external laser pulses [CZ95]. For an electronic spin injected or optically excited in a semiconductor, $T_2 \sim 1\mu s$ [KA00], and for an exciton created in a quantum dot $T_2 \sim 10^{-9}s$ [B+98]; in both situations $1/\Gamma \sim 10^{-9} - 10^{-13}s$. and femtosecond ($10^{-15}s$) laser pulses are available. In the case of NMR physics $T_2 \sim 10^{-6} - 10^3s$, $1/\Gamma \sim 1\mu s$, and the duration of pulses can vary between 1ps and 1µs [Sli90, EBW87, A+03].

In all above examples the response time $1/\Gamma$ of the bath is much shorter than the internal time $1/\Omega$ of the spin. Sometimes it is argued that such a separation is related to the large size of the bath and is something generic by itself. This is clearly incorrect, since as seen from the derivation in section 4.1, the dimensionless parameter $\Omega/\Gamma$ has to do with the bath-spin interaction, rather than with the size of the bath. Moreover, several examples of the interaction of the spin with the bath are known and were analyzed both experimentally and theoretically, where $\Omega/\Gamma \sim 1$. 
For example, Ref. [DSDA77] focuses on relaxation of nuclear spins with hyperfine frequencies $\Omega \simeq 700$ MHz. $T_2 < 90$ MHz, and the ratio $\Omega/T$ may vary between 10 and 0.1.

Another important parameter that characterizes our setup is the initial polarization $\langle \hat{\sigma}_z \rangle$ of the spin. It is known in NMR and ESR physics that the response of magnetic atoms (nucleus) to external dc magnetic field is best characterized by the frequency ratio, which for electronic magnetic moments can reach few MHz per gauss of applied magnetic field, while for nuclear magnetic moments it is $\simeq 2 \times 10^3$ times smaller due to the difference between atomic and nuclear Bohr magnetons. For example, one has $\frac{\text{frequency}}{\text{field}} = 4.2$ kHz/ gauss for a single proton.

### 4.4.1 Why do we insist on the exact solvability of the model

The model as stated above—that is, with the Hamiltonian presented in chapter 3, Eq.(3.1) together with Eq.(4.34) is exactly solvable for all temperatures and all bath-spin coupling constants. It is useful at this point to recall for the reader what are the specific reasons to insist on this feature. The model with interaction Hamiltonian Eq.(3.4) is a particular case of a more general spin-boson model, where the influence of the $T_1$-time is retained either via an additional term $\propto \hat{\sigma}_r$ in the Hamiltonian of the spin, or via an additional coupling in the interaction Hamiltonian. This model is in general not solvable, and, what is worse, there are no realizable approximate methods which apply for a fixed (may be weak) coupling to the bath and all temperatures including very low ones. The standard weak coupling theories—both markovian leading to well-known Bloch equations, and non-markovian ones—are satisfactory only for sufficiently high temperatures, while at low-temperatures weak-coupling series are singular, and different methods of their resummation produce different results. This situation becomes even more problematic under driving by external fields. The objects studied by us—such as work, the energy of the spin—can be rather fragile to various not very well-controlled approximations, since there is a general limitation governing their behavior: Thomson's formulation of the second law. It is derived from the first principles of quantum mechanics [Thi83, BK77, BK79, Bas78, PW78, Len78, AX02] and has to be respected in any particular model.

### 4.5 The spectral density

The coupling with the bath can be parametrized via the spectral density $J(\omega)$:

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k). \quad (4.54)$$

In the thermodynamical limit the number of bath oscillators goes to infinity, and $J(\omega)$ becomes a smooth function, whose form is determined by the underlying physics of the system-bath interaction.
4.5.1 Ohmic spectrum of the bath

We shall be mainly working with the ohmic spectrum:

\[ J(\omega) = \gamma \omega e^{-\omega/\Gamma}. \]  (4.55)

where \( \gamma \) is a dimensionless coupling constant, and where \( \Gamma \) is the maximal characteristic frequency of the bath's response. This spectrum and its relevance for describing open quantum systems was extensively discussed in literature; see, e.g., [L78].

Quantum noise correlation function and decay times

The correlation function of the quantum noise in the ohmic case, using Eqs. (4.22, 4.54, 4.55), is given by:

\[ K(t) = \int_0^\infty d\omega J(\omega) \coth \left( \frac{\hbar \omega}{2T} \right) \cos \omega t \]

\[ = \gamma \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2T} \right) e^{-\omega/\Gamma} \cos \omega t. \]  (4.57)

Recall that the decay factor \( \xi(t) \) is related to \( K(t) \) via Eq. (4.24): \( \dot{\xi}(t) = K(t) \). Properties of these functions are worked out in Appendix 4.B. In particular, for \( \xi(t) \) one gets from Eqs. (4.B.13, 4.B.14) the following explicit expression:

\[ \xi(t) = \gamma \ln \left[ \frac{\Gamma^2}{\Gamma (1 + \frac{T}{\hbar \Gamma} - i \frac{T}{\hbar}) \Gamma (1 + \frac{T}{\hbar \Gamma} + i \frac{T}{\hbar})} \right]. \]  (4.58)

where \( \Gamma \) is Euler's Gamma function. It is seen that the temperature enters via the dimensionless parameter \( T/(\hbar \Gamma) \).

Let us now determine the behavior of this quantity for low and large temperatures. Using Eq. (4.B.7) one obtains for \( \hbar \Gamma/T \gg 1 \) (low temperatures):

\[ \xi(t) = \gamma \ln \left[ \frac{\hbar \beta}{\pi t} \sinh \left( \frac{\pi t}{\hbar \beta} \right) \right] + \frac{\gamma}{2} \ln \left[ 1 + \Gamma^2 t^2 \right]. \]  (4.59)

This implies two regimes of decay: power-law and exponential.

\[ t \ll \hbar \beta : \quad e^{-\xi(t)} = (1 + \Gamma^2 t^2)^{-\gamma/2}. \]  (4.60)

\[ t \gg \hbar \beta : \quad e^{-\xi(t)} = e^{-t/T_2}, \quad T_2 = \frac{\hbar}{\gamma T \pi}. \]  (4.61)

For \( \hbar \Gamma/T \ll 1 \) (high temperatures) one uses Eq.(4.B.12) to get

\[ \xi(t) = \frac{2\gamma T}{\hbar \Gamma} \left[ \Gamma t \arctan(\Gamma t) - \frac{1}{2} \ln(1 + \Gamma^2 t^2) \right]. \]  (4.62)
This time the possible regimes of decay can be approximated as gaussian and exponential.

\[ t \lesssim 1/\Gamma : \quad e^{-\xi(t)} \simeq e^{-t^2/T_2^2}. \quad T_2 = \sqrt{\frac{\hbar^2}{\gamma T \Gamma}}. \quad (4.63) \]

\[ t \gg 1/\Gamma : \quad e^{-\xi(t)} = e^{-t/T_2}. \quad T_2 = \frac{\hbar}{2\gamma T}. \quad (4.64) \]

In this latter case, as seen from Eq. (4.B.12), \( K(t) \) behaves approximately as a delta-function: \( K(t) \simeq \frac{2\gamma T}{\hbar (1 + \frac{\hbar^2 T^2}{\gamma})} \) with strength \( 2\gamma T/\hbar \) determined by the parameters \( \gamma \) and \( T \). Note that in all the above cases the characteristic decay times become shorter upon increasing the temperature \( T \) or the coupling constant \( \gamma \), as expected.

**The G-factor**

Finally we will indicate the form of the function \( G(t) \) in the ohmic case (see Eq. (4.8)) which characterizes the reaction of the spin to the bath. As seen below, this function and its primitive are rather important for our purposes. Using

\[ \int_0^\infty d\omega e^{-\omega/\Gamma} \cos(\omega t) = \Gamma \frac{1}{1 + \Gamma^2 t^2}. \quad (4.65) \]

one gets for the ohmic case:

\[ G(t) = \gamma \Gamma \left(1 - \frac{1}{1 + \Gamma^2 t^2}\right). \quad (4.66) \]

It is seen that \( G(t) \) becomes equal to a constant on the characteristic time \( 1/\Gamma \). It is therefore justified to call the latter the response time of the bath. For future references note as well that the function \( F \) defined in Eq.(4.14) takes the form

\[ F(t) = \gamma [\Gamma t - \arctan(\Gamma t)]. \quad (4.67) \]

**4.5.2 Other types of environment**

The main characteristic of the spectral density function is its behaviour at low frequencies. That is characterized by the power dependence on \( \omega \). Generalizing Eq. (4.55)

\[ J(\omega) = \gamma \omega^\alpha e^{-\omega/\Gamma}. \quad (4.68) \]

This equation brings 3 qualitatively different regimes:

- \( \alpha > 1 \) Superohmic situation: Underdamped case
- \( \alpha = 1 \) Ohmic situation: Critical case
II.4.5 The spectral density

- $\alpha < 1$ Subohmic situation: Overdamped case

where the overdamped, critical, underdamped language comes from the damped harmonic oscillator jargon. Throughout this thesis we will only consider the ohmic case since it is the most relevant situation and brings the most interesting physics. When dealing with the cooling experiment an example of subohmic spectral density function, the so called $1/f$ noise, will be briefly presented and studied. In $1/f$ noise one has $\alpha = 1/2$. This interaction has been found lately to be the relevant interaction, for instance, to the background charges in a Cooper pair box [N+02].

Though the superohmic situation is also relevant, for instance $\alpha = 3$ or 5 for a defect in a three dimensional solid interacting with acoustic phonons, it will not be considered here since it does not bring any new qualitative phenomena for the purposes studied here.
Appendix 4.A
Correlated versus factorized initial conditions

Here we shortly outline how the two-temperature state (4.25) can be prepared starting from the overall equilibrium state

\[ \rho(0) = \frac{1}{Z} \exp \left[ -\beta_S \hat{H}_S - \beta_\Lambda (\hat{H}_1 + \hat{H}_B) \right]. \quad Z = \text{tr} e^{-\beta_S \hat{H}_S - \beta_\Lambda (\hat{H}_1 + \hat{H}_B)}. \] (4.4.1)

which has equal temperatures of the spin and the bath.

Assume that the bath was subjected to another much larger thermal bath (superbath) at temperature \( T \) different from \( T_S \), so that the total Hamiltonian of the spin, bath and superbath reads:

\[ \hat{H}_\text{total} = \hat{H} + \hat{H}_\text{sup}. \] (4.4.2)

where the latter operator \( \hat{H}_\text{sup} \) characterizes the weak interaction of the bath with the superbath and contains also self-Hamiltonian of the superbath. Thus

\[ [\hat{H}_S, \hat{H}_\text{sup}] = 0. \] (4.4.3)

Now the statement of this appendix is that under the action of the superbath at temperature \( T \), the common state of the spin and the bath will relax to the state Eq. (4.4.11 or 4.25) with different temperatures for the spin and the bath. The reason is that due to Eq. (4.4.3), \( \hat{\sigma}_z \) is conserved during the whole evolution generated by the superbath, so that \( \hat{\sigma}_z \) does not relax and keeps its value given by Eq. (4.4.1). In contrast, the variables of the bath including \( \hat{X} \) do not have such a protection and they relax under influence of the superbath. Let us now substantiate this statement.

Since \( [\hat{H}_S, \hat{H}] = 0 \), the initial equilibrium state \( \rho(0) \) of the spin and the bath can be represented as

\[ \rho(0) = \sum_{j=\pm 1} p_{jj} \rho_{jj}(0) |j\rangle \langle j|, \] (4.4.4)

where

\[ p_{jj} = \frac{e^{-j^2 \beta_S \xi / 2}}{2 \cosh(j^2 \beta_S \xi / 2)}, \quad j = \pm 1. \] (4.4.5)

are probabilities for the spin to be up \((j = +1)\) or down \((j = -1)\) respectively. \(|j\rangle\)
is the eigenstate of \( \hat{H}_S = \frac{\xi}{2} \hat{\sigma}_z \) with eigenvalue \( j = \pm 1 \), and where

\[ \rho_{jj}(0) = \frac{1}{Z_j} \exp \left[ -\beta_S \left( \frac{j}{2} \hat{X} + \hat{H}_\text{B} \right) \right]. \quad Z_j = \text{tr}_B e^{-\beta_S \left( \frac{j}{2} \hat{X} + \hat{H}_\text{B} \right)}, \quad j = \pm 1. \] (4.4.6)

are conditional states of the bath.

The total initial state of the spin, bath and superbath thus reads:

\[ \rho_{\text{total}}(0) = \sum_{j=\pm 1} p_{jj} \rho_{\text{sup}}(0) \otimes \rho_{jj}(0) |j\rangle \langle j|. \] (4.4.7)
where $\rho_{\text{sup}}(0)$ is the initial equilibrium state of the superbath. Note that due to weak coupling between the bath and superbath, their initial states can be assumed to be factorized.

As follows from (4.A.3, 4.A.4), the time-dependent state of the total system consisting of spin, bath and superbath can be presented as

$$\rho_{\text{total}}(t) = \sum_{j=\pm 1} p_{jj} \Omega_{jj}(t) |j\rangle \langle j|.$$  \hspace{1cm} (4.A.8)

where $\Omega_{jj}(t)$ — the conditional state of the bath and superbath — satisfies the von Neumann equation

$$i\hbar \dot{\Omega}_{jj} = \left[ \frac{\hat{J}_z}{2} \hat{X} + \hat{H}_B + \hat{H}_{\text{sup}}, \Omega_{jj} \right].$$  \hspace{1cm} (4.A.9)

with the initial condition:

$$\Omega_{jj}(0) = \rho_{\text{sup}}(0) \otimes \rho_{jj}(0).$$  \hspace{1cm} (4.A.10)

Thus, $\Omega_{jj}$ moves according to the Hamiltonian $\frac{\hbar}{2} \hat{X} + \hat{H}_B + \hat{H}_{\text{sup}}$. It is now clear that in the weak coupling limit of the bath-superbath interaction the marginal conditional state $\text{tr}_{\text{sup}} \Omega_{jj}(t)$ will — for sufficiently long times $t$ — relax to Gibbs distribution at temperature $T$ (equal to the one of the superbath) and with Hamiltonian $\frac{\hbar}{2} \hat{X} + \hat{H}_B$. Thus the (unconditional) marginal state of the spin and the bath will indeed relax to

$$\rho \propto \exp\left[-\beta_\lambda \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)\right],$$  \hspace{1cm} (4.A.11)

as was claimed before.
Appendix 4.B
Quantum noise generated by ohmic bath

Here we discuss properties of the function:

\[ K(t) = \gamma \int_0^\infty d\omega \, \coth(\hbar \omega \beta / 2) e^{-\omega / \Gamma} \cos \omega t. \]  

In the given integration domain one can use

\[ \coth(h\omega\beta/2) = 1 + 2 \sum_{n=1}^{\infty} e^{-h\omega\beta n} \]  

and can get from (4.B.1):

\[ K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + 2\gamma \sum_{n=1}^{\infty} \frac{(\Gamma^{-1} + h\beta n)^2 - t^2}{((\Gamma^{-1} + h\beta n)^2 + t^2)^2} \]  

With help of a standard relation:

\[ \sum_{n=1}^{\infty} \frac{1}{t^2 + y^2(n + \kappa)^2} = \frac{i}{2ty} \left[ \psi(1 + \kappa - i \frac{t}{y}) - \psi(1 + \kappa + i \frac{t}{y}) \right]. \]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \), one obtains

\[ \sum_{n=1}^{\infty} \frac{(\kappa + n)^2 - t^2}{((\kappa + n)^2 + t^2)^2} = \frac{1}{2}[\psi'(1 + \kappa - it) + \psi'(1 + \kappa + it)]. \]

Combining (4.B.5) with (4.B.3) and \( \kappa = 1/(h\beta \Gamma) \) one ends up with the following formula

\[ K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{\gamma T^2}{\hbar^2} \left[ \psi' \left( 1 + \frac{1}{\hbar \Gamma \beta} - i \frac{t}{\hbar \beta} \right) \right. \]
\[ \left. \quad + \psi' \left( 1 + \frac{1}{\hbar \Gamma \beta} + i \frac{t}{\hbar \beta} \right) \right] \]  

Let us now consider separately the cases of low and high temperatures. For \( h\Gamma \beta \gg 1 \) one uses the known relation

\[ \Gamma \left( 1 - i \frac{t}{\hbar \beta} \right) \Gamma \left( 1 + i \frac{t}{\hbar \beta} \right) = \frac{\pi t}{\hbar \beta} \frac{1}{\sinh[\pi t/(h\beta)]} \]  

and obtains from (4.B.6):

\[ K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{\gamma T^2 \pi^2}{\hbar^2} \frac{1}{\sinh^2[\pi t/(h\beta)]}. \]
For small $t$ ($t \ll 1/\Gamma$) $K(t)$ is positive as it should be:

$$K(t) = \gamma \Gamma^2 + \frac{\gamma T^2 \pi^2}{3\hbar^2}. \quad (4.B.9)$$

In contrast, for $t \sim \hbar\beta \gg 1/\Gamma$ it becomes negative, namely the noise is anticorrelated:

$$K(t) = 3\gamma \frac{1}{\Gamma^2 t^4} - \frac{\gamma T^2 \pi^2}{\hbar^2} \frac{1}{\sinh^2[\pi t/(\hbar\beta)]}. \quad (4.B.10)$$

At the end it is again correlated in the limit of very large times $t \gg \hbar\beta$ where the first term in the r.h.s. of Eq. (4.B.10) does dominate (this domain is small for low temperatures).

In the high-temperature limit $\hbar \beta \Gamma \ll 1$ one can use in Eq. (4.B.6) the Stirling formula:

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + ..., \quad z \geq 1 \quad (4.B.11)$$

and then the quasiclassical limit for the quantum noise reads (after some more simplifications):

$$K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{2\gamma T\Gamma}{\hbar} \frac{1}{1 + t^2\Gamma^2}. \quad (4.B.12)$$

In the purely classical limit the first term in the r.h.s. can be neglected and we return (for $t\Gamma \gg 1$) to the classical white noise with the strength $2\pi\gamma T$ (recall that what enters in the expressions is $\hbar K(t)$).

Finally in the context of Eq. (4.B.6) we notice the following useful relations:

$$\dot{\xi}(t) = \int_0^t dt' K(t') = \gamma \frac{t\Gamma^2}{1 + \Gamma^2 t^2} + \frac{i\gamma T}{\hbar} \left[ \psi \left( 1 + \frac{1}{\hbar\Gamma\beta} - i\frac{t}{\hbar\beta} \right) - \psi \left( 1 + \frac{1}{\hbar\Gamma\beta} + i\frac{t}{\hbar\beta} \right) \right]. \quad (4.B.13)$$

$$\xi(t) = \int_0^t dt' \int_0^{t'} dt'' K(t'') = -\gamma \ln \left[ \frac{\Gamma \left( 1 + \frac{1}{\hbar\Gamma\beta} - i\frac{t}{\hbar\beta} \right) \Gamma \left( 1 + \frac{1}{\hbar\Gamma\beta} + i\frac{t}{\hbar\beta} \right)}{\Gamma^2 \left( 1 + \frac{1}{\hbar\Gamma\beta} \right) \sqrt{1 + t^2\Gamma^2}} \right]. \quad (4.B.14)$$

which are used in the main text.