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Line-profile variations of non-radial adiabatic pulsations of rotating stars

II. The diagnostic value of amplitude and phase diagrams derived from time series of spectra

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Abstract. We discuss the possibilities to derive the pulsation parameters $\ell$ and $m$ of non-radially pulsating rotating stars from spectroscopic observations. We model the line-profile variations caused by the oscillatory velocity field and temperature variations at the surface of the star. In our description of the velocity field of the pulsations, we use the expressions for linear adiabatic pulsations, and include terms that account for the Coriolis force. For various stellar and pulsational parameters we generate time series of spectra, and analyse the phase and amplitude diagrams resulting from a Fourier analysis of the time series.

We find that for stars with $V_\infty \sin i$ larger than approximately five times the half-width (HWHM) of the intrinsic profile, one can derive both the degree $\ell$ and the order $|m|$ from the phase diagrams of the line-profile variations. We present linear relations between observable phase differences and the parameters $\ell$ and $|m|$. These relations can be used to identify pulsation modes. This method works for spheroidal and toroidal, sectoral and tesseral modes; it is possible to derive values of $\ell \lesssim 15$ and values of $|m| \lesssim 10$. The method is also applicable to multi-periodic multi-mode pulsations.

We apply the method to analyse spectroscopic data sets of $\zeta$ Oph and $\epsilon$ Per, and present values of $\ell$ of the pulsation modes in these stars. We use harmonic phase diagrams to constrain values of $|m|$ of some of these modes.

We argue that the presence of the complex pattern of frequencies in the periodogram of the line-profile variations of $\epsilon$ Per, is consistent with the expectations for profile variations which are dominated by the oscillatory velocity field, rather than the oscillatory temperature variations.

Key words: line: profiles – stars: oscillations; rotation – stars: variable: $\delta$ Scuti – stars individual: $\zeta$ Oph, $\epsilon$ Per

1. Introduction

The study of pulsations in stars provides direct tests for the validity of stellar evolution models. With observed pulsation frequencies one can constrain these models, provided that the pulsation modes can be identified. For non-radial pulsations the relevant parameters are the degree $\ell$ and azimuthal order $m$, which specify the tangential shape of the pulsation mode. The intrinsic pulsation frequency is physically linked with the degree of the pulsation (e.g., Dziembowski & Pamyatnykh 1993, Gautschy & Saio 1993). For rotating stars the azimuthal order affects the apparent frequency, since the modal pattern is rotating with the star. Hence, for asteroseismological purposes one needs accurately determined values of $\ell$, $m$ and the observed frequency. Heynderickx et al. (1994) and Cugier et al. (1994) showed how to identify low-degree pulsation modes of $\beta$ Cephei stars from multi-passband photometry. In this paper we focus on the possibility to spectroscopically determine $\ell$ and $m$ in rotating early-type stars; in many cases, our results are applicable to $\delta$ Scuti stars as well.

Stellar pulsations are reflected as line-profile variations in absorption lines formed in the photosphere; for rotating stars patterns of alternating absorption and emission features cross the profiles from blue to red on a time scale of hours to days. Such profile variations have been successfully modelled as the result of non-radial pulsations (see e.g. Smith 1978, Vogt & Penrod 1983, Baade 1984, Gies & Kullavanijaya 1988, Kambe et al. 1990, Reid et al. 1993). The non-radial pulsations divide the stellar surface in regions with different velocity fields and temperatures. The velocity fields give rise to local Doppler shifts, the temperature variations cause local brightness and equivalent-width changes. Due to the rotation of the star these variations are Doppler mapped to the absorption line profiles, creating a moving pattern of peaks and troughs (see Fig. 1).

Various methods to spectroscopically identify pulsation parameters have been proposed (Smith 1977; Balona 1986; Ken-
Fig. 1. Line-profile variations due to the 3-dimensional velocity field of non-radial pulsations. For different values of the pulsation degree \( \ell \) we show from top to bottom: radial (vertical) part of the eigenfunction \( V_r \) – superposed line profiles of pulsating and non-pulsating case – difference of line profiles of pulsating and non-pulsating case – grey scale representation of residual spectra (mean subtracted) of 3 pulsation cycles – distribution of the amplitudes of the variations with input pulsation frequency \( I_0(\lambda) \) (thick line) and first harmonic \( I_1(\lambda) \) (thin line) expressed in units of average central line depth; numbers refer to the maximum values of the amplitudes \( I_0(\lambda) \) and \( I_1(\lambda) \) – distribution of the phase of the variations with input pulsation frequency \( \Psi_0(\lambda) \) (thick line) and first harmonic of the pulsation frequency \( \Psi_1(\lambda) \) (thin line) expressed in units of \( \pi \) radians; numbers refer to the blue-to-red phase differences \( \Delta \Psi_0 \) and \( \Delta \Psi_1 \). Note that the slope of the main phase distribution \( \Psi_0(\lambda) \) changes as a function of \( \ell \), while the slope of the harmonic phase distribution \( \Psi_1(\lambda) \) stays rather constant.

Gies & Kullavanijaya (hereafter G&K88) take the phase difference of the variations in the blue and the red edge of the line profile as a direct measure of the absolute value of the pulsation parameter \( m \), which refers to the number of meridional nodal great circles of the eigenfunction (see Fig. 1 where we used \( |m|=4 \)). At any instant, the pulsational variations are distributed for periodicity as the Intensity Period Search (IPS), and to the subsequent mode identification as the IPS method.
over the azimuthal angle $\phi$ as $e^{im\phi}$, and consequently the full velocity range of the absorption line profile samples a total phase difference of $m/2$ cycles (or $m$ times $\pi$ radians). However, this relation is only valid for sectoral modes ($\ell = m$), $\ell$ referring to the total number of nodal lines of the eigenfunction), which have the source of most of the variability focused at the equator of the star. For modes other than sectoral, the main source of variability lies off the equator, and the $m/2$ cycles phase difference is sampled by a smaller part of the absorption line. Therefore one can expect steeper phase versus wavelength relations for non-sectoral modes (see Fig. 1). The question then arises whether the observed phase difference, and consequently the number of bumps and troughs in the line profiles, is an estimator for the degree $\ell$ rather than the azimuthal order $m$. Merryfield & Kennelly (1993) already suggested that this is the case, but did not explore nor verify the validity of their suggestion in the literature. The applicability of the IPS method for modes other than sectoral has not been discussed before. For the few reported mode identifications performed with the IPS method the authors assumed that the detected pulsation modes are sectoral.

The strength of the IPS method is that for multi-periodic multi-mode stars the line-profile variabilities caused by the individual pulsation modes are separated in frequency as a result of the Fourier analysis. This allows the characteristics of each mode to be studied separately. The superposition of the individual three-dimensional surface velocity fields of the modes leads to beating of the variations in the line profiles. Mathias et al. (1994) have incorporated these beatings in the “moment method” (see also Balona 1986, Aerts et al. 1992, De Pauw et al. 1993), but the effects of beatings on the amplitude and phase diagrams resulting from the IPS method have never been studied before.

In this paper we investigate the IPS phase diagrams for all possible spheroidal and toroidal modes: sectoral, zonal ($|m|=0$, also called axisymmetric modes) and tesseral (any mode that is neither sectoral nor zonal), in order to find out whether one can retrieve the input parameters $\ell$ and $m$. We also discuss the influence on the amplitude and phase diagrams of beatings that are caused by multi-periodic pulsations. For this purpose, we use a model for adiabatic non-radial pulsations, including terms that account for the Coriolis force, to generate time series of line profiles.

This paper is part of a series on line-profile variability caused by adiabatic non-radial pulsations. Schrijvers et al. (1996, hereafter Paper I) describe the model we use to synthesize line profiles, and present a survey of the relevant stellar and pulsational parameters. Telting & Schrijvers (1996, Paper III) discuss the cancellation effects that occur for near equator-on stars with pulsation modes for which $\ell - m$ is an odd number. Preliminary results of the work presented here are given by Telting & Schrijvers (1995). Other theoretical work on line-profile variations due to non-radial pulsations has been presented by Kambe & Osaki (1988), Unno et al. (1989), Lee & Saio (1990), Lee et al. (1992), Aerts & Waelkens (1993), Reid & Aerts (1993), Clement (1994) and Townsend (1996).

In Sect. 2 we briefly discuss the model of non-radial adiabatic oscillations and the synthesis of time series of spectra. In Sect. 3 we summarize how to derive amplitude and phase diagrams from time series of spectra. In Sect. 4 we present the results of our computations and discuss in detail the effects of all relevant parameters on the expected phase diagrams. In Sect. 4.4 we present a statistical study on the chances of correctly identifying the pulsation parameters $\ell$ and $m$. In Sect. 5 we discuss multi-periodic stars, and in Sect. 6 we apply our results to observations of $\epsilon$ Per and $\zeta$ Oph. In Sect. 7 we summarize our conclusions.

2. Modelling non-radial pulsations of slowly rotating stars

We model line-profile variability caused by adiabatic non-radial pulsations of a rotating star, with the rotation axis as the symmetry axis of the pulsation. Our model is essentially the same as the one described by Aerts & Waelkens (1993), with a few improvements which have been discussed in Paper I.

The model describes the three-dimensional surface field of adiabatic oscillations, and is derived from a linear perturbation analysis of the equations of stellar structure, including terms describing the effects of the Coriolis force. The tangential dependence of the eigenfunction can be written as the sum of a spheroidal and two toroidal terms. At the surface of the star the Lagrangian displacement vector $\xi = (\xi_r, \xi_\theta, \xi_\phi)$ can be expressed as

$$
\xi = a_{sph,\ell} \left( \frac{1}{k} \frac{\partial}{\partial \theta}, k \frac{\partial}{\partial \phi} \right) N^m_\ell Y^m_\ell (\theta, \phi) e^{i\omega t} + a_{tor,\ell+1} \left( 0, \frac{1}{k} \frac{\partial}{\partial \theta}, k \frac{\partial}{\partial \phi} \right) N^m_\ell Y^m_{\ell+1} (\theta, \phi) e^{i(\omega t + \frac{\pi}{2})},
$$

(1)

$$
+ a_{tor,\ell-1} \left( 0, \frac{1}{k} \frac{\partial}{\partial \theta}, k \frac{\partial}{\partial \phi} \right) N^m_\ell Y^m_{\ell-1} (\theta, \phi) e^{i(\omega t - \frac{\pi}{2})},
$$

where $a_{sph,\ell}$ is the spheroidal radial (vertical) amplitude and $k$ is the ratio of horizontal to radial amplitudes of the spheroidal part of the eigenfunction. The spherical harmonics $Y^m_\ell$ specify the $\theta$ and $\phi$ dependence of the eigenfunction, and are normalized by the factor $N^m_\ell$. The oscillation frequency $\omega$ is defined in the frame that is corotating with the star. The toroidal terms are due to the Coriolis force; the amplitudes $a_{tor,\ell-1}$ and $a_{tor,\ell+1}$ are proportional to $\Omega/\omega^{(0)}$, with $\Omega$ the rotation frequency of the star and $\omega^{(0)}$ the pulsation frequency in the zero-rotation approximation. The expressions for $Y^m_\ell$, $N^m_\ell$, $a_{tor,\ell-1}$ and $a_{tor,\ell+1}$ are specified in Paper I. For a discussion of the limitations of this model we refer to Saio (1981), Martens & Smeyers (1982), Aerts & Waelkens (1993) and Paper I.

At the surface of the star the ratio of horizontal to radial amplitudes $k$ can be written as $k = k^{(0)}(1 + \Omega/\omega^{(0)})$, where we use an expression for $k^{(0)}$ equivalent to the formulation given in Paper I. As in Paper I, we only generate spectra for combinations of $\Omega/\omega^{(0)}$ and $k$ that correspond to an equatorial rotation velocity of less than 50% of the break-up velocity.
2.1. Temperature effects

In this paper, we account for local brightness and equivalent-width changes, which are induced by the oscillatory temperature variations (Buta & Smith 1979, Lee et al. 1992). To enhance the general applicability of our results, we will not strictly follow the adiabatic treatment of these temperature effects. In the adiabatic approximation, the perturbation of the surface temperature follows the perturbation of the density, and is in phase with the radial displacement for \( g \) modes, and in anti-phase for \( p \) modes. In our work we use the positive root of the equation

\[
(\ell(\ell + 1)) k^2 - 4k - 1 = 0
\]

(see e.g. Buta & Smith) to discriminate between \( p \) (low \( k \)) and \( g \) (high \( k \)) modes, and change the phase lag between temperature variations and radial displacement accordingly. In the non-adiabatic case, however, an additional phase lag between the radial displacement and temperature variations can be present (Saio & Cox 1980), and the amplitude of the temperature variations can differ from the case described by Buta & Smith.

To model these effects we introduce three parameters: the maximum amplitude of the temperature variations \( \delta T/T \), the phase lag \( \phi \) between the displacement field and the temperature variations, and the response of the equivalent width to the temperature variations \( \alpha_{W_E} \), where we assume \( dW_E/W_E = \alpha_{W_E} \delta T/T \). The visual brightness of each surface element is scaled proportional to \( 1.8 \delta T/T \). For simplicity, we change the equivalent width of the local intrinsic line profile by changing only the depth of the profile.

We have set up our parameter study such that we do not have to specify the fundamental stellar parameters such as mass, radius and rotation period. To allow general applicability of our results we also do not explicitly specify the pulsation frequency. Hence, we cannot use the adiabatic relation between the amplitude of the temperature variations and the velocity amplitude (e.g. Buta & Smith), since this involves knowledge of the pulsation frequency and the stellar radius. Instead we use \( \delta T/T \) as a free parameter, to maintain the general applicability of our results.

2.2. Line-profile synthesis

The velocity field of the oscillation is found by taking the time derivative of the Lagrangian displacement vector, and is calculated on a sphere with typically more than 5000 visible equally sized surface elements. We attribute a Gaussian intrinsic profile with constant width \( W \) (=1.20 HWHM) to each visible surface element (see Paper I). Line profiles are then generated by a weighted integration of the Doppler-shifted intrinsic profiles

\[ a \text{ weighted integration of the Doppler-shifted intrinsic profiles} \]

of all visible surface elements, each having different equivalent widths due to the pulsational temperature variations. The weights are given by the aspect angle of each element, the local oscillatory brightness changes, and a linear limb-darkening correction with a value 0.35 for the coefficient. We neglect the change of aspect angles corresponding to the deformation of the star due to the oscillatory displacements.

For our mono-mode calculations we choose to model the line-profile variability with the apparent frequency as if observed in the corotating frame of the star, in order to get clear distinction between prograde (negative \( m \), pulsation pattern moves with the rotation of the star, bumps move from blue to red through the line profile) and retrograde (positive \( m \), pulsation patterns moves against the stellar rotation, profile bumps move from red to blue) travelling waves. This way we can easily study the effects of rotation, which affect the line profiles of prograde and retrograde modes differently. In the observer’s frame, however, both prograde and retrograde modes will give rise to variability migrating from blue to red through the absorption line profile, if the stellar rotation rate is high enough. To transform our calculations to the observer’s frame, one simply has to calculate the apparent frequency from

\[
\omega_{\text{obs}} = \omega - m \Omega
\]

and reverse time sequence of the spectra if the observed frequency \( \omega_{\text{obs}} \) changes sign with respect to the pulsation frequency \( \omega \). Equivalently, the slope of the resulting phase diagrams and the blue-to-red phase difference should be negated in the case of retrograde modes.

In this paper, we aim to investigate how accurate one can retrieve the input parameters \( \ell \) and \( m \), in case of a perfect spectroscopic data set. For this reason we do not account for noise in the spectra, and generate series of spectra which are perfectly sampled in time.

We scale the pulsational displacement amplitude \( a_{\text{ph}} \) such that the maximum vector velocity of the pulsation \( V_{\text{max}} = \left( V_{\text{rpm}}^2 + V_{\text{rpm}}^2 + V_{\text{rpm}}^2 \right)_{\text{max}} \) equals a specified value. We treat the unknown surface quantities \( V_{\text{max}}, k_0 \), the degree \( \ell \) and order \( m \), and the rotation parameter \( \Omega/\omega_0 \) as free and independent parameters. Together with the inclination \( i \) of the star, the width \( W \) of the intrinsic profile and the three temperature related parameters \( \delta T/T \), \( \phi \), and \( \alpha_{W_E} \), they form a set of 10 parameters that determine the shape of the line profiles.

3. Analysis of time series of spectra

To investigate the effects of the relevant parameters on the phase and amplitude distributions across the absorption line, we generate time series of absorption line profiles and analyse these with an IPS technique equivalent to that proposed by G&K88. For each wavelength bin in the line profile, we decompose the variable intensity signal into its sinusoidal components; in the mono-mode case by fitting sinusoids with known (input) frequencies, in the multi-mode case (Sect. 5) by computing a Fourier transform.
For mono-periodic oscillations we generate time series of 24 spectra covering one complete pulsation cycle in constant time steps. Since the line-profile variations are in general not strictly sinusoidal (see Gies 1991, Reid & Aerts 1993, and Paper I), we fit the variable intensity with a function of the form

\[ I(\lambda, t) = I_{\text{mean}}(\lambda) + I_0(\lambda) \sin(\omega_{\text{obs}} t + \Psi_0(\lambda)) + I_1(\lambda) \sin(2\omega_{\text{obs}} t + \Psi_1(\lambda)) + I_2(\lambda) \sin(3\omega_{\text{obs}} t + \Psi_2(\lambda)) \]  

for each wavelength bin in the line profile. For both the pulsation frequency and its first harmonic frequency, the amplitudes \((I_0(\lambda), I_1(\lambda))\) and phases \((\Psi_0(\lambda), \Psi_1(\lambda))\) of the variability are plotted across the line profile (see Fig. 1).

In Fig. 2 we show examples of phase diagrams \(\Psi_0(\lambda)\). The blue-to-red phase differences \(\Delta \Psi_0\) and \(\Delta \Psi_1\) are obtained by reading off the maximum phase change between the outermost wavelength/velocity positions at which the corresponding amplitude of variations (respectively \(I_0(\lambda)\) and \(I_1(\lambda)\)) exceeds \(d_{\text{mean}}/10000\), where \(d_{\text{mean}}=1-I_{\text{mean}}(\lambda=\lambda_0)\) is the time-averaged central depth of the absorption line profile (see Paper I). For real observations (with noise in the data), this cut-off velocity will be limited within the part of the line profile where significant pulsational variability is detected. For monotonic phase diagrams, our values for the blue-to-red phase changes will therefore be upper limits to the observable ones. Note that we do not impose a priori knowledge of a value of \(V_\star \sin i\) to be able to read off the blue-to-red phase differences.

In Fig. 1 we present an example of such an analysis for four time series of generated line profiles. From left to right the degree of the spherical harmonic \(\ell\) is increased while the order \(m\) is held constant \((m=-4)\). The other parameters are: \(V_{\text{max}}=0.15 V_\star \sin i, k^{(0)}=0.3, i=55^\circ, W=0.15 V_\star \sin i, \Omega/\omega^{(0)}=0.0\) (zero-rotation model), and \((\delta T/T)_{\text{max}}=0.0\). In the bottom part of the figure the phase and amplitude diagrams of the line profile variations with frequency equal to the pulsation frequency \((I_0(\lambda), \Psi_0(\lambda))\) are depicted by heavy lines. Those lines depict those for the variations appearing with the first harmonic of the pulsation frequency \((I_1(\lambda), \Psi_1(\lambda))\).

In Fig. 1 we see that the slope of the \(\Psi_0(\lambda)\) phase diagram steepens with increasing \(\ell\), while the slope of the \(\Psi_1(\lambda)\) phase diagram is more or less constant. We also see that the absolute value of the phase difference \(\Delta \Psi_0\) (expressed in units of \(\pi\) radians) is close to the input value of the degree \(\ell\) of the pulsation mode. Therefore we conclude that the assumption that the phase difference \(\Delta \Psi_0\) is a measure for \(|m|\) (G&K88) is only correct for sectoral modes, which have \(|m|=\ell\). In the next sections we will show that in general the phase difference \(\Delta \Psi_0\) is a measure of \(\ell\) (rather than \(|m|\)), and that the phase difference \(\Delta \Psi_1\), if detectable, is a reasonable estimator of the value of \(|m|\).

## 4. Relation between blue-to-red phase differences and pulsation parameters \(\ell\) and \(m\)

In Figs. 3–8 we present the results of the analyses of phase diagrams derived from generated time series of absorption line profiles of a mono-periodic non-radially pulsating early-type star. For all modes with \(0 \leq \ell \leq 15\) and \(-\ell \leq m \leq \ell\) there is one entry in each of the panels in the plots; for Fig. 8 only positive values of \(m\) are computed. In the top row of panels we plot the blue-to-red phase difference \(\Delta \Psi_0\) as a function of input \(\ell\). In the bottom row we plot the blue-to-red harmonic phase difference \(\Delta \Psi_1\) as a function of input \(m\). In Figs. 4 and 8 we plot an additional row of panels with the blue-to-red harmonic phase difference \(\Delta \Psi_1\) as a function of input \(\ell\).

### 4.1. Single spheroidal modes

From Figs. 3–7 we derive that in many cases there is sufficient information in the line-profile variations to estimate \(\ell\) and \(m\). One can immediately see that there is at least a tendency towards the linear relations

\[ |\Delta \Psi_0| \approx \pi \ell, \quad |\Delta \Psi_1| \approx 2\pi |m|. \]  

In the following we will discuss for what range of the relevant parameters these relations can actually be used to estimate the pulsation parameters \(\ell\) and \(|m|\).

**Width of the intrinsic line profile** (see Fig. 3). Unambiguous Doppler mapping of the pulsational variations onto the absorption line profile is only the case if the \(V_\star \sin i\) of the star is
sufficiently larger than the intrinsic line width $W$. For higher values of $W/(V_\text{e} \sin i)$ the variability caused by a particular surface element is smoothed out over a large part of the line profile, overlapping and cancelling the line-profile variability caused by other surface elements. If the overlap is too large the characteristic pattern of bumps moving from blue to red is not an unambiguous measure of the pulsation parameters $\ell$ and $|m|$ anymore, and the identification of the pulsation mode can better be carried out using the “moment method” (Balona 1986 and Aerts et al. 1992), which is intrinsically independent of $W$.

From Fig. 3 we find that the identification of $\ell$ from the blue-to-red phase difference $\Delta \Psi_0$ is possible for nearly all investigated $\ell$ values if $W/(V_\text{e} \sin i) \leq 0.1$. For $W/(V_\text{e} \sin i) = 0.2$, $\ell$-values up to $\approx 10$ can be identified.

Identification of $m$ from the $\Delta \Psi_1$ diagnostic becomes hazardous if $W/(V_\text{e} \sin i) > 0.1$. If $W/(V_\text{e} \sin i) = 0.1$ values of $|m|$ up to $\approx 10$ can be derived, for $W/(V_\text{e} \sin i) = 0.2$ only values of $|m|$ lower than $\approx 7$. Note that for increasing $W$, the contribution of harmonics to the line-profile variability decreases (Paper I), which means that values for $m$ are best derived from lines with little intrinsic broadening.

Inclination (see Fig. 4). We find that for all inclination angles the phase differences $\Delta \Psi_0$ fit a linear relation with $\ell$. For inclinations close to $i = 90^\circ$, however, profile variations of tesseral modes with a nodal circle along the equator suffer from severe cancellation effects. This can result in very small amplitudes of the line-profile variations, and in non-monotonic phase diagrams $\Psi_0(\lambda)$ (see Sect. 4.3). For the exact equator-on case these modes do not follow the $\Delta \Psi_0$ versus $\ell$ relation (see Reid & Aerts 1993 and Paper III).

The reverse is true for the harmonic phase differences: the closer the inclination is to equator-on, the clearer is the relation between $m$ and $\Delta \Psi_1$. From the middle row of panels in Fig. 4 we find that for low inclination angles modes with low $m$ tend to follow a relation between $\Delta \Psi_1$ and $\ell$, rather than $|m|$.

Pulsation amplitudes (see Fig. 5). In general we can say that the larger the pulsation amplitudes the better one can detect and interpret the pulsational line-profile variability. However, if the pulsation amplitude is more than (approximately) $0.5 V_\text{e} \sin i$, the variability of a particular surface element will be Doppler-imaged over a relatively large part of the line profile. Such smearing of the variability hampers the identification of $\ell$ of high-degree modes.

For high values of $V_{\text{max}}/(V_\text{e} \sin i)$ the relative contribution of line-profile variability with harmonic frequencies becomes important. In the bottom panels of Fig. 5 one can see that for high pulsation amplitudes the disturbing effects of rotation (visible as the different behaviour of prograde and retrograde modes) are less prominent than for low pulsation amplitudes.
Fig. 4. Blue-to-red phase differences $\Delta \Psi_0$ and $\Delta \Psi_1$ for different values of the inclination of the star. We plot the phase difference $\Delta \Psi_0$ as a function of $\ell$, and the phase difference $\Delta \Psi_1$ as a function of both $\ell$ and $m$; all spheroidal modes with $0 \leq \ell \leq 15$ and $-\ell \leq m \leq \ell$ have one entry in each panel. The top half of the figure shows our computations for modes with $k(0)=0.2$, the bottom half shows that for $k(0)=1.8$. The other parameters are: $\Omega/\omega(0)=0.0$, $(\delta T/T)_{\text{max}}=0.0$, $V_{\text{max}}=0.15V_e \sin i$, $W=0.1V_e \sin i$. See Fig. 3 for further explanation. The number of outliers of the relation between $\Delta \Psi_0$ and $\ell$ is large for inclinations near $i=90^\circ$, because of cancellation effects for modes with a nodal line on the equator. For other values of the inclination there are only few outliers. We find that, especially for high inclination angles, the harmonic phase difference $\Delta \Psi_1$ is related to the value of $m$. 
Fig. 5. As Fig. 3, but for different values of the maximum vector velocity amplitude $V_{\text{max}}$ of the pulsation; we chose $V_{\text{max}}=0.1V_c \sin i$ and $V_{\text{max}}=0.3V_c \sin i$, and used $\delta V=0.1V_c \sin i$. See Fig. 3 for the other relevant parameters. With $V_{\text{max}}(V_c \sin i)$ increasing to very high values, the ability to derive $\ell$ decreases whereas the ability to derive $m$ increases.

If the pulsation velocity amplitude is very high, one expects that the linear description of the oscillation behaviour is not valid, and that the pulsation eigenfunctions will be periodic but not sinusoidal. This will increase the amplitude of the line-profile variations with harmonic frequencies with respect to the case of linear oscillations, but from our model we cannot make predictions for the phase relations $\Psi_0(\lambda)$ and $\Psi_1(\lambda)$ in the non-linear case.

Ratio of horizontal to vertical pulsation amplitudes (see Figs. 4 and 7). For high values of $k^{(0)}$ (say $k^{(0)} \gtrsim 0.4$) the $\ell$ versus $\Delta \Psi_0$ diagram shows two parallel arms, one $\sim 2\pi$ apart from the other. This occurs for modes that show no or very little variability at line centre in the zero-rotation model, which enables secondary sources of line-centre variability (Coriolis terms, temperature effects) to connect the variations in the red side of the profile with those in the blue side (see Paper I for a discussion of this effect).

With increasing $k^{(0)}$ the $m$ versus $\Delta \Psi_1$ relation becomes more ambiguous. Any derived value of $m$ will have a relatively large error for pulsation modes with large $k$ ($g$ modes). For $p$ modes the relation between $m$ and $\Delta \Psi_1$ is fairly strict.

Ratio of rotation frequency and pulsation frequency. Fig. 6 displays the results of our computations for different values of the rotation parameter $\Omega/\omega^{(0)}$, which governs the relative contribution of the Coriolis terms to the eigenfunction. In the diagrams in Fig. 6 we mark all modes with $|m| \leq 1$, which shows that virtually all outliers of the $\ell$ relation are zonal modes and low-$m$ tesseral modes. We see that the effects of rotation causes asymmetry between the prograde and retrograde arms of the diagrams, but that within the limits of our pulsation model ($\Omega/\omega^{(0)} \leq 0.5$) the effects of the Coriolis force do not significantly change the relations between the blue-to-red phase differences and pulsation parameters $\ell$ and $m$.

Temperature effects. In Fig. 7 we show that for $p$ modes the oscillatory brightness variations do not affect the relations between the blue-to-red phase differences and $\ell$ and $m$. For $g$ modes the scatter around these relations increases, because the temperature effects can, similar to the effects of the Coriolis force, cause deviations of $\sim 2\pi$ in the blue-to-red phase difference.

Lee et al. (1992) found that the changes of the equivalent width ($W_E$) of the local intrinsic profiles either enhance or decrease the effects of brightness changes on the line-profile variability. We investigate values of the equivalent width response of $-2 < \Delta W_E < 2$. We confirm the results of Lee et al. (1992), and find that the derived phase differences are distributed in the diagrams as in the case of brightness variations.

We investigate the values of the phase differences $\Delta \Psi_0$ and $\Delta \Psi_1$ for moderate (non-adiabatic) phase-lags between the temperature variations and the radial displacement ($-10^\circ < \phi_{\text{lag}} < 10^\circ$ for $g$ modes, $170^\circ < \phi_{\text{lag}} < 190^\circ$ for $p$ modes). We find no substantial changes in the derived phase differences with respect to cases with fixed (adiabatic) phase-lags ($\phi_{\text{lag}}=0^\circ$ for $g$ modes, $\phi_{\text{lag}}=180^\circ$ for $p$ modes).
4.2. Single toroidal modes

Papaloizou & Pringle (1978) found that in rotating stars toroidal oscillations can be excited, that have pulsation frequencies in the corotating frame

\[ \omega = \frac{2m\Omega}{\ell(\ell + 1)}, \]

and therefore propagate retrograde (counter to the stellar rotation) in the corotating frame. For the toroidal velocity vector we use

\[ \mathbf{V}_{\text{tor}} \propto \left(0, \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) Y^{m}_{\ell} e^{i\omega t}. \]

Fig. 8 shows the phase differences \( \Delta \Psi_{0} \) and \( \Delta \Psi_{1} \), for all toroidal modes with \( 1 \leq \ell \leq 15 \) and \( 1 \leq m \leq \ell \), for different values of the inclination of the star. As for spheroidal modes, we also find for toroidal modes a relation between \( \ell \) and \( \Delta \Psi_{0} \), and between \( m \) and \( \Delta \Psi_{1} \). However, for low inclination the latter relation does not hold. Note that for low inclination retrograde tesseral modes appear as prograde in the line profile (in the corotating frame): the variations at the “far” side of the star contribute dominantly to the line-profile variability in these cases.

In the remainder of this paper we will concentrate on spheroidal modes only.

4.3. Outliers

4.3.1. Outliers of the \( \Delta \Psi_{0} \) versus \( \ell \) relation

Figs. 3–7 display the diagnostic value of the blue-to-red phase differences, which relate to the pulsation parameters \( \ell \) and \( m \).
Fig. 8. As Fig. 4, but for toroidal oscillation modes \((1 \leq m \leq 15)\). We find also for toroidal modes a relation between \(\ell\) and \(\Delta \Psi_0\), and between \(m\) and \(\Delta \Psi_1\). For low-degree modes seen at a low inclination the latter relation does not hold. In the bottom right panel the relation between \(m\) and \(\Delta \Psi_1\) is so strict that many circles are plotted on top of each other. Note that for low inclination retrograde tesseral modes can appear as prograde in the line profile (in the corotating frame), which is because in this case the variability at the “far” side of the star dominates the line-profile variations

Within the parameter ranges in Table 1 (see Sect. 4.4), sectorial modes with \(3 \leq \ell \leq 12\) do not show sign changes of the slope of \(\Psi_0(\lambda)\) in the part of the profile outside the velocity interval \([-0.25V_e \sin i, 0.25V_e \sin i]\). This means that phase diagrams \(\Psi_0(\lambda)\) with slope changes outside this interval, and with a blue-to-red phase difference \(|\Delta \Psi_0| \geq 2\), can only be caused by tesseral or zonal modes. Less than 1% of the investigated tesseral modes with \(3 \leq |m| \leq 12\) and \(\ell \leq 12\) have slope reversals which result in outliers in \(\ell\) versus \(\Delta \Psi_0\). Considering that the outliers can be detected by inspection of the phase diagram, we conclude that \(\ell\) versus \(\Delta \Psi_0\) relation is fairly strict.

4.3.2. Outliers of the \(\Delta \Psi_1\) versus \(m\) relation

The number of tesseral modes with sign changes in the slope of the phase diagram \(\Psi_1(\lambda)\) is quite large: \(\sim 30\%\) of the investigated cases. An example of this is given in the third column of Fig. 1. Contrary to the \(\ell\) versus \(\Delta \Psi_0\) diagram, the untidy nature of the relation between \(m\) and \(\Delta \Psi_1\) can not be removed by excluding modes for which slope reversals occur.

4.4. Monte-Carlo approach

For diagnostic purposes we quantify the relation between \(\ell\) and \(|\Delta \Psi_0|\), and that between \(|m|\) and \(|\Delta \Psi_1|\), for spheroidal modes. We also estimate the percentage of outliers of these relations.

In the frame of the observer the line-profile variability caused by non-radial spheroidal oscillations will appear prograde in many cases, because the rotation of the star adds \(|m|\Omega\) to the observed pulsation frequency. On the contrary, in particular cases with low inclination, when the oscillations at the “far” side of the star contribute significantly to the line-profile variations, the profile variations appear retrograde by the same effect (see Baade 1984 for the case of the Be star \(\mu\) Cen). To by-pass such ambiguities we model \(\ell\) and \(|m|\) as a function of the absolute blue-to-red phase differences \(|\Delta \Psi_0|\) and \(|\Delta \Psi_1|\) respectively.

Assuming that each possible combination of \(\ell\) and \(m\) is equally probable, we compute 60 time series of line profiles for each combination of \(\ell\) and \(m\) with \(\ell \leq 15\). The other relevant parameters \((i, W, V_{\text{max}}, \Omega/\omega(0), k(0), (\delta T/T)_{\text{max}}, \phi_{\text{lag}}, \alpha_{W_0})\) are chosen at random within the ranges specified in Table 1. Values for the inclination are drawn according to the probability \(p(i) = \sin i\). Values for \(\log(k(0))\) are drawn such that the combi-
Fig. 9. Results of Monte-Carlo calculations. We plot the blue-to-red phase difference $|\Delta \Psi_0|$ against $\ell$, and $|\Delta \Psi_1|$ against $\ell$ and $|m|$, with the number of occurrences as a grey value. For each vertical bin in the plots the grey scale is normalized to the total number of computed modes in that bin. Top: All 15360 modes, with stellar and pulsation parameters as specified in Table 1. Middle: Selection of all modes, with $k>0.7$ (mainly $g$ modes), $i>45^\circ$ and without slope reversals (see Sect. 4.3) in the main phase diagram $\Psi_0(\lambda)$. Bottom: Selection of all modes, with $k<0.3$ (mainly $p$ modes), $i>45^\circ$ and without slope reversals in $\Psi_0(\lambda)$. 

To quantify the relation between the phase differences and $\ell$ and $|m|$ we perform a least-squares fit of a straight line to the data in Fig. 9, with $\ell$ as a function of $|\Delta \Psi_0|$ and $|m|$ as a function of $|\Delta \Psi_1|$.

$$\ell = p\ell + q|\Delta \Psi_0|/\pi, \quad |m| = pm + qm|\Delta \Psi_1|/\pi,$$  \hspace{1cm} (8)

and use an iterative rejection algorithm (sigma-clipping with 2$\sigma$ threshold) to discard the outlying points. After the rejection iterations, we iteratively increase the number of fit points $N_{fit}$ to all modes within 3$\sigma$ distance from the fit, where we estimate $\sigma^2$ as the sum of the squared distances from the fit divided by $N_{fit}$. Finally, we record how many modes lie within the intervals $[\ell-1, \ell+1]$, $[\ell-2, \ell+2]$, $[\ell-3, \ell+3]$ from the fit (and similar for $|m|$), to estimate the probability that the fit can successfully model real data. We use all modes with $\ell \leq 15$ to fit the coefficients $p\ell$ and $q\ell$; to derive the coefficients $pm$ and $qm$, we use only modes with $\ell \leq 12$. 

In Fig. 9 we overplot the fits in the grey-scale images. In Tables 2 and 3 we list the fit results for several regions in parameter space, including entries for “easy” detectable modes and modes with large temperature variations relative to velocity variations. The category of easy detectable modes consists of those $\sim 50\%$ of all modes that have the highest ratio of mean amplitude of
Table 2. Straight line fits to $\ell$ as a function of $|\Delta \Psi_0|$. $N_{\text{tot}}$ is the number of modes before rejection. $N_{\text{fit}}$ is the number of modes within 3$\sigma$ distance from the fit. The rightmost three columns list the percentage of $N_{\text{fit}}$ modes that lie within the indicated intervals around the fit.

| $|m|$ | $p_m$ | $q_m$ | $N_{\text{tot}}$ | $N_{\text{fit}}$ | $|\ell-1\ell+1|$ | $|\ell-2\ell+2|$ | $|\ell-3\ell+3|$ |
|-------|-----|-----|----------------|----------------|----------------|----------------|----------------|
| all modes | 0.099(27) | 1.090(3) | 15360 | 11006 | 71% | 78% | 94% |
| “easy” detectable modes (see text) | 0.065(35) | 1.098(4) | 7198 | 6055 | 85% | 88% | 99% |
| modes with $|m| > 2$ | 0.227(38) | 1.082(4) | 10920 | 8558 | 79% | 82% | 99% |
| modes without slope reversals in $\Psi_0(\lambda)$ outside $[-0.25,0.25]$ sin $i$: | | | | | | | |
| all modes | 0.089(29) | 1.091(3) | 13444 | 10490 | 78% | 83% | 99% |
| “easy” detectable modes | 0.089(37) | 1.096(4) | 6950 | 5921 | 86% | 88% | 100% |
| modes with $|m| > 0$ | 0.117(30) | 1.089(3) | 12943 | 10101 | 78% | 83% | 99% |
| modes with $|m| > 1$ | 0.163(33) | 1.086(3) | 11941 | 9417 | 79% | 83% | 100% |
| modes with $|m| > 2$ | 0.226(38) | 1.082(4) | 10644 | 8481 | 80% | 83% | 100% |
| modes with $\ell - |m| < 2$ | 0.076(42) | 1.110(5) | 3448 | 2997 | 88% | 90% | 100% |
| modes with $\ell - |m| < 6$ | 0.015(32) | 1.109(3) | 8720 | 7268 | 84% | 86% | 100% |
| modes with $25^\circ < i < 50^\circ$ | 0.028(54) | 1.103(5) | 3971 | 3233 | 83% | 86% | 100% |
| modes with $i > 65^\circ$ | 0.186(42) | 1.075(4) | 6107 | 4711 | 75% | 82% | 100% |
| modes with $\ell(0) < 0.3$ | -0.098(52) | 1.100(5) | 3647 | 3452 | 94% | 98% | 100% |
| modes with $\ell(0) > 0.7$ | 0.188(45) | 1.078(5) | 6496 | 4318 | 67% | 73% | 99% |
| modes with $\Omega/\omega(0) < 0.2$ | 0.055(38) | 1.093(4) | 7517 | 6261 | 83% | 89% | 100% |
| modes with $\Omega/\omega(0) > 0.2$ | 0.124(44) | 1.091(5) | 5927 | 4230 | 72% | 78% | 99% |
| modes with $(\delta T/T)_{\text{max}} V_\ell sin i/V_{\text{max}} > 0.4$ | 0.046(77) | 1.095(8) | 1980 | 1537 | 78% | 84% | 100% |

Table 3. Straight line fits to $|m|$ as a function of $|\Delta \Psi_1|$.

| $|m|$ | $p_m$ | $q_m$ | $N_{\text{tot}}$ | $N_{\text{fit}}$ | $|m| - 1$ | $|m| + 1$ | $|m| - 2$ | $|m| + 2$ | $|m| - 3$ | $|m| + 3$ |
|-------|-----|-----|----------------|----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| all modes | -1.334(24) | 0.537(2) | 10140 | 9969 | 46% | 83% | 91% |
| “easy” detectable modes (see text) | -1.001(37) | 0.484(3) | 5359 | 5240 | 53% | 85% | 97% |
| modes with $|m| > 0$ | -1.120(27) | 0.542(2) | 9360 | 8887 | 46% | 81% | 92% |
| modes with $|m| > 1$ | -0.937(33) | 0.534(3) | 7920 | 7664 | 45% | 84% | 93% |
| modes with $|m| > 2$ | -0.650(42) | 0.522(3) | 6600 | 6482 | 48% | 87% | 95% |
| modes with $\ell - |m| < 2$ | -1.028(43) | 0.613(3) | 2880 | 2692 | 83% | 95% | 100% |
| modes with $\ell - |m| < 6$ | -1.289(29) | 0.567(2) | 7200 | 7140 | 44% | 87% | 97% |
| modes with $25^\circ < i < 50^\circ$ | -1.321(46) | 0.519(4) | 2913 | 2837 | 48% | 81% | 91% |
| modes with $i > 65^\circ$ | -1.361(34) | 0.586(3) | 4767 | 4408 | 50% | 87% | 92% |
| modes with $\ell(0) < 0.3$ | -1.097(48) | 0.614(4) | 2585 | 2033 | 77% | 82% | 90% |
| modes with $\ell(0) > 0.7$ | -1.475(35) | 0.527(3) | 5045 | 4990 | 50% | 82% | 92% |
| modes with $\Omega/\omega(0) < 0.2$ | -1.360(32) | 0.571(3) | 5532 | 5231 | 48% | 82% | 92% |
| modes with $\Omega/\omega(0) > 0.2$ | -1.354(36) | 0.518(3) | 4608 | 4548 | 47% | 80% | 90% |
| modes with $(\delta T/T)_{\text{max}} V_\ell sin i/V_{\text{max}} > 0.4$ | -1.261(62) | 0.521(5) | 1507 | 1480 | 47% | 82% | 91% |

Line-profile variations (expressed in units of average line depth) to pulsation velocity $V_\text{max}/V_\ell \sin i$.

We find that throughout parameter space the fitted coefficients $p_m$, $q_m$, $p_m$, and $q_m$ are remarkably stable, and that in general the phase differences relative to the pulsation parameters as

$$\ell \approx 0.10 + 1.09|\Delta \Psi_0|/\pi$$

(9)

$$|m| \approx -1.33 + 0.54|\Delta \Psi_1|/\pi .$$

(10)

From the stability of the coefficients we conclude that it is possible to derive good estimates for the pulsation parameters $\ell$ and $|m|$ from the evaluation of the phases of the variability across the line profile. The percentages of modes in the vicinity of the fits (Tables 2 and 3) indicate that reasonable error estimates for a derivation of $\ell$ or $|m|$ with the IPS method are $\pm 1$ and $\pm 2$ respectively. For $p$ modes values of $|m|$ with accuracy $\pm 1$ can be derived. If stellar and pulsational parameters can be constrained, better accuracies can be achieved. The values of $\ell$ and $|m|$, as derived from the phase diagrams, can then be used as initial guesses in more detailed modelling of the line-profile variability.

4.5. Line-centre phases of variability at the apparent pulsation frequency and at its first harmonic

If we compare the arguments of the main and first harmonic sinusoid describing the line-centre variability (see Eq. 4), and allow an arbitrary choice of time reference $t_0$

$$2(\omega_{\text{obs}}(t - t_0) + \Psi_0) - (2\omega_{\text{obs}}(t - t_0) + \Psi_1)$$

(11)
we find that the phase difference between the line-centre phases $\Psi_0$ and $\Psi_1$ written as

$$\Psi_{01} = 2\Psi_0 - \Psi_1.$$

(12)
is invariant for a translation in time, provided that the observed harmonic frequency $2\omega_{\text{obs}}$ is exactly twice the observed frequency $\omega_{\text{obs}}$. For our 15360 time series we determine the value of $\Psi_{01}$, and plot the results as a histogram in Fig. 10. We find that the value of $\Psi_{01}$ does not depend on any of the parameters that we vary in our Monte-Carlo simulations, and that $\Psi_{01}=1.5\pi\pm0.06\pi$. In principle one can use this information to check whether an observed phase diagram is due to harmonic variability or to a different pulsation mode, provided that the apparent frequencies and phases in the line centre are well established.

### 4.6. The origin of the derived relations

In the previous subsections we have shown that the phase difference $\Delta\Psi_0$ is a measure of $\ell$, that the phase difference $\Delta\Psi_1$ is a measure of $m$, and that the line-centre phase difference $\Psi_{01}=1.5\pi$. These relations were found empirically by careful examination of the results of our modelling.

One of the implications of these relations is that for stars pulsating in a single mode the number of bumps and troughs in the line profiles is a measure of $\ell$, but only when the harmonic contribution is relatively unimportant. If the line-profile variability at the first harmonic frequency is larger than that at the pulsation frequency, the number of bumps and troughs gives twice the value of $\ell$ (see Paper III). In Sect. 1 we gave a qualitative explanation for the fact that the phase difference $\Delta\Psi_0$ is a measure of $\ell$, rather than $m$ (see Fig. 1).

The pulsational velocity field gives rise to profile variations in the velocity direction, which consequently leads to non-sinusoidal intensity variations in the profiles. The larger the pulsational velocity, the larger the harmonic component in the line-profile intensity variations. Brightness variations give rise to sinusoidal intensity variations, since there are no velocity shifts involved other than the rotational broadening. Brightness variations in combination with equivalent width variations also give rise to harmonic variability. At present we do not understand why the harmonic phase difference $\Delta\Psi_1$ is a measure of $m$.

From the phase relation $\Psi_{01}=1.5\pi$ it follows that at the line centre the “absorption” troughs go deeper below the mean spectrum than that the “emission” bumps reach above it, which is evident in the grey-scale images in Fig. 1. Blue-shifted and red-shifted surface areas are mapped on top of each other in the line profile, leading to deep absorption features. The absorption troughs are narrow and the emission bumps are wide, which reflects the fact that the pulsational velocity field gives rise to conservative redistribution of the flux in the line profile.

We stress that a mathematical proof is needed to improve our understanding of the above derived relations.

### 5. Amplitude and phase diagrams of spheroidal multi-mode pulsations

Although linear pulsation theory allows the addition of the velocity fields of individual pulsation modes, the mapping from the three dimensional velocity field to line-profile variability is not additive; one cannot simply add up generated line-profile variations of individual pulsation modes to create line-profile variability of a multi-mode situation. In fact, the superposition of pulsation modes leads to beatings in the line-profile variations. The number of bumps and troughs in the line profiles is determined by all pulsation modes that are present, and may vary in time due to the beatings that appear in the time series of profiles.

This can be understood easiest by considering the Doppler velocity of a single surface element. For a single oscillation mode the velocity variation of a particular surface element is mapped onto a small region of the line profile. A second pulsation mode, if simultaneously present in the star, will add its share to the Doppler velocity of the element, and consequently the velocity variations of the element will be Doppler mapped onto a wider range in the line profile. Since the two pulsation modes will generally not have the same apparent frequency, the contributions of each of the modes to the Doppler velocity of the surface element will not be in phase with each other. Looking at a particular position in the line profile, one would see modulated signals of the two sources of variability. Hence, for multi-mode pulsations we expect to find variability with the pulsation frequencies and their harmonics, and also with sum and beat frequencies of all these.
Fig. 11. Analysis of generated spectral time series of a star with 4 simultaneous non-radial spheroidal pulsation modes. The frequencies, and parameters are given in the text and in Table 4. Left Periodogram of our computations with $k=0.4$. For each investigated frequency, the grey-coded amplitudes of the variations are plotted as a function of wavelength. The amplitudes are cut from $1 \times 10^{-4}$ to $6 \times 10^{-4}$ (continuum units) as white to black. The input pulsation frequencies are marked at both sides of the plot. The bottom panel gives the average line profile. Middle Amplitude and phase diagrams for our simulations with $k=0.1$. The amplitudes of the variations are given in continuum units; the phases are given in $\pi$ radians. For each mode the amplitude $I_0(\lambda)$ and phase $\Psi_0(\lambda)$ diagram are plotted as thick lines, and the harmonic amplitude $I_1(\lambda)$ and phase $\Psi_1(\lambda)$ diagram are plotted as thin lines. The dashed-dotted lines give those for the mono-mode case. The top panel gives the average line profile. Right Computations with $k=0.4$. Note that in the multi-mode case variational power leaks to sum and beat frequencies, and that therefore the amplitude diagrams of the multi-mode and mono-mode case differ. Also note that although the amplitude diagrams in the multi-mode case are different from the mono-mode case, the phase diagrams $\Psi_0$ are the same.

Note that the pulsational brightness variations do not give rise to sum and beat frequencies in the line-profile variations. Only for brightness variations in combination with large variations of the equivalent width of the intrinsic profiles can harmonic, sum and beat frequencies occur. However, these beatings are never as prominent as those of the velocity fields that accompany such large temperature variations. (We intend to elaborate on this subject in a separate paper.)

To investigate the importance of the beatings for the amplitude and phase diagrams we generate time series of 1000 line profiles as seen in the observer’s frame, spread over one continuous data set of 10 days with constant time step (all beat frequencies should be properly sampled). Since we do not know in advance which of the apparent pulsation frequencies, harmonics and beat frequencies have significant amplitudes, we choose to analyse these data sets with the IPS technique described by G&K88. For each velocity bin we compute the Fourier transform, and CLEAN the result from the window function (Roberts et al. 1987) using a gain of 0.2 and 400 CLEAN iterations. For all investigated frequencies $0<f<25$ cycles/day we then transform the variational powers $p(f,\lambda)$ to amplitudes $a(f,\lambda)$ using $a=2\sqrt{p}$. From the resulting amplitudes and phases we create similar diagrams as for our mono-mode analyses.

In Fig. 11 we present our analyses of spectral time series of a star with four simultaneous pulsation modes (two sectoral and two tesseral). The stellar and pulsational parameters that we used are: $i=75^\circ$, $V_e \sin i=150$ km/s, $W=0.1V_e \sin i$, $\Omega/\omega^{(0)}=0.0$ (zero-rotation model), and $(\delta T/T)_{\text{max}}=0.0$. We present calcula-
tions for two values of the ratio of horizontal to vertical oscillatory motions: $k=0.1$ and $k=0.4$. The apparent frequencies and the velocity amplitudes of the modes are given in Table 4.

In the left side of Fig. 11 we present the periodogram of the time series computed with $k=0.4$. For each investigated frequency, the amplitude (not power) diagram is plotted as a grey value across the line profile. We see that the line-profile variations do not only appear with the input pulsation frequencies and their harmonics, but also with sum and beat frequencies of all these. We stress here that this is due to the intrinsic summation of the pulsation fields, and not due to the method of analysis; the window function of our generated time series hardly has sidelobes, and is effectively removed by the CLEAN algorithm.

In Fig. 11 (middle and right) we also present amplitude and phase diagrams of both sets of multi-mode calculations, and compare these diagrams with those of modes in a mono-periodic star. As discussed above, the summation of the Doppler-velocities associated with the pulsations leads to a different mapping to the absorption line than in the case of a single pulsation mode. In the multi-mode case variational power leaks to sum and beat frequencies, and therefore the amplitude diagrams of the multi-mode and mono-mode case differ. However, for the identification of the pulsation modes it is important to know that although the amplitude diagrams of the mono and multi-mode case are different, the phase diagrams $\Psi_0(\lambda)$ are the same.

In principle, the multi-mode harmonic phase diagrams $\Psi_1(\lambda)$ are also not affected with respect to the mono-mode case. However, a phase diagram might be affected by that of neighbouring frequencies. In Fig. 11 one can see that the amplitude and phase diagrams of the weak first harmonic of $f_3$ are influenced by the variability at the sum frequency of the stronger $f_2$ and $f_4$ modes. Only with a data set with a very long time coverage, these two combined variations can be resolved to distinct frequencies. In our examples the difference between the apparent frequencies $2f_1$ and $f_2+f_3$ is 0.06 cycle/day, which means that one needs a time base of $\sim$15 days to completely separate these variational frequencies.

We conclude that if all apparent frequencies can be resolved, the phase diagrams modelled for mono-periodic stars also apply to multi-periodic multi-mode stars.

6. Application of the IPS method to real data

We apply the outcome of our modelling to analyse and interpret two time series of line profiles, kindly made available to us by Drs. D.R. Gies and A.H.N. Reid.

6.1. $\epsilon$ Per

The B0.7 III star $\epsilon$ Per was the first subject of the IPS method as described by G&K88 and Gies (1991), and might be part of a triple system (Tarasov et al. 1995). For the observational history of this star we refer to these papers and the references therein. G&K88 found four frequencies in the line-profile variability, and attributed these to four coexisting pulsation modes.

We compute a periodogram (see Fig. 12) using the data of the SiIII $\lambda$ 4552 line that was first presented by G&K88. Like G&K88, we use the IPS technique, but with different parameters to CLEAN (Roberts et al. 1987) the periodograms: 400 iterations with a gain of 0.2 (G&K88 used 15 iterations with gain 0.9). The result is very similar to that obtained by G&K88, except for different contributions of power at some of the one-day aliases. The observed frequencies are listed in Table 5; frequencies labelled with subscripts $a$ and $b$ denote one-day aliases. We adopt the HWHM of the main power peak of the window function as estimate for the systematic error in the derived frequencies. For a few detected frequencies we plot the amplitudes and phases of the variations as a function of position in the line profile in Fig. 13.

From Fig. 12 we find, besides variability at the four previously noted frequencies and their one-day aliases, evidence for variational power at a complex pattern of frequencies. The pattern consists of many discrete patches of power in the periodogram, some of which extend throughout the line profile. We note that this pattern is very similar to what we find in our multi-mode calculations (see Fig. 11), where harmonics and sum and beat frequencies of the pulsation frequencies are apparent. This can be an indication that the line-profile variations in $\epsilon$ Per are...
Table 5. Apparent pulsation frequencies (in cycles/day), derived blue-to-red phase differences and phases in line centre (λ 4552.7) of ε Per. Phases are given in π radians. The frequencies labelled with a and b are one-day aliases. The last 3 columns lists the values of ℓ or |m| that we derive from the phase differences ∆Ψ. 1σ errors are listed in parentheses. For scenario B the frequencies f₄ and f₅ are harmonics of f₁ and f₂ respectively.

<table>
<thead>
<tr>
<th>ω₀Di</th>
<th>∆Ψ</th>
<th>Ψ centre</th>
<th>scenario A</th>
<th>scenario B</th>
<th>Gies &amp; Kullavanjaya (1988)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f₁</td>
<td>5.37(9)</td>
<td>2.8(0.5)</td>
<td>0.7(1)</td>
<td>ℓ=3.1(1.0)</td>
<td>ℓ=3.1(1.0)</td>
</tr>
<tr>
<td>f₂</td>
<td>6.26(9)</td>
<td>3.8(0.5)</td>
<td>1.7(1)</td>
<td>ℓ=4.2(1.0)</td>
<td>ℓ=4.2(1.0)</td>
</tr>
<tr>
<td>f₃ₐ</td>
<td>6.94(9)</td>
<td>4.5(0.5)</td>
<td>1.0(1)</td>
<td>ℓ=5.0(1.0)</td>
<td>ℓ=5.0(1.0)</td>
</tr>
<tr>
<td>f₃₆</td>
<td>7.91(9)</td>
<td>3.3(1.0)</td>
<td>1.6(1)</td>
<td>ℓ=3.6(2.0)</td>
<td>ℓ=3.6(2.0)</td>
</tr>
<tr>
<td>f₄</td>
<td>10.61(9)</td>
<td>5.0(0.5)</td>
<td>0.1(1)</td>
<td>ℓ=5.5(1.0)</td>
<td>ℓ=2.0(1.0)</td>
</tr>
<tr>
<td>f₅</td>
<td>12.53(9)</td>
<td>4.5(1.0)</td>
<td>0.1(1)</td>
<td>ℓ=4.6(2.0)</td>
<td>ℓ=1.7(2.0)</td>
</tr>
</tbody>
</table>

mainly due to Doppler redistribution of flux caused by the velocity field of the pulsations; for line-profile variability that is mainly caused by pulsational temperature variations beatings are not expected to be so prominent (see Sect. 5).

In principle, one can use the observed sum, difference and harmonic frequencies, to identify which of the frequencies are one-day aliases. However, the crowdedness of the observed frequency pattern in combination with the limited time base of the observations (5 nights), makes such a procedure very difficult in this case. Nevertheless, we pick out one frequency at which we find power, since this frequency corresponds to the harmonic of one of the four frequencies found by G&K88: f₅=12.53(0.09)=2×6.26(0.09)=2f₂. Furthermore, we note that within the accuracy of the frequency determination f₅≈2f₁, which gives another harmonic frequency candidate.

From the data in Figs. 12 and 13 we read off the phases of the variability at line centre, for each of the frequencies in Table 5. We test whether f₄ and f₅ are harmonic frequencies of f₁ and f₂ respectively, using this line-centre phase information (see Sect. 4.5). For the frequency pair f₁, f₂ we find Ψ₀₁=2Ψ₀−Ψ₁=1.3π(0.2π), for pair f₂, f₅ we also find Ψ₀₁=1.3π(0.2π) (see Table 5). Here the estimated errors should account for possible misplacement of the line centre and inaccuracy in the frequency determinations. Within their accuracies, these values are (marginally) consistent with that expected for first harmonic frequencies, i.e. Ψ₀₁=1.50π(0.06π).

For each of the detected frequencies, we read off the phase diagrams in the way described in Sect. 3, to obtain blue-to-red phase differences (see Table 5). Note that the blue-to-red phase differences of the variations at frequencies f₃₆ and f₅ are probably underestimated due to limited variational power in the line wings.

We interpret the derived phase differences for two scenarios. For the first case (scenario A) we ignore that, given the large line-profile variations, we expect considerable contributions of harmonics: we assume that none of the detected frequencies are due to harmonic variability. Contrary to G&K88, we do not assume that the modes are sectoral; we convert the blue-to-red phase differences to values of ℓ using Eq. (9). The results of this procedure are listed in Table 5, and are consistent with the results of G&K88.

For scenario B we assume that frequencies f₄ and f₅ are indeed the first harmonics of f₁ and f₂, respectively. In this
case there are only three intrinsic frequencies left, and consequently the star has only three coexisting pulsation modes. The amplitude diagrams of the detected frequencies are consistent with those expected for p modes. Hence, we use the entry in Table 3 for modes with low k values and Eq. (8), to constrain the value of |m| of the modes responsible for the variations at frequencies f₁ and f₃. We find that the mode that gives rise to the detected frequencies f₁ and f₄ has ℓ=3.1(1.0) and |m|=2.0(1.0), and that the detected frequencies f₂ and f₅ are due to a mode with ℓ=4.2(1.0) and |m|=1.7(2.0). The latter of these |m| values (derived from f₅) might be underestimated due to limited variational power with respect to the noise in the wings of the profile. Although the values of ℓ and |m| of the first of these two modes are consistent with a sectoral mode, the amplitude ratio of the variations at f₁ and f₄ fits a tesseral mode better (see Paper I). We conclude that if frequencies f₄ and f₅ are harmonic frequencies, the corresponding pulsations modes might not be sectoral (see Table 5).

We argue that only if more accurate frequency determinations become available, one can determine which of the two scenarios is favourable.

6.2. ζ Oph

The photospheric line-profile variations of the O9.5 V star ζ Oph have been the subject of many studies (e.g. Vogt & Penrod 1983, Kambe et al. 1990, 1993ab, Reid et al. 1993, Gies 1995). Vogt & Penrod found that the line-profile variations of this star can be successfully modelled as the result of non-radial pulsations. Kambe et al. found evidence for two pulsation modes. From an extensive spectroscopic data set of ζ Oph, Reid et al. (hereafter Rea93) derived that the line-profile variations show periodicity at four different frequencies (a superset of the frequencies found by Kambe et al.), three of which are severely affected by aliasing. Recently Kambe et al. (1995) have presented an almost aliasing-free spectroscopic dataset, from which they conclude that different aliases than the ones found by Rea93 might be the true apparent pulsation frequencies. Kambe et al. (1995) also find evidence for harmonic variability. In their mode identifications, all these authors (except Kambe et al. (1995) who do not give a mode identification) assumed that the pulsation modes are sectoral, and attempted to derive values for the pulsation parameter m; we will discuss below what pulsation parameters can be derived using the outcome of our modelling (Sects. 3 and 4).

In Fig. 14 we show the IPS phase diagrams as derived by Rea93. In Table 6 we list the apparent pulsation periods from Rea93, and the line-centre phases and blue-to-red phase differences that we derive from the phase diagrams in Fig. 14. We read off the blue-to-red phase differences at the assumed V sin i=400 km/s, since the line-blending in this star makes it difficult to read off the phase diagrams in the way described in Paper I and Sect. 3.1. We use the HWHM of the main power peak of the window function as estimate for the systematic error in the derived frequencies.

![Fig. 14. Phase diagrams of the O9.5 V star ζ Oph, from Reid et al. (1993). We show the region around the Si III λ4552 line. Note that this line is blended with the Heti λ4541 line, which can be the reason why some of the phase diagrams seem to extend to shorter wavelengths. The dashed vertical lines indicate the assumed V sin i=400 km/s. The phase diagrams are shifted by multiples of 2π for clarity. The slope reversals seen at high velocities in the steepest phase diagrams might be real, but are probably due to limited variational signal to noise in the wings of the profile.](image)
frequency $f_{3ab}$ and $\ell=8.4(2.0)$ for frequency $f_{ab}$, where subscripts $a$ and $b$ denote the one-day aliases. The values of $\ell$ that we find for the frequencies $f_{3ab}$ and $f_{ab}$ are smaller than those found by Rea93 (see Table 6). We stress that this difference is not due to our choice of the velocity at which to read off the phase diagrams ($V=400\ km/s$), nor to the inclusion of rotational terms in our model, but due to the fact that Rea93 used an oversimplified model to fit and interpret the phase diagrams.

We now discuss a possibility that has been rejected by Rea93, i.e. that some of the observed frequencies might be harmonics of others (scenario B). Knowing that one can expect non-sinusoidal line-profile variability, one can determine which of the one-day aliases is the true apparent frequency by matching either of the aliases to the harmonic frequencies and vice-versa. We note that within the accuracy of the pulsation frequency determinations of Rea93, one cannot exclude the possibility that the apparent frequency $f_{ab}$ is the first harmonic of $f_{2a}$: $17.57(5)\approx 2\times(8.85(5))=17.70(10)$. Furthermore, since the amplitude of the variations expressed in average line depth is quite large ($\sim 10\%$ for the variations at frequencies $f_1$ and $f_{2ab}$, see Rea93), we expect considerable contribution of harmonics in the line-profile variations, if these are caused by redistribution of flux due to the pulsational velocity field (see Paper I). The almost double peaked shape of the amplitude distributions (see Rea93), suggests that indeed the velocity effects dominate over temperature effects, and that the value of $k$ might be larger than would be expected for $p$ modes (see e.g. Kambe & Osaki 1988, Lee & Siao 1990, Paper I).

In Sect. 4.5 we have shown that one may expect a relation between the phases in line centre of the variability at the apparent pulsation frequency and its harmonic. If we compute for frequencies $f_{2a}$ and $f_{ab}$ the line-centre phase difference $\Psi_{01}=2\Psi_{01}/\Psi_{1}$ we find $\Psi_{01}=1.2\pi(0.2\pi)$. Here the error is an estimated readoff error; since the derived value of $f_{ab}$ is not precisely twice the value of $f_{2a}$, an additional error $\Psi_{01}$ can be expected. Nevertheless, we cannot conclude that the derived value of $\Psi_{01}$ is consistent with the expected $\Psi_{01}=1.50\pi\pm 0.06\pi$ (see Sect. 4.5). However, as we stressed before, the apparent pulsation frequencies should be very well established in order to get reliable estimates of $\Psi_{01}$, and therefore we urge for new pulsation frequency determinations of this star (such as given by Kambe et al. 1995), based on an even longer time stretch of data (Rea93 collected data on a time base of 10 days), such that errors in the derived frequencies and line-centre phases can be decreased.

Assuming the detected frequency $f_{ab}$ is due to harmonic variability, we can constrain the value of $|m|$ for the mode giving rise to frequencies $f_{2a}$ and $f_{ab}$. In this scenario (B) the star $\zeta$ Oph exhibits only three pulsation modes, instead of four. Assuming that the star has an inclination $i>65^\circ$, we find according to Eq. (8) and Table 3 a value of $|m|=3.1(2.0)$ for the mode with $\ell=5.3(1.0)$. Rea93 found from the strength of the profile variations of different atoms, that the pulsations in $\zeta$ Oph are probably focused towards the equator. If we assume that $\ell-|m|<2$ (see Table 3), we find $|m|=3.6(1.0)$ from the phase diagram at $f_{ab}$.

### 7. Summary of conclusions

We have modelled the line-profile variability caused by non-radial pulsations in rotating early-type stars. In our model we account for pulsational temperature effects and for the effects of the Coriolis force on the oscillatory displacement field. We investigated the line-profile variability of sectoral, tesseral and zonal, spheroidal modes, and of sectoral and tesseral toroidal modes.

We have generated time series of spectra and analysed these with IPS techniques to obtain the variational behaviour as a function of the position in the line profile. This method of analysis relies on the assumption that the surface of the star is Doppler mapped to the spectral line by the stellar rotation, and hence is only applicable for stars with intrinsic profile widths which are narrow in comparison with the rotational broadening: $W/V_s \sin i \approx 0.25$. Similarly, this method can only benefit from the rotational broadening if the pulsation pattern supplies enough structure in the azimuthal direction, and hence the method only works well for modes with $|m|>2$. However, modes that do not fulfill the latter requirement can often be recognized by slope reversals in the phase diagrams.

In a wide range in parameter space, the spectral time series contain sufficient information to derive the pulsation parameters $\ell$ and $|m|$. We presented a simple linear relation between the blue-to-red phase difference of the variations with the apparent pulsation frequency, $\Delta \Psi_{0}$, and the value of $\ell$. Provided that the rotational broadening is large enough, this relation is valid over the full range in parameter space that we explored, and is also valid for subsets of this parameter space.

The order $|m|$ of the pulsations can be estimated from a similar linear relation with the blue-to-red phase difference of the variations at the first harmonic of the apparent pulsation frequency, $\Delta \Psi_{1}$. For $p$ modes this relation is strict.
The phases at line centre of the line-profile variations with the pulsation frequency and its first harmonic relate as $2\Psi_0 - \Psi_1 = 1.50\pi \pm 0.06\pi$. This relation can be used to check whether variations seen at a harmonic frequency are really due to harmonic line-profile variability or to another pulsation mode, provided that the apparent frequencies are well known.

We have found that for multi-periodic stars the velocity fields of the modes give rise to beatings that show up as sum and difference frequencies in the periodograms. We advocated that these observable frequencies, as well as the observable harmonic frequencies, can be used to distinguish real pulsation frequencies from their aliases, if the data stretch is long enough.

If, for a multi-periodic star, all apparent frequencies can be resolved, the phase diagrams of the variations seen at the pulsation frequencies can be interpreted as if in a mono-mode situation.

We presented applications of the IPS analysis to previously published observations of $\epsilon$ Persei and $\zeta$ Ophiuchi, and interpreted the phase diagrams derived from these observations using the results of our modelling. We argued that the multitude of frequencies seen in the periodogram of data of $\epsilon$ Per suggests that the line-profile variations are primarily due to the pulsational velocity field rather than pulsational temperature variations. We found evidence for at least one first-harmonic frequency in this data. From the IPS phase diagrams, we derived values for the pulsation degree $\ell$, and we constrained values of $|m|$ using the harmonic phase diagrams.

Our results suggest that for two of the four frequencies detected in line-profile variability of $\zeta$ Ophiuchi, the degree of pulsation $\ell$ was previously overestimated. We also discussed the possibilities that one of the frequencies of the observed variations, is actually the first harmonic of one of the other frequencies, and hence that the phase diagrams at this frequency supplies information on the order $|m|$ of the corresponding pulsation.

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