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DOI

[10.1016/j.jedc.2019.04.003](https://doi.org/10.1016/j.jedc.2019.04.003)

Publication date

2019

Document Version

Final published version

Published in

Journal of Economic Dynamics & Control

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[Link to publication](#)

Citation for published version (APA):

Bolt, W., Demertzis, M., Diks, C., Hommes, C., & van der Leij, M. (2019). Identifying booms and busts in house prices under heterogeneous expectations. *Journal of Economic Dynamics & Control*, 103, 234-259. <https://doi.org/10.1016/j.jedc.2019.04.003>

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Identifying booms and busts in house prices under heterogeneous expectations

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ARTICLE INFO

Article history:

Received 28 November 2018

Revised 21 March 2019

Accepted 4 April 2019

Available online 30 April 2019

JEL classification:

C53

R21

R31

Keywords:

House prices

Heterogeneous agents model (HAM)

Bounded rationality

Bubbles

Early warning signal

Stabilizing policies

ABSTRACT

We introduce heterogeneous expectations in a standard housing market model linking housing rental levels to fundamental buying prices. Using quarterly data we estimate the model parameters for eight different countries. We find that the data support heterogeneity in expectations, with temporary endogenous switching between fundamental mean-reverting and trend-following beliefs based on their relative performance. For all countries we identify temporary, long lasting house price bubbles amplified by trend extrapolation and crashes reinforced by mean-reverting expectations. The average market sentiment may be used as an early warning signal of a (temporary) bubble regime. The qualitative predictions of such non-linear models are very different from standard linear benchmarks with important policy implications. The fundamental price becomes unstable when the interest rate is set too low or mortgage tax deductions are too high, giving rise to multiple non-fundamental equilibria and/or global instability.

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1. Introduction

Do house prices exhibit expectations-driven temporary bubbles and crashes? The aim of this paper is to develop a stylized behavioral model to assess the empirical relevance of bubbles and crashes in house prices across different countries. This is important, since housing market bubbles are considered to be leading indicators of financial instability and crises (Leamer, 2008). Financial crises and recessions are often preceded by a decline in housing investments (Reinhart and Rogoff, 2009). For this reason, a good understanding of house price dynamics and the booms and busts they can generate are crucial for central banks. Unfortunately, a good understanding of the housing market and the business cycle are still lacking. Even state-of-the art dynamic stochastic general equilibrium (DSGE) models with housing consumption and production

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(Davis and Heathcote, 2005) are unable to match house price fluctuations and do not capture the phenomenon that house investment leads GDP. Since the 1980s, e.g., Shiller has repeatedly warned that changes in fundamentals cannot account for the large swings in home prices observed empirically (Case and Shiller, 2003).

Standard housing models enable one to derive fundamental relations between house prices, rents and user costs (Himmelberg et al., 2005; Poterba and Sinai, 2008). But fundamental factors apparently are insufficient to explain the observed large booms and busts in house prices. What could be a reasonable alternative model for house price dynamics? Glaeser and Nathanson (2015) argue that many non-rational explanations for housing bubbles exist, but the most promising theories emphasize some form of trend-chasing, which in turn reflects boundedly rational learning. In this paper we construct a model in which house price fluctuations are partly driven by almost self-fulfilling expectations (animal spirits) of boundedly rational heterogeneous agents. During a housing boom expectations regarding future house prices are typically optimistic, while they are pessimistic during a bust. There is anecdotal as well as empirical evidence that both short run (1 year) and long run (10 years) house price expectations were unrealistically high in the US during the housing boom (Case et al., 2012; Cheng et al., 2014). Shiller (2007) and Piazzesi and Schneider (2009) conducted surveys among home owners and showed that the survey data are characterized by heterogeneity and mutual feedback between house price expectations and realized house prices. These findings are in line with laboratory experimental markets. Gjerstad and Smith (2014) emphasize how easily speculative bubbles form and subsequent crashes lead to collapse in real speculative asset markets for durable goods, such as housing markets, and this pattern is consistent with empirical evidence of bubble formation in laboratory experiments. Bao and Hommes (2015) design an experimental housing market and find expectations-driven bubbles and crashes very similar to those observed in other experimental asset markets, such as Hommes et al. (2005, 2008).

The purpose of this paper is to develop a nonlinear heterogeneous agent model for the housing market with *endogenous switching* between optimistic and pessimistic expectations and estimate the resulting nonlinear model using empirical house price data from different countries. Our point of departure is the standard ‘user cost of capital (housing)’ model (Himmelberg et al., 2005; Poterba and Sinai, 2008). We extend this standard housing model by introducing heterogeneous expectations feedback relations, as in Brock and Hommes (1998; 1997), that may drive the price-to-rent ratio temporarily away from its long-run fundamental value, while at other times expectations may reinforce mean-reversion back to fundamentals. A convenient feature of our general setup is that the fundamental price-to-rent ratio of the standard housing model is nested as a special case within our nonlinear model. Our heterogeneous expectations housing market model thus provides an empirical test whether behavioral heterogeneity and expectations-driven booms and busts deviating from fundamental value are economically and statistically significant.

Our goal is to develop a stylized but *general* structural model that can be estimated using house price data from *different* countries. This stylized housing model lacks country specific institutional detail but has the advantage that it can be used to compare the occurrence of housing bubbles and crashes across different countries. Since our model is formulated in deviations from a fundamental benchmark country specific details about housing markets could be added and the model could be reestimated in deviations from more detailed country level fundamentals. Agnello et al. (2015) recently also analyzed house price dynamics across various countries following a time series econometrics approach. Our approach is complementary to theirs, since we estimate a structural behavioral model with boundedly rational heterogeneous agents. Another difference is that we rely on the notion of a benchmark fundamental price based on rental price levels. Although both approaches lead to nonlinear models for house price dynamics an advantage of estimating a structural behavioral model is that the estimated model parameters allow for simple behavioral interpretations.

Our paper focuses on the empirical relevance of performance-based strategy switching in the housing market. We develop a stylized 2-type heterogeneous expectations model with a trend-extrapolation versus a mean-reverting forecasting rule and estimate the model for eight different countries: United States (US), Japan (JP), United Kingdom (UK), The Netherlands (NL), Switzerland (CH), Spain (ES), Sweden (SE) and Belgium (BE) for the period 1970–2017. We arrived at this set of countries *ex ante* by starting with the US and adding countries until we had a fairly mixed list of countries that have recently seen a housing bubble (US, CH, JP), are currently near the peak of a bubble (UK, SE and BE) or are in a price corrective regime (NL, ES). For all countries the estimated parameters measuring the heterogeneity and switching behavior turn out to be significant and lie in – or close to – the region where the fundamental equilibrium is *unstable* (and hence it does not prevail in the long-run). For all eight countries we identify long-lasting periods of temporary housing bubbles amplified by an explosive market sentiment. Because of the simple generic features of our 2-type HAM our estimation results for house prices can be compared to similar estimated 2-type HAMs for other data sets, such as stock prices, commodity prices, exchange rates and macro data. In this way we are able to compare the duration of expectations-driven bubbles and crashes across different markets. Housing markets exhibit the strongest and longest bubbles, often over many years and much longer than in other markets. The longest bubble that has been detected in other asset markets using a 2-type HAM is the dot-com bubble between approx. 1995–2000 in the stock market (e.g., Boswijk et al., 2007; Hommes and in’t Veld, 2017). For most housing markets we find much longer booms (US, ES) or busts (JP) of 10 years or longer. This stresses the importance of identifying bubbles in housing markets as potential early indicators of an upcoming financial-economic crisis.

Our structural nonlinear switching model provides an *early warning signal* for identifying housing bubbles, when the average market sentiment, that is, agents’ average extrapolation factor, exceeds 1 indicating that house price dynamics and deviations from fundamentals become temporarily explosive. The bubble is reinforced by trend-extrapolation and, as the bubble bursts, a crash in house prices is reinforced by switching back to the mean-reverting fundamental forecast rule.

However, while the model can tell us whether we are in a period of instability (bubble), it cannot necessarily predict the timing of market switches/corrections. Nevertheless, the derived conditions for instability are linked explicitly to policy variables, so we are able to discuss which policies – e.g., interest rate policies or reduction of mortgage tax deduction – can stabilize booms and busts or may destabilize house prices. In this respect our model can inform policy makers about the possibility of the economy approaching or even being at, what Blanchard described as the “dark corners” of the economy: places where variables react in very non-linear ways, such that even small, and otherwise innocuous shocks may produce very large and unpredictable effects (Blanchard, 2014). Our nonlinear heterogeneous agents switching model thus can provide important insights for policy makers to avoid the “dark corners” of the economy.

The paper is organized as follows. In Section 1.1 we review earlier housing market models with boundedly rational heterogeneous agents. Section 2 develops a standard housing model with heterogeneous expectations, derives the local stability conditions for the fundamental house price and shows that a pitchfork bifurcation may lead to multiple non-fundamental steady states and global instability. In Section 3 we estimate the heterogeneous expectations model using data from eight different countries. We find evidence of temporary bubbles in all countries and discuss the average market sentiment as an early warning signal of (temporary) housing bubbles. In Section 4 we discuss how interest rate and/or mortgage policies may prevent instability and housing bubbles. Finally, Section 5 concludes.

1.1. Related literature

Heterogeneous agents models (HAMs) provide a new tool, which can be used to study temporary deviations from economic fundamentals. Rather than a single, representative, fully rational agent, HAMs allow for bounded rationality and heterogeneity of expectations among agents. HAMs were originally introduced by Brock and Hommes (1998; 1997) to describe financial asset price fluctuations. They showed that heterogeneous beliefs, together with switching between beliefs based on recent past performance, can lead to situations where the fundamental price is locally unstable and asset prices show multiple equilibria or chaotic fluctuations, with irregular bubbles and crashes, around the fundamental value; see Hommes (2006) and more recently Dieci and He (2018) for an up to date overview. HAMs have also been applied to exchange rates (DeGrauwe and Grimaldi, 2006; Spronk et al., 2013) and macroeconomics, particularly within the New Keynesian framework (Branch and Evans, 2006; Branch and McGough, 2010; 2009; Cornea et al., 2019; DeGrauwe, 2011; Hommes and Lustenhouwer, 2019; Kurz, 2011; Massaro, 2013).

HAMs with performance-based endogenous switching have been successfully estimated for different asset markets. Boswijk et al. (2007) estimated (Brock and Hommes, 1998) HAM with fundamentalists versus chartists using annual S&P 500 stock market data from 1871 to 2003. They found evidence for the presence of heterogeneity and endogenous switching for a fundamental price based on both the price-to-earnings ratio and the price-to-dividend ratio. Their model explains the dot-com bubble as being triggered by economic fundamentals (good news about the economy, because of a new internet technology), strongly amplified by investors' switching to trend-following behavior. Hommes and in't Veld (2017) follow a similar approach, using both the dynamic Gordon present-discounted-value and the Campbell-Cochrane consumption habit fundamental benchmarks, for quarterly S&P500 data 1950–2016 and conclude that the financial crises has been amplified by switching between mean-reverting and trend-following strategies. Lof (2015) estimates a HAM with different VAR-model specifications to the S&P500 index and finds temporary switching between fundamentalists and rational and contrarian speculators. Lux (2009) estimated the parameters of a dynamic opinion formation process with social interactions based on survey data on business expectations (sentiment index data). Franke and Westerhoff (2011, 2012) estimate HAMs with structural stochastic volatility using S&P 500 index data. De Jong et al. (2009) estimated a HAM for the EMS exchange rate dynamics and for Asian stock markets during the Asian crisis (De Jong et al., 2010). Lux and Zwinkels (2018) provide an up-to-date review of the empirical validation and estimation of HAMs.

House price fluctuations have been studied extensively in the literature and we are not the first to apply bounded rationality and heterogeneous expectations in housing market models. Glaeser and Nathanson (2017) consider an extrapolative model of house price dynamics. Their model displays three features that are present in the data but usually missing from perfectly rational models: momentum at one-year horizons, mean reversion at 5-year horizons, and excess longer-term volatility relative to fundamentals. Adam et al. (2012) consider a housing market model with Bayesian learning of an “internally rational” representative agent. Gelain and Lansing (2014) consider a housing model with adaptive learning. Agents' perceived law of motion (PLM) is an AR(1) process of rent growth and agents update the parameters for the mean, the autocorrelation and the volatility over time. Their model generates time varying risk-aversion, volatility and persistence similar to what is observed in house price data, especially when agents update their beliefs using only recent data.

Models for house price dynamics involving HAMs have been pioneered by Dieci and Westerhoff (2013, 2012). More recent housing HAMs include Dieci and Westerhoff (2016), Ascari et al. (2018) and Schmitt and Westerhoff (2019). Another related theoretical housing HAM is Burnside et al. (2016). Their approach differs in two important ways from ours. Firstly, their agents disagree about the fundamental value of housing, whereas we assume that agents agree on the fundamental value of houses but disagree on how prices return to it. Secondly, their model is epidemiological in nature, in that agents infect each other, while in our approach strategy switching is based upon relative performance.

In related empirical work Ambrose et al. (2013) examined a long time series of house price data of Amsterdam from 1650 to 2005, and found that substantial deviations from fundamentals persisted for decades and are corrected mainly

through price adjustments and to a lesser extent through rent adjustments. Based on the same data set, Eichholtz et al. (2015) found that there is evidence for switching in expectation formation between fundamental and trend following beliefs. Kouwenberg and Zwickels (2014) estimated a HAM model specifically for the US housing market using quarterly data from 1960 until 2012. An important difference with our approach is that their model (following Dieci and Westerhoff, 2013; Dieci and Westerhoff, 2012) uses a price adjustment rule based on excess demand, while we use a temporary equilibrium pricing model (as well as that we estimate the model for a number of different countries).

Geanakoplos et al. (2012) develop an agent-based model to explain the housing boom and crash, 1997–2009 in the Washington DC area. Their ABM simulations show that leverage, and not interest rates, played the dominant role in the U.S. housing boom and bust from 1997–2009. Baptista et al. (2016) develop an agent-based model (ABM) of the UK housing market to study the impact of macro-prudential policies on key housing market indicators. For example, they study the effect of a loan-to-income portfolio limit and find that this policy attenuates the house price cycle.

An alternative recent approach used in the literature consists of using fully-fledged DSGE models that have been adjusted to study the macroeconomic effects of housing (Iacoviello and Neri., 2010; Piazzesi et al., 2007). Typically, in those models, households receive utility from consumption of non-durable goods and housing services and they maximize expected lifetime discounted utility subject to a budget constraint. This budget constraint may include (convex) transaction and adjustment costs and/or liquidity and debt (i.e. mortgage) restrictions. The first-order Euler condition equates the marginal rate of substitution of housing services for non-durable consumption to the ‘shadow price’ or expected user cost of owner-occupied housing services which then comprises current transaction costs, the foregone return to housing equity and/or the cost of mortgage payments plus future expected transaction costs, maintenance cost and property taxes minus expected capital gains (see Diaz and Luengo-Prado, 2008). In other words, this Euler condition brings us back again to the standard ‘user cost of capital’ housing model. However what these types of models do not have is the possibility of differentiated expectations in ways that allow for bounded rationality, heterogeneity, herding behavior and sudden stops or indeed the existence of more than one equilibrium. These features are in our view crucial to the housing markets, very much like in stock markets, despite the fact that they operate at lower frequency (in line with most macro variables). Our approach adds such bounded rationality and heterogeneity features to the standard user cost of capital housing model.

2. A housing market model with heterogeneous beliefs

In this section, we develop a standard housing pricing model based on user costs of capital (see e.g. Poterba and Sinai, 2008), which we extend by incorporating heterogeneous beliefs, following Brock and Hommes (1998, 1997). Our purpose is to develop a stylized, general model that can be applied to different countries. Agents are boundedly rational and have different views about the expected capital gains of housing. At the same time, agents are allowed to switch from one period to the next between a number of available forecasting strategies, $h \in \{1, \dots, H\}$, based on how well they have performed in the recent past.

2.1. Model description

The point of departure is a standard user cost of capital model where home buyers and/or investors choose between either buying or renting a house. In equilibrium the annual cost of home ownership –in the literature known as the “imputed rent” (e.g. Himmelberg et al., 2005)– must equal the housing rent. Agents base their decisions at time t on their expectations regarding the ex post excess return R_{t+1} on investing in housing relative to renting during the period between time t and $t + 1$. Let P_t denote the price of one unit of housing at time t . Let the price for renting one unit of housing in the period between times t and $t + 1$ be given by Q_t . Since rents are typically payed up-front (at time t), to express the rent at time t in terms of currency at time $t + 1$, it should be inflated by a factor $(1 + r^{rf})$, where r^{rf} denotes the risk free mortgage rate. Therefore, the cost of renting in the period between time t and $t + 1$, expressed in terms of currency at time $t + 1$, is given by $(1 + r^{rf})Q_t$ rather than Q_t . The ex post excess return R_{t+1} on investing in housing during the period between time t and $t + 1$ then is given by the sum of the capital gain minus mortgage/maintenance costs and the saving on rent (cf. Ambrose et al., 2013; Campbell et al., 2009)

$$R_{t+1} = \frac{(P_{t+1} - (1 + r_t)P_t) + (1 + r_t^{rf})Q_t}{P_t} = \frac{P_{t+1} + (1 + r_t^{rf})Q_t}{P_t} - (1 + r_t),$$

where $r_t = r_t^{rf} + \omega_t$, with r_t^{rf} the risk-free mortgage rate and ω_t the maintenance costs/tax rate.

The demand, $z_{h,t}$, of agents of belief type h is determined by myopic mean-variance maximization, i.e. agents maximizing one-period ahead expected excess returns adjusted for risk:

$$\mathbb{E}_{h,t}(R_{t+1}z_{h,t}) - \frac{a}{2}\text{Var}_{h,t}(R_{t+1}z_{h,t}), \tag{1}$$

where a is a measure of risk aversion. The investors and/or home buyers agree on fundamentals, but since they can have different opinions on the dynamics that govern price fluctuations around the fundamental price, they may have heterogeneous

expectations about future excess returns $\mathbb{E}_{h,t}((P_{t+1} + (1 + r_t^{\text{rf}})Q_t)/P_t - (1 + r_t))$, while they are assumed to have homogeneous expectations regarding the conditional variance of the excess return, that is, $\text{Var}_{h,t}((P_{t+1} + (1 + r_t^{\text{rf}})Q_t)/P_t - (1 + r_t)) = V$. Maximizing Eq. (1) leads to the demand for housing:

$$z_{h,t} = \frac{(\mathbb{E}_{h,t}(P_{t+1} + (1 + r_t^{\text{rf}})Q_t)/P_t - (1 + r))}{aV} = \frac{\mathbb{E}_{h,t}(R_{t+1})}{aV}, \quad (2)$$

of agents of type $h \in \{1, \dots, H\}$.

Upon aggregation of the demand across the H types of agents, the market clearing condition is:

$$\sum_{h=1}^H n_{h,t} \frac{\mathbb{E}_{h,t}(P_{t+1} + (1 + r_t^{\text{rf}})Q_t)/P_t - (1 + r_t)}{aV} = S_t, \quad (3)$$

where S_t is the stock of housing and $n_{h,t}$ is the fraction of agents in period t that hold expectations of type h . Solving the market clearing condition for the house price P_t leads to the following price equation:

$$P_t = \frac{1}{1 + r_t + \alpha} \sum_{h=1}^H n_{h,t} \mathbb{E}_{h,t}(P_{t+1} + (1 + r_t^{\text{rf}})Q_t), \quad (4)$$

where $\alpha \equiv aV \times S_t$ is assumed to be constant. Taking into account time variation in α would be possible, but is considered to be beyond the scope of the present paper. Including this would require a model and data concerning the supply of new houses¹ and also concerning demographic and cultural changes over time, such as the tendency for families to decrease in size. Although this would be possible, it is important to note that changes in housing supply and cultural changes take place on much longer time scales than the relatively fast changes in demand for housing that we are trying to capture with the model. Our goal here is to develop a simple demand-side driven stylized heterogeneous expectations housing model so that the *same* model can be estimated easily using readily available OECD housing data from *different* countries². In the model, agents require a rate of return on housing equal to $r_t + \alpha = r_t^{\text{rf}} + \omega_t + \alpha$ rather than $r_t = r_t^{\text{rf}} + \omega_t$. Therefore the parameter α can be interpreted as a risk premium for investing in housing; treating α as a constant in the model allows for estimating the average extra required rate of return.

Fundamental price

The model will be formulated in terms of deviations from a benchmark fundamental price. We will use the discounted sum of expected future rents as the benchmark fundamental.³ More precisely, we take the dynamic Gordon model, with time varying interest rates and growth rates of rent⁴. Following Boswijk et al. (2007), we assume that the fundamental process underlying the model, i.e. the rent Q_t , follows a geometric Brownian motion with drift, i.e.

$$\log Q_{t+1} = \mu + \log Q_t + v_{t+1}, \quad \{v_t\} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma_v^2),$$

with commonly known parameters μ and σ_v^2 , from which one obtains

$$\frac{Q_{t+1}}{Q_t} = (1 + g)\varepsilon_{t+1},$$

with $g = e^{\mu + \frac{1}{2}\sigma_v^2} - 1$ and $\varepsilon_{t+1} = e^{v_{t+1} - \frac{1}{2}\sigma_v^2}$, such that $\mathbb{E}_t(\varepsilon_{t+1}) = 1$.

We define the fundamental price as the price that would prevail under homogeneous rational expectations $\mathbb{E}_t(R_{t+1})$ about the conditional mean of R_t , while taking into account the risk premium α . Incorporating the risk premium in the fundamental price is convenient, as it will provide an equilibrium fundamental price from which the market price will deviate by an amount which averages out to zero in long time series.

Under rational expectations we can re-write the price Eq. (4) as

$$(1 + r_t + \alpha)P_t = \mathbb{E}_t(P_{t+1} + (1 + r_t^{\text{rf}})Q_t).$$

¹ The importance of the supply side of the housing market was forcefully stressed by Glaeser and Nathanson (2015) and discussed in recent work of Dieci and Westerhoff (2016) and Schmitt and Westerhoff (2019).

² Another shortcoming of our model is that we do not impose short-selling constraints, which occur in the housing market as one can only sell a house that one owns. The (de-)stabilizing effects of short-selling constraints in an asset pricing HAM has been studied in Anufriev and Tuinstra (2013) and in Veld (2016), who showed that the stabilizing effects of short-selling constraints are limited. In fact, when the asset is overvalued the costs for short-selling may increase mispricing and price volatility.

³ In our model housing rents follow an exogenous stochastic process. Dieci and Westerhoff (2016) and Schmitt and Westerhoff (2019) develop housing market models where the rent is determined endogenously by market clearing.

⁴ An alternative general and easy to implement fundamental benchmark for empirical heterogeneous agents models (HAMs) may be a moving average of past prices. See Ellen et al. (2018) for a general discussion of empirical HAMs comparing different asset classes.

By applying the law of iterated expectations and imposing the transversality condition, we obtain the fundamental price at time t , given by

$$\begin{aligned}
 P_t^* &= \mathbb{E}_t \left(\frac{(1+r_t^{\text{rf}})Q_t}{1+r_t^{\text{rf}}+\omega+\alpha} \right) + \mathbb{E}_t \left(\frac{(1+r_t^{\text{rf}})Q_t(1+g_{t+1})}{(1+r_t^{\text{rf}}+\omega+\alpha)(1+r_{t+1}^{\text{rf}}+\omega+\alpha)} \right) \\
 &\quad + \mathbb{E}_t \left(\frac{(1+r_t^{\text{rf}})Q_t(1+g_{t+1})(1+g_{t+2})}{(1+r_t^{\text{rf}}+\omega+\alpha)(1+r_{t+1}^{\text{rf}}+\omega+\alpha)(1+r_{t+2}^{\text{rf}}+\omega+\alpha)} \right) + \dots \\
 &= \mathbb{E}_t \left(1 + \sum_{j=1}^{\infty} \left(\prod_{i=1}^j \frac{1+g_{t+i}}{1+r_{t+i}^{\text{rf}}+\omega+\alpha} \right) \right) \frac{(1+r_t^{\text{rf}})}{1+r_t^{\text{rf}}+\omega+\alpha} Q_t. \tag{5}
 \end{aligned}$$

The dynamic Gordon model takes into account time variation in the variables r_t^{rf} and g_t by using a Taylor approximation of the interest rate and growth rate around their respective mean values r^{rf} and g . Following this approach, which was first proposed by [Poterba and Summers \(1998\)](#), we find that fundamental price is, up to first order in $(r_t^{\text{rf}} - r^{\text{rf}}, g_t - g) \equiv (\Delta r_t^{\text{rf}}, \Delta g_t)$, given by

$$P_t^* \approx \left(1 + c_r \Delta r_t^{\text{rf}} + c_g \Delta g_t \right) \frac{(1+r^{\text{rf}})}{r^{\text{rf}}+\omega+\alpha-g} Q_t, \tag{6}$$

where

$$c_r = -\frac{r^{\text{rf}}+\alpha-g}{(1+r^{\text{rf}}+\omega+\alpha)(r^{\text{rf}}+\omega+\alpha-g)} \frac{\gamma\rho}{1-\gamma\rho} + \frac{1}{1+r^{\text{rf}}} - \frac{1}{1+r^{\text{rf}}+\omega+\alpha} \tag{7}$$

and

$$c_g = \frac{r^{\text{rf}}+\alpha-g}{(1+g)(r^{\text{rf}}+\omega+\alpha-g)} \frac{\gamma\phi}{1-\gamma\phi}, \tag{8}$$

with $\gamma = \frac{1}{R+\bar{\alpha}}$ (see [Appendix A.1](#) for a derivation).

The static Gordon model is obtained under the absence of time variation in the interest rate and growth rate, $\Delta r_t \equiv 0$ and $\Delta g \equiv 0$, leading to

$$P_t^{*,\text{static}} = \frac{1+r^{\text{rf}}}{r^{\text{rf}}+\omega+\alpha-g} Q_t, \tag{9}$$

which can also be derived directly (see [Appendix A.1](#) for details).

2.1.1. Deviations from the fundamental price

Define $X_t = \frac{P_t}{P_t^*} - 1$ as the relative deviation of the price from the fundamental price. We specify the dynamics of X_t by separating behavioral heterogeneity from fundamental factors. Specifically, we assume that the agents use a constant discount factor $\frac{1}{1+r+\alpha}$ (as in the static Gordon model, where $r_t^{\text{rf}} = r^{\text{rf}}$ and $g_t = g$) when determining their demand as a function of X_t , even if they agree on a fundamental price based on the dynamic Gordon model.

Substituting $P_t = (1+X_t)P_t^*$ into the price [Eq. \(4\)](#) with constant $r_t = r$ and $g_t = g$ and subtracting that same relation for the fundamental price, gives the dynamics of the relative deviation X_t from the fundamental:

$$X_t = \frac{1}{R+\bar{\alpha}} \sum_{h=1}^H n_{h,t} \mathbb{E}_{h,t}(X_{t+1}), \tag{10}$$

where $R = \frac{1+r}{1+g}$ and $\bar{\alpha} = \frac{\alpha}{1+g}$. Since the model is formulated in terms of deviations from the fundamental price, it can be used with various benchmark fundamentals. Also note that in the special case where all belief types have expectations $\mathbb{E}_{h,t}(X_{t+1}) \equiv 0$, $h = 1, \dots, H$, the house price always equals its fundamental value. Hence, the special homogeneous rational expectations benchmark is nested as a special case. The model for the deviation from the fundamental price holds regardless of the agent types considered and the fundamental price adopted for P_t^* . This is a convenient setup for testing empirically whether observed deviations from a benchmark fundamental price are significant.

By means of illustration, [Figs. 1](#) and [2](#) show the fundamental house price, according to the dynamic Gordon model, together with the realized house prices and deviations from the fundamental for eight countries, as discussed in more detail in the data [Section 3](#). These plots show excess volatility, that is, house prices fluctuate much more than underlying fundamentals. The next subsection introduces a two-type heterogeneous expectations switching model with boundedly rational agents to capture the observed excess volatility in house prices.

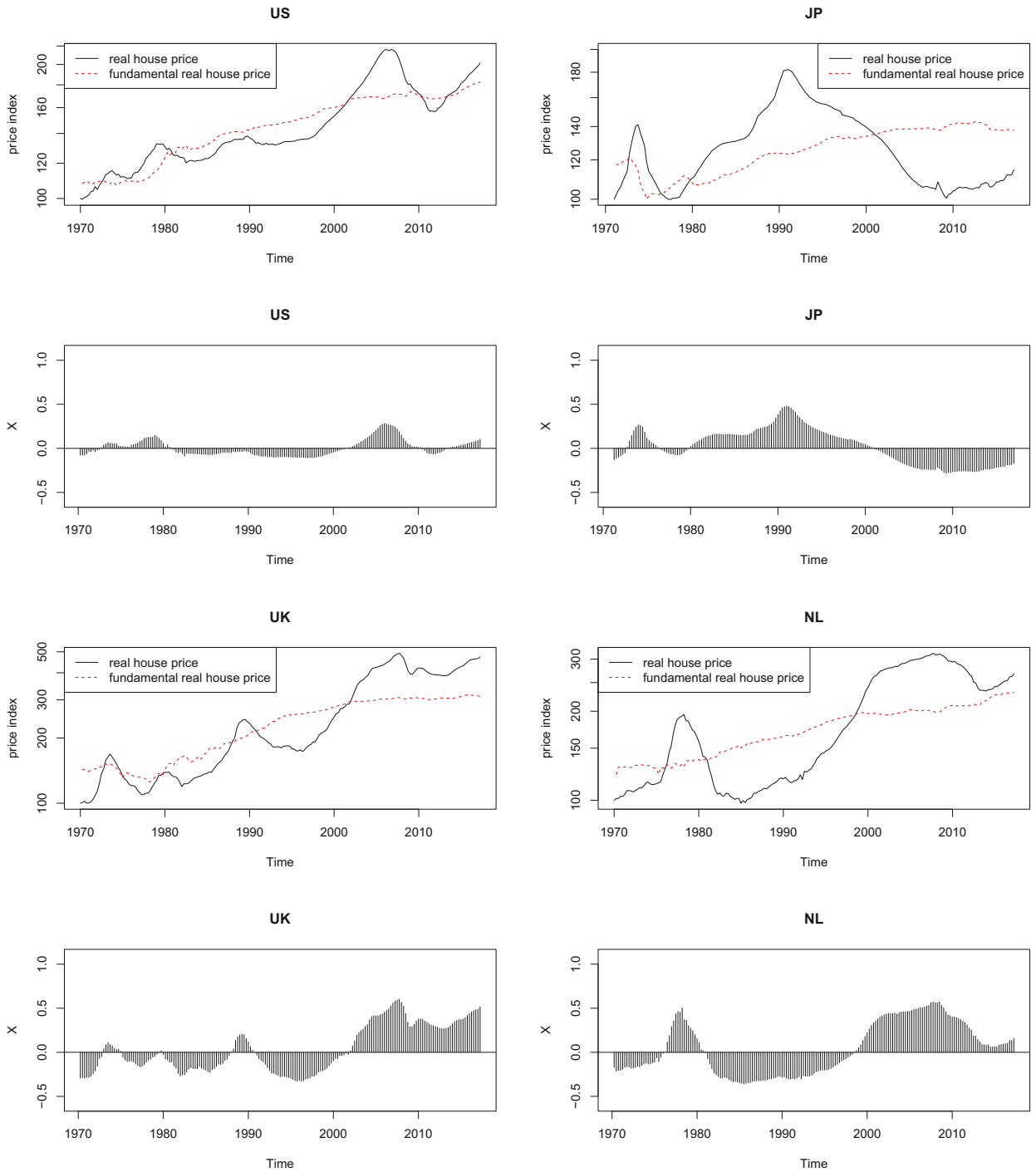


Fig. 1. Real house price indices (top sub-panels, solid lines, 1970Q1=100), estimated fundamental real house prices for dynamic Gordon model (dashed lines) and corresponding relative deviation X_t from fundamental (bottom sub-panels) for US, JP, UK and NL.

2.2. Two types of agents

Following [Boswijk et al. \(2007\)](#), henceforth BHM, we assume that each of the two types of agents have simple linear beliefs about X_{t+1} , but with different values of the coefficient ϕ :

$$\begin{aligned} \mathbb{E}_{1,t}(X_{t+1}) &= \phi_1 X_{t-1}, \\ \mathbb{E}_{2,t}(X_{t+1}) &= \phi_2 X_{t-1}. \end{aligned}$$

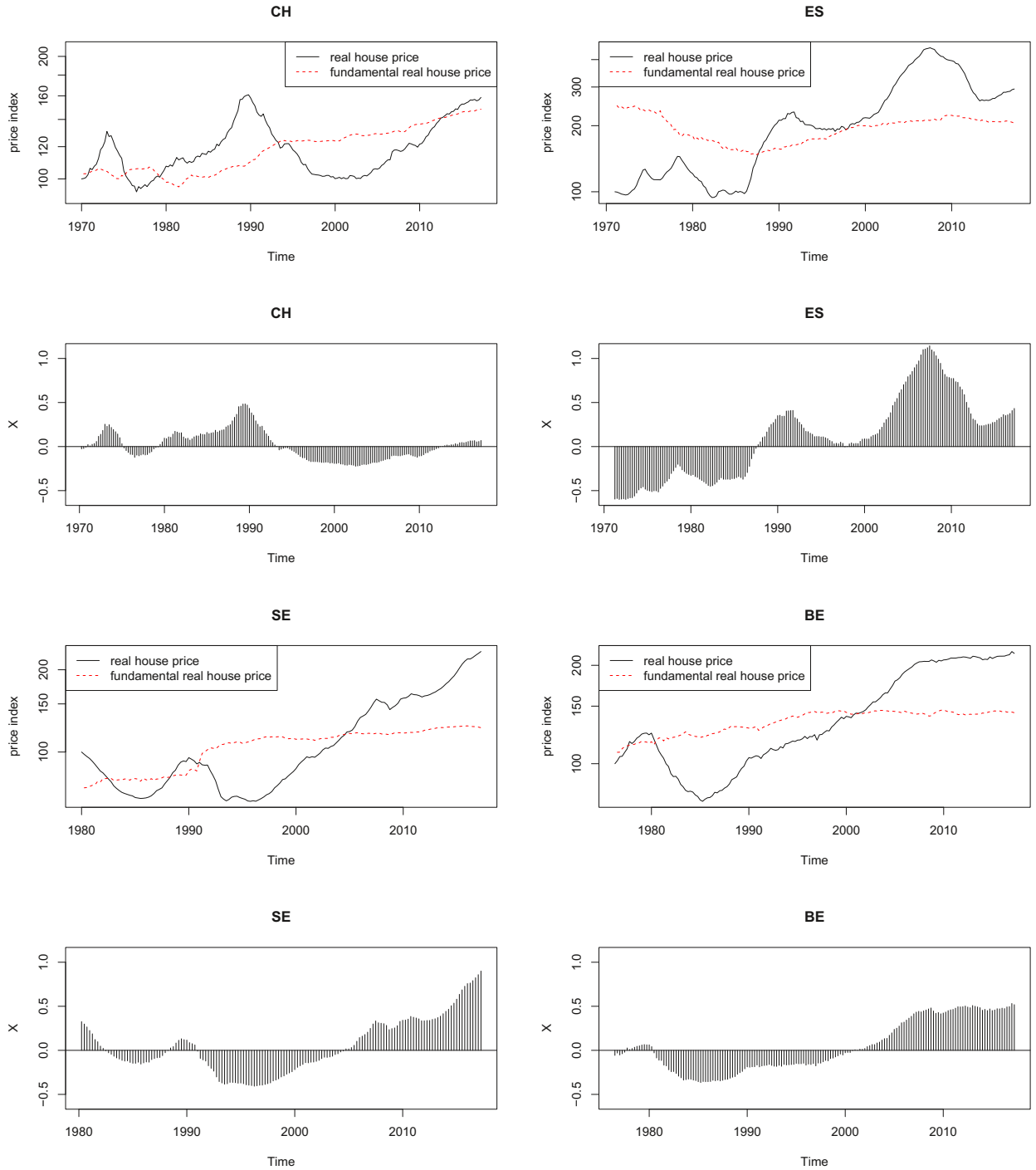


Fig. 2. Real house price indices (top sub-panels, solid lines, 1970Q1=100), estimated fundamental real house prices for dynamic Gordon model (dashed lines) and corresponding relative deviation X_t from fundamental (bottom sub-panels) for CH, ES, SE and BE.

Hence, the two types disagree about the speed of convergence to or divergence from the fundamental benchmark. In particular, $\phi_1 < 1$ corresponds to believing in mean-reversion towards the fundamental, while $\phi_2 > 1$ corresponds to trend-followers believing that prices further divert from the fundamental.

We have assumed the presence of two belief types here, but for the sake of argument, consider homogeneous beliefs: $\phi_1 = \phi_2$. The homogeneous case $\phi_1 = \phi_2 < R + \bar{\alpha}$ would lead to the price converging to the fundamental price, whereas homogeneous beliefs $\phi_1 = \phi_2 > R + \bar{\alpha}$ would imply an explosive bubble, where prices would deviate more and more from

the fundamental price.⁵ Next consider the heterogeneous case $\phi_1 \neq \phi_2$. If one of the belief parameters, ϕ_1 say, is smaller than $R + \bar{\alpha}$ and the other, ϕ_2 , larger than $R + \bar{\alpha}$, the fractions $n_{1,t}$ and $n_{2,t} = 1 - n_{1,t}$ of agents of belief types 1 and 2, determine whether prices are temporarily converging to the fundamental price or diverging from it. Since agents are allowed to switch between the two different types of beliefs, the fractions themselves are changing over time. This in turn implies that the system may temporarily be in an explosive bubble regime, where prices deviate further from fundamentals, or in a correction or mean-reversion regime with prices converging back to the fundamental.

The endogenous switching between the two types of beliefs is based on the recent past performance of the strategies measured in terms of realized profits, $\pi_{h,t-1}$, as in Brock and Hommes (1998, 1997).⁶ We derive the realized profits $\pi_{h,t-1}$ at time $t - 1$ along the lines of Boswijk et al. (2007), starting from

$$\pi_{h,t-1} = R_{t-1} z_{h,t-2} = R_{t-1} \frac{\mathbb{E}_{h,t-2}(R_{t-1})}{aV}.$$

We can express R_{t-1} as (see Appendix A.2)

$$R_{t-1} \approx (1 + g)Y_{t-2}(X_{t-1} + \bar{\alpha} - RX_{t-2}),$$

where $Y_{t-2} = \frac{P_{t-2}^*}{P_{t-2}}$.

Note that $\bar{\alpha}$ represents an endogenously determined risk premium for home owners. To see this, suppose X_{t-1} and X_{t-2} are zero, that is, house prices are at fundamental value. If the stock of housing S , the risk aversion parameter a and the perceived variance V are positive, $\bar{\alpha}$ is positive, and the excess return on housing R_{t-1} is positive even if prices evolve according to the fundamental price.

The performance measure is the product of the excess return R_{t-1} and the demand $z_{h,t-2}$, which by (2) is proportional to the expected excess return

$$z_{h,t-2} = \frac{\mathbb{E}_{h,t-2}(R_{t-1})}{aV},$$

where

$$\mathbb{E}_{h,t-2}(R_{t-1}) = (1 + g)Y_{t-2}(\mathbb{E}_{h,t}(X_{t-1}) + \bar{\alpha} - RX_{t-2}).$$

Taking together the expectations on returns and conditional variance gives

$$\pi_{h,t-1} = z_{h,t-2}R_{t-1} = \frac{(1 + g)^2}{a\eta^2}(X_{t-1} + \bar{\alpha} - RX_{t-2})(\mathbb{E}_{h,t-2}(X_{t-1}) + \bar{\alpha} - RX_{t-2}),$$

i.e. a constant involving the risk aversion times the realized excess return on housing at time $t - 1$, times the expected (at time $t - 2$) one-step-ahead excess return. We define the latter product as the fitness measure at time $t - 1$ for type h :

$$U_{h,t-1} = (X_{t-1} + \bar{\alpha} - RX_{t-2})(\mathbb{E}_{h,t-2}(X_{t-1}) + \bar{\alpha} - RX_{t-2}). \quad (11)$$

The fractions are determined by a logistic switching model with a-synchronous updating:

$$\begin{aligned} n_{1,t} &= \delta n_{1,t-1} + (1 - \delta) \frac{e^{\beta U_{1,t-1}}}{e^{\beta U_{1,t-1}} + e^{\beta U_{2,t-1}}} \\ &= \delta n_{1,t-1} + (1 - \delta) \frac{1}{1 + e^{-\beta(X_{t-1} + \bar{\alpha} - RX_{t-2})(\phi_1 - \phi_2)X_{t-3}}} \\ n_{2,t} &= 1 - n_{1,t}. \end{aligned} \quad (12)$$

The term a-synchronous updating refers to the fact that only a fraction $(1 - \delta)$ of agents re-evaluates and updates beliefs according to the logit model in each given period. Parameter β , referred to as the intensity of choice, represents the sensitivity of agents' to small changes in past performance $\pi_{h,t-1}$.⁷

The price equation with two types of agents is given by

$$X_t = \frac{n_{1,t}\phi_1 + n_{2,t}\phi_2}{R + \bar{\alpha}} X_{t-1}, \quad R = \frac{1 + r}{1 + g}, \quad \bar{\alpha} = \frac{\alpha}{1 + g}. \quad (13)$$

⁵ A house price bubble occurs when agents have unreasonably high expectations about future capital gains, leading them to perceive their user cost to be lower than it actually is and thus pay "too much" to purchase a house today.

⁶ Empirical evidence shows that switching based on past performance is relevant for real financial markets. For example, Ippolito (1989), Chevalier and Ellison (1997), Sirri and Tufano (1998) and Karceski (2002) found that money flows out of past poor performers into good performers in mutual funds data. Pension funds also switch away from bad performers (DelGuercio and Tkac, 2002).

⁷ Note that by working with $\beta U_{h,t-1}$ rather than $\beta \pi_{h,t-1}$ in Eq. (12) a factor $(1 + g)/(a\eta^2)$ has been absorbed in the definition of β . This has the advantage that we do not need to estimate this term, but it should be kept in mind that it makes direct comparisons between β -estimates for different countries difficult.

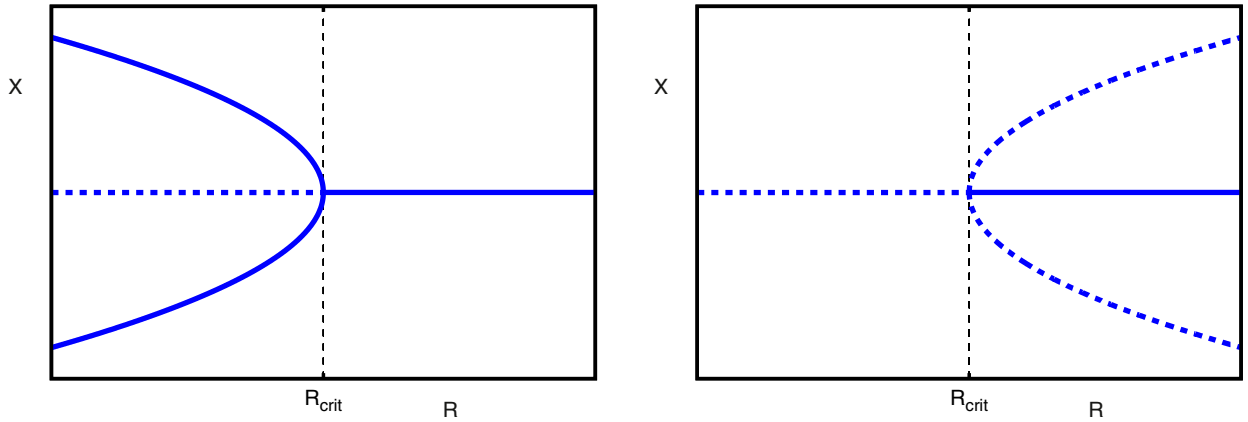


Fig. 3. Pitchfork bifurcation w.r.t. the policy parameter R at $R_{crit} = \frac{\phi_1 + \phi_2}{2}$ for $\bar{\alpha} = 0$. Bold curves denote stable steady states, dotted curves are unstable steady states. **Left Panel:** supercritical pitchfork bifurcation with stable fundamental steady state for $R > R_{crit}$ and unstable fundamental steady state surrounded by two stable non-fundamental steady states for $R < R_{crit}$. **Right Panel:** subcritical pitchfork bifurcation with stable fundamental steady state surrounded by two unstable non-fundamental steady states for $R > R_{crit}$ and (globally) unstable fundamental steady state with exploding dynamics for $R < R_{crit}$.

2.3. Dynamics of the deterministic model

The structural model in (12) and (13) is completely deterministic. Before allowing for some forecast error or noise term required to estimate the model it is useful to discuss the system without noise; the so-called deterministic ‘skeleton’. The first proposition describes the existence and the local stability of the fundamental steady state:

Proposition 1. *The fundamental steady state of the model is given by $X^* = 0$ and $n_1^* = n_2^* = \frac{1}{2}$. The fundamental steady state is locally stable if*

$$\left| \frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})} \right| = \left| \frac{(1 + g)(\phi_1 + \phi_2)}{2(1 + r + \alpha)} \right| < 1. \tag{14}$$

Proof. see Appendix A.3 □

At the fundamental steady state $X = 0$ the fractions of both types are equal: $n_1 = n_2 = \frac{1}{2}$. The local stability condition in (14) states that the fundamental steady state is locally stable when the discounted value of the average extrapolation factor over both types is less than 1. Stated differently, the fundamental steady state is locally stable when the discounted average expected mean reversion is less than 1.

Besides the fundamental steady state the model may have additional non-fundamental steady states created in a so-called *pitchfork bifurcation* as summarized in:⁸

Proposition 2. *For $\bar{\alpha} = 0^9$ and $\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})} = 1$, or equivalently $R = R_{crit} = \frac{\phi_1 + \phi_2}{2}$, the fundamental steady state exhibits a pitchfork bifurcation in which two additional non-fundamental steady states $\pm \bar{X}$ are created (or disappear). There are two cases:*

- (i) for $R_{crit} > 1$ the pitchfork bifurcation is supercritical and the (stable) non-fundamental steady states occur for $R < R_{crit}$ when the fundamental steady state is unstable (Fig. 3, left panel), and
- (ii) for $R_{crit} < 1$ the pitchfork bifurcation is subcritical and the (unstable) non-fundamental steady states occur for $R > R_{crit}$ when the fundamental steady state is stable (Fig. 3, right panel)¹⁰.

Proof. see Appendix A.4. □

Fig. 3 shows bifurcation diagrams of the nonlinear housing model. A bifurcation is a qualitative change of the dynamics, such as a change in the existence or stability of steady states. The model exhibits *pitchfork bifurcations*, where the fundamental steady state becomes unstable and additional non-fundamental steady states are created or disappear. There are two

⁸ In the housing models of Dieci and Westerhoff (2016) and Schmitt and Westerhoff (2019) the primary bifurcation of the fundamental steady state is a Neimark-Sacker bifurcation. This may be due to the supply side or investors’ trend-extrapolating behavior with a forecasting rule with two lags in their model.

⁹ For $\bar{\alpha} = 0$ the model is symmetric w.r.t. $X = 0$ and a pitchfork bifurcation occurs. In the non-symmetric case $\bar{\alpha} \neq 0$ the non-fundamental steady states are created in a saddle-node bifurcation; see the Proof of Proposition 2.

¹⁰ Since $R = (1 + r)/(1 + g)$ we have $R > 1$. Nevertheless, the subcritical pitchfork bifurcation in case (ii) for $R = R_{crit} < 1$ will be relevant to explain the dynamics and some of the empirical estimations in Section 3.

different cases for the pitchfork bifurcation, a supercritical and a subcritical, and both cases are illustrated in Fig. 3.¹¹ In both cases, for high values $R > R_{crit}$ the fundamental steady state is (locally) stable, while for low values $R < R_{crit}$ the fundamental steady state is unstable. In the case of a supercritical pitchfork (left panel) when the fundamental steady state becomes unstable two additional non-fundamental steady states are created, one above and one below the fundamental, both of which are stable. These non-fundamental steady states lie on a “parabola” emanating from the stable fundamental steady state to the left.

The second case of a subcritical pitchfork bifurcation is illustrated in Fig. 3 (right panel). As before, the fundamental steady state becomes unstable as the parameter R decreases below its critical value R_{crit} . In the subcritical case, however, the two non-fundamental steady states exist for $R > R_{crit}$, lying on a “parabola” emanating from the fundamental steady state to the right. Both non-fundamental steady states are unstable and they form a *corridor of stability* around the stable fundamental steady state. Initial states between the two non-fundamental steady states converge to the fundamental steady state; initial states outside this corridor of stability diverge (possibly to infinity, depending on higher order nonlinearities).

It should be stressed that our simple nonlinear housing model and the pitchfork bifurcation are by no means artificial or exceptional. On the contrary, the pitchfork bifurcation, supercritical as well as subcritical, is a *generic* phenomenon that can arise in many (higher dimensional) nonlinear systems for example in a more detailed model of the housing market.¹²

2.4. The stochastic model

Since we are interested in empirical estimates of our nonlinear housing model we now discuss the model with stochastic error terms. We will estimate the stochastic model in two steps. First we estimate the fundamental parameters and the corresponding relative deviations, X_t , of the realized prices from their fundamental value, using house price and rent indices. Second we use the deviations X_t to estimate the behavioral parameters of the agent-based model. The price Eq. (13) is interpreted as providing a conditional forecast of the price deviation X_t given the available information up to and including $t - 1$. This allows for forecast errors u_t in the model, leading to the price equation with error

$$X_t = \frac{n_{1,t}\phi_1 + n_{2,t}\phi_2}{R + \bar{\alpha}} X_{t-1} + u_t, \quad R + \bar{\alpha} = \frac{1 + r_t}{1 + g_t}.$$

Assuming that the errors u_t in the price equation consist of white noise this corresponds to a nonlinear time-varying AR(1) model, the parameters of which can be estimated using nonlinear least squares (NLS) ($n_{1,t}$ and $n_{2,t}$ depend non-linearly on the model parameters). The term u_t represents random exogenous shocks not taken into account by the model plus any systematic model error that happens to be present. *A priori*, therefore, there is no guarantee that u_t is white noise or homoskedastic. To acknowledge this, in the empirical section we perform diagnostic model checks by investigating the properties of the residuals. We will refer to the time varying AR(1) coefficient

$$\frac{n_{1,t}\phi_1 + n_{2,t}\phi_2}{R + \bar{\alpha}}, \quad (15)$$

as the (time-varying) *market sentiment*, representing the average mean-reverting or mean-diverting beliefs in the housing market. When the market sentiment exceeds 1 the market is explosive and we will say that the market is in a (temporary) bubble regime. When the market sentiment exceeds 1 and the house price is above (below) fundamental we will say that the market exhibits a positive (negative) bubble. The market sentiment (15) may thus be interpreted as an early warning signal of a temporary bubble when it exceeds 1.

3. Data and empirical results

We use an OECD housing data set similar to that described in Rousová and Van den Noord (2011), but extended to include more recent observations. This data set contains quarterly data for nominal and real house prices for 20 countries, starting from 1970Q1 for most countries (see Rousová and Van den Noord, 2011, Appendix 1-2, for the list of countries and corresponding data sources). We use data downloaded in November 2017, with data until 2017Q2, with the exception of Japan, for which 2017Q2 was not yet available (see Table 1 for the exact start and end quarters of our sample per country). The real house price is indexed using 2010 as base year. The price-to-rent ratio is defined as the nominal house price index divided by the rent component of the consumer price index (CPI), made available by the OECD. Short term interest rates are also retrieved from the OECD Economic Outlook 89 database. Country-specific quarter-to-quarter CPI series were used to convert all rates to real rates.

¹¹ The first order condition for a pitchfork bifurcation is that the linearized system has an eigenvalue +1. Whether the pitchfork is super- or subcritical depends on higher order derivatives of the system at bifurcation. See Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory. See also the proof of Proposition 2 in A.4.

¹² To be mathematically precise, one should say that pitchfork bifurcations are generic in systems that are symmetric w.r.t. to a coordinate axis. When symmetry breaks down, the pitchfork bifurcation “breaks up” into the generic non-symmetric case with two curves, an equilibrium curve and a saddle-node bifurcation curve, with very similar dynamics as for the pitchfork. We refer once more to Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory and its importance in applications.

Table 1

Start and end of sample period for each country, as well as the rental yield (rent-to-price ratio's) used to calibrate the price-to-rent ratio time series, and the corresponding reference quarter.

Country	Start	End	Rent/price	Reference quarter
US	1970Q1	2017Q2	0.0429	2013Q1
JP	1970Q1	2017Q1	0.0553	2013Q1
UK	1970Q1	2017Q2	0.0209	2013Q1
NL	1970Q1	2017Q2	0.0568	2013Q2
CH	1970Q1	2017Q2	0.0294	2013Q2
ES	1971Q1	2017Q2	0.0392	2013Q2
SE	1980Q1	2017Q2	0.0700	2005Q3
BE	1976Q2	2017Q2	0.0436	2013Q2

Note that since we are using price and rent indices rather than prices and rents, we are only able to calculate the ratio Q_t/P_t from the data up to an unknown factor from the indices of Q_t and P_t . To overcome this, we calibrated the series Q_t/P_t by using observed rent-to-price ratios at particular, country-specific, reference dates for each of the countries [source: GlobalPropertyGuide.com]. The rent-to-price ratios used for calibration are given in Table 1.

In what follows we estimate the fundamental price and the two-type heterogeneous beliefs model and present the results for the housing markets of the US, JP, UK, NL, CH, ES, SE and BE. We arrived at this subset of countries ex ante by starting our analysis with the US and adding countries until we had a fairly mixed list of countries that have recently seen the collapse of a housing bubble (US, NL, ES), are currently near the peak of a bubble (UK, SE, BE), in a price corrective regime (JP) or close to the fundamental price (CH).

3.1. Estimation of the fundamental model parameters

Before estimating the behavioral model we calibrate the fundamental model parameters, namely $R(= \frac{1+r}{1+g} = \frac{1+r^{rf}+\omega}{1+g})$ and $\bar{\alpha}(= \frac{\alpha}{1+g})$. We estimate these first using the static Gordon model in Eq. (9), after which we account for time variation in the interest rate and the rental growth rate using the dynamic Gordon model given in Eq. (6).

Static Gordon model

The fundamental relation between quarterly prices and rents in the static Gordon model Eq. (9) can be re-written as

$$R = 1 + \frac{(1+r^{rf})Q_t}{(1+g)P_t^{*,static}} - \bar{\alpha} \approx 1 + \frac{Q_t}{P_t^{*,static}} - \bar{\alpha}, \tag{16}$$

where we used the approximation $(1+r^{rf})/(1+g) \approx 1$, which is reasonable since quarterly interest and growth rates are small relative to 1. Note that even with this simplification the fundamental parameters R and $\bar{\alpha}$ cannot be estimated independently, since we have one fundamental equation and two unknown fundamental parameters.¹³ We address this problem by fixing $\bar{\alpha}$, which is hard to obtain empirically for individual countries, at a plausible value.

Himmelberg et al. (2005) estimated the risk premium of house owning relative to renting to be about 2% per year for the US, which would correspond to $\bar{\alpha} \approx 0.005$. However, they also noted that this is probably an over-estimation, since it ignores relevant factors such as the insurance value of owning a house in hedging risk associated with future changes in rent. Based on this, we consider 0 to be a lower limit and 0.005 an upper limit for $\bar{\alpha}$. It turns out that the exact choice of $\bar{\alpha}$ within this range has relatively small effects on the estimated behavioral parameters.

Based on Eq. (16), for a given value of $\bar{\alpha}$ we estimate R as $\hat{R} = 1 + \bar{y} - \bar{\alpha}$, where \bar{y} is the mean quarterly rental yield (average of Q_t/P_t) over the sample period. Since the data involved yearly rental yields, \bar{y} was converted to quarterly rental yields prior to estimating R . The estimates of \bar{y} and $R + \bar{\alpha}$ are given in Table 2. For completeness we also give the estimated inflation rate π (mean quarter-to-quarter inflation rate) and the nominal and real growth rates g (mean growth rates in rents) and $r = (1+g)R - 1$.

Figs. 1 and 2 present the house price indices P_t with the corresponding estimated fundamental values P_t^* of the dynamic Gordon model (top panels) and the log-difference between the two, $X_t = \log P_t - \log P_t^*$ (bottom panels). All eight countries exhibit long lasting periods of persistent under- or overvaluation of house prices compared to fundamentals, ranging from over-valuations of 25% for the US, around 50% for JP, UK, NL, CH, SE and BE, to 100% for ES. House prices in US, NL and ES have been increasing rapidly since the mid-1990s and have peaked around 2008. Since then house prices have dropped considerably for those countries, but are again/still above the fundamental value by the end of the sample. The JP and CH house prices peaked earlier, around 1990, and subsequently declined to levels below the fundamental values, after which price levels started to recover towards the fundamental price. House prices in the UK and BE are still around 50% above their fundamental values at the end of the sample, while those in SE still are around 100% above the fundamental value.

¹³ Recall that R and α appear independently in the fitness measure for strategy switching in Eq. (11).

Table 2

Empirically observed mean quarterly inflation rate π , mean quarterly rental yield \bar{y} , nominal and real growth rate g , nominal and real value of $r + \alpha$ and corresponding values of $R + \bar{\alpha} = (1 + r + \alpha)/(1 + g)$. All quarterly rates are multiplied by 100, except $R + \bar{\alpha}$.

Country	π	\bar{y}	Nominal		Real		$R + \bar{\alpha}$
			g	$r + \alpha$	g	$r + \alpha$	
US	0.986	1.038	1.270	2.312	0.285	1.326	1.0104
JP	0.630	1.028	0.719	1.748	0.089	1.118	1.0103
UK	1.325	0.676	1.747	2.425	0.422	1.100	1.0068
NL	0.794	1.620	1.107	2.733	0.313	1.938	1.0162
CH	0.583	0.748	0.776	1.526	0.193	0.943	1.0075
ES	1.625	1.321	1.517	2.837	-0.108	1.212	1.0132
SE	1.105	1.883	1.448	3.337	0.342	2.231	1.0188
BE	0.906	1.672	1.065	2.740	0.160	1.835	1.0167

Table 3

Fundamental parameters (fixed during estimation of the behavioral parameters), estimated behavioral model parameters for $\bar{\alpha} = 0$ and corresponding implied behavioral parameters. Below the estimates the corresponding standard errors are given in parentheses. The labels ‘*’, ‘**’ and ‘***’ denote significance at the 10%, 5% and 1% level of significance, respectively.

Country	US	JP	UK	NL	CH	ES	SE	BE
Fundamental parameters								
$\bar{\alpha}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$R + \bar{\alpha}$	1.0104	1.0103	1.0068	1.0162	1.0075	1.0132	1.0188	1.0167
Estimated behavioral parameters								
ϕ_1	0.8874*** (0.0213)	0.9396*** (0.0118)	0.8660*** (0.0321)	0.9558*** (0.0245)	0.8790*** (0.0220)	0.9568*** (0.0071)	0.9823*** (0.0114)	0.9880*** (0.0159)
$\Delta\phi$	0.2086*** (0.0305)	0.1530*** (0.0223)	0.2652*** (0.0661)	0.1331** (0.0484)	0.2282*** (0.0366)	0.1134*** (0.0121)	0.0996*** (0.0156)	0.0770* (0.0340)
$\beta (\times 10^3)$	25.04 (17.71)	3.655* (1.681)	0.510 (0.321)	2.577 (2.699)	3.341 (2.098)	2.702° (1.604)	94.9 (210.5)	9.173 (14.93)
δ	0.5309*** (0.1233)	0.0124 (0.1826)	0.0000 (0.2008)	0.4857* (0.2073)	0.4821*** (0.1302)	0.4455*** (0.1252)	0.3346* (0.1646)	0.7334*** (0.1496)
BIC	-1120.404	-986.922	-825.315	-792.981	-925.839	-802.779	-687.732	-825.698
Implied behavioral parameters/local stability								
ϕ_2	1.0960*** (0.0154)	1.0925*** (0.0131)	1.1316*** (0.0358)	1.0889*** (0.0259)	1.1072*** (0.0192)	1.0702*** (0.0077)	1.0819*** (0.0083)	1.0650*** (0.0197)
$\phi_2 - (R + \bar{\alpha})$	0.0856*** (0.0154)	0.0822*** (0.0131)	0.1249*** (0.0258)	0.0727** (0.0259)	0.0997*** (0.0192)	0.0570*** (0.0077)	0.0630*** (0.0083)	0.0483** (0.0197)
$\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$	0.9815*** (0.0106)	1.0057*** (0.0055)	0.9923*** (0.0079)	1.0060*** (0.0070)	0.9858*** (0.0094)	1.0003*** (0.0043)	1.0130*** (0.0061)	1.0096*** (0.0055)
loc. stability	stable	unstable	stable	unstable	stable	unstable	unstable	unstable
$R_{crit.}$	0.9917*** (0.0107)	1.0160*** (0.0056)	0.9990*** (0.0079)	1.0223*** (0.0071)	0.9931*** (0.0095)	1.0135*** (0.0043)	1.0321*** (0.0062)	1.0265*** (0.0056)

Dynamic Gordon model

In this paper, we assume that agents agree on the fundamental price P_t^* as given by the dynamic Gordon model, which is related to the static Gordon model fundamental price through (see Eqs. (6) and (9))

$$P_t^* \approx (1 + c_r \Delta r_t^{rf} + c_g \Delta g_t) P_t^{*,static} \quad (17)$$

where c_r and c_g are constants depending on the fundamental parameters of the static Gordon model and on the first order autocorrelations ρ and γ of the real interest rate and the real growth rate, respectively. We estimate c_r and c_g by plugging in the first order sample autocorrelation of these series into Eqs. (7) and (8), after which we use Eq. (17) to calculate P_t^* . Overall, the corrections implied by the dynamic Gordon model relative to the static Gordon model were observed to be relatively small, leading to differences between P_t^* and $P_t^{*,static}$ of the order of 1% up to about 3%.

3.2. Estimation of the heterogeneous agents model

The behavioral model parameters are estimated based on the time series $X_t = \frac{P_t}{P_t^*} - 1 \approx \log P_t - \log P_t^*$, that is, the relative deviation of the house price from the estimated fundamental ratio according to the dynamic Gordon model. When presenting the empirical results we focus on the case $\bar{\alpha} = 0$ and NLS estimation, unless stated otherwise explicitly.¹⁴

The estimated behavioral model parameters ϕ_1 , $\Delta\phi$, β and δ are given in Table 3. The estimated values of $\Delta\phi = \phi_2 - \phi_1$ are significant for all countries confirming the presence of time-varying behavioral heterogeneity in the way agents

¹⁴ As discussed below, the results are rather robust with respect to fixing the risk premium to the estimate $\bar{\alpha} = 0.005$ of Himmelberg et al. (2005) and with respect to using weighted NLS, allowing for GARCH(1,1) structure on the innovations u_t .

Table 4

Fundamental parameters (fixed during estimation of the behavioral parameters), estimated behavioral model parameters for $\bar{\alpha} = 0.005$ and corresponding implied behavioral parameters. Below the estimates the corresponding standard errors are given in parentheses. The labels **, ***, **** and ***** denote significance at the 10%, 5% and 1% level of significance, respectively.

Country	US	JP	UK	NL	CH	ES	SE	BE
Fundamental parameters								
$\bar{\alpha}$	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050
$R + \bar{\alpha}$	1.0104	1.0103	1.0068	1.0162	1.0075	1.0132	1.0188	1.0167
Estimated behavioral parameters								
ϕ_1	0.8691*** (0.0600)	0.9214*** (0.0330)	0.8476*** (0.0682)	0.9660*** (0.0233)	0.8854*** (0.0717)	0.9397*** (0.0190)	0.9790*** (0.0125)	0.9965*** (0.0259)
$\Delta\phi$	0.1798* (0.0886)	0.1702** (0.0629)	0.2003* (0.1005)	0.0887* (0.0435)	0.1926 (0.1384)	0.1194*** (0.0281)	0.0955*** (0.0158)	0.0498 (0.0443)
$\beta (\times 10^3)$	13.77 (38.09)	1.194 (1.171)	38.2 (537.1)	5.063 (8.117)	1.050 (2.223)	37.0 (318.3)	86.4 (181.8)	21.3 (133.1)
δ	0.6404* (0.2487)	0.1027 (0.2474)	0.6970*** (0.1911)	0.4914 ^c (0.2785)	0.4743 (0.3203)	0.7781*** (0.0912)	0.2710 (0.2478)	0.8201** (0.2642)
BIC	-896.417	-897.207	-467.244	-758.755	-764.870	-616.748	-664.961	-709.147
Implied parameters/local stability								
ϕ_2	1.0489*** (0.0360)	1.0917*** (0.0314)	1.0478*** (0.0394)	1.0547*** (0.0227)	1.0780*** (0.0695)	1.0592*** (0.0128)	1.0745*** (0.0125)	1.0463*** (0.0209)
$\phi_2 - (R + \bar{\alpha})$	0.0385 (0.0360)	0.0814** (0.0314)	0.0411 (0.0394)	0.0385* (0.0227)	0.0705 (0.0695)	0.0459*** (0.0128)	0.0556*** (0.0082)	0.0296 ^c (0.0209)
$\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$	0.9491*** (0.0218)	0.9963*** (0.0069)	0.9413*** (0.0239)	0.9943*** (0.0074)	0.9744*** (0.0139)	0.9864*** (0.0080)	1.0077*** (0.0069)	1.0046*** (0.0077)
loc. stability	stable	stable	stable	stable	stable	stable	unstable	unstable
$R_{crit.}$	0.9540*** (0.0220)	1.0015*** (0.0069)	0.9427*** (0.0241)	1.0054*** (0.0075)	0.9767*** (0.0140)	0.9945*** (0.0081)	1.0217*** (0.0070)	1.0164*** (0.0079)

form expectations. We also find that β is not significantly different from zero. Note, however, that we cannot put $\beta = 0$, firstly because β is restricted to be strictly larger than zero in the model since otherwise the fractions $n_{1,t}$ and $n_{2,t}$ will converge to the constant 0.5. Secondly, setting $\beta = 0$ leads to the problem that $\Delta\phi$ is not identified in that case. To avoid such identification problems and to be able to identify the significance of the differences in the forecast rules β should be nonzero. From this perspective, the fact that β is found to be insignificant is merely an indication that the model's forecast accuracy is not very sensitive to the exact value of β and the other parameters can to a large extent compensate for changes in β ¹⁵.

The asynchronous updating parameter δ is significant for the US, NL, CH, ES, SE and BE, with roughly half of the agents re-evaluating their strategies per period. For JP and UK no significant evidence for asynchronous updating is found.

The bottom part of Table 3 provides a number of coefficients expressed as linear functions of the estimated parameters and hence their standard errors could be easily calculated based on the variance-covariance matrix of the estimated parameters. These coefficients are important for the underlying dynamics of the nonlinear model. The first of these is $\phi_2 = \phi_1 + \Delta\phi$, which appears to be significantly larger than 1 for all countries. The second is $\phi_2 - (R + \bar{\alpha})$. If this is positive, the dynamics allow for temporary explosive bubbles around the fundamental price, when a sufficiently large fraction of agents is of type 2. We find this coefficient to be significantly larger than zero for all countries considered. The third coefficient is $(\phi_1 + \phi_2)/(2(R + \bar{\alpha}))$, which occurs in the left-hand-side of the stability condition (14). This is the implied value of the AR-coefficient at equilibrium, determining whether the estimated model has a stable or an unstable fundamental equilibrium. Whether the equilibrium was found to be stable or not is indicated in the row immediately below. The implied ratios $(\phi_1 + \phi_2)/(2(R + \bar{\alpha}))$ on the left-hand-side of the stability condition (14) can be seen to be surprisingly close to 1 for all countries, which implies that the fundamental equilibrium is very close to the border of (in)stability. This means that in all cases the dynamics are very close to a unit root process (a random walk) around the fundamental equilibrium (i.e. for X_t small). The fourth and final implied parameter is $R_{crit.} = (\phi_1 + \phi_2)/2 - \bar{\alpha}$, the critical value of R below which the fundamental equilibrium would become unstable if the behavioral model parameters were held fixed at their estimated values. Recall from Proposition 2 that for $R_{crit.} > 1$ the pitchfork bifurcation is supercritical, while for $R_{crit.} < 1$ the pitchfork bifurcation is subcritical. The role of $R_{crit.}$ will be discussed in more detail in Section 4, where we discuss multiple equilibria and policy implications.

As a robustness check we have repeated the estimation for $\bar{\alpha} = 0.005$, which, as discussed above in Section 3.1, we consider to be the upper bound of the risk premium $\bar{\alpha}$. The results, shown in Table 4 qualitatively similar to those obtained for $\bar{\alpha} = 0$ for most countries. Although, as judged from changes in the value of $\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$, JP, NL and ES move from unstable to stable, only for ES the estimated value of $\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$ is significantly smaller than 1. For all countries except CH and BE, $\Delta\phi$ remains significantly different from zero, although the significance has become less pronounced for the US, JP, UK and NL.

¹⁵ Hommes and in't Veld (2017) estimate a similar 2-type switching model on the relative deviations of the S&P500 stock market index from two benchmark fundamentals, the Gordon growth model and the Campbell-Cochrane consumption-habit model and show that the likelihood function is very flat w.r.t. the intensity of choice parameter β . They use quarterly data 1950–2016 and show by Monte-Carlo simulations that the test to reject the null of a switching model with estimated parameter values has essentially zero power for small samples of 250 observations. This explains the large standard deviations in the estimates for β and its non-significance.

Table 5

Fundamental parameters (fixed during estimation of the behavioral parameters), estimated behavioral model parameters for $\bar{\alpha} = 0$, with GARCH(1,1) innovations, and corresponding implied behavioral parameters. Below the estimates the corresponding standard errors are given in parentheses. The labels ‘*’, ‘**’ and ‘***’ denote significance at the 10%, 5% and 1% level of significance, respectively.

Country	US	JP	UK	NL	CH	ES	SE	BE
Fundamental parameters								
$\bar{\alpha}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$R + \bar{\alpha}$	1.0104	1.0103	1.0068	1.0162	1.0075	1.0132	1.0188	1.0167
Estimated behavioral parameters								
ϕ_1	0.9095*** (0.0169)	0.9468*** (0.0061)	0.9029*** (0.0285)	0.9508*** (0.0276)	0.8780*** (0.0255)	0.9611*** (0.0070)	0.9727*** (0.0078)	0.9867*** (0.0150)
$\Delta\phi$	0.1904*** (0.0263)	0.1364*** (0.0130)	0.1988*** (0.0559)	0.1439** (0.0550)	0.2534*** (0.0472)	0.1052*** (0.0116)	0.1109*** (0.0106)	0.0779* (0.0321)
$\beta (\times 10^3)$	28.05° (15.63)	5.802*** (1.589)	0.9104 (0.5967)	1.702 (1.639)	1.946° (1.069)	2.472° (1.402)	72.2 (142.6)	8.78 (13.28)
δ	0.5251*** (0.1397)	0.3438° (0.1495)	0.0000 (0.1969)	0.6680*** (0.1197)	0.5120*** (0.1300)	0.4231** (0.1549)	0.4465** (0.1479)	0.7401*** (0.1449)
BIC	-1197.889	-1148.065	-871.043	-918.352	-984.764	-813.779	-710.942	-837.908
Implied behavioral parameters/local stability								
ϕ_2	1.1000*** (0.0128)	1.0832*** (0.0080)	1.1017*** (0.0282)	1.0947*** (0.0280)	1.1314*** (0.0237)	1.0663*** (0.0070)	1.0836*** (0.0059)	1.0645*** (0.0185)
$\phi_2 - (R + \bar{\alpha})$	0.0895*** (0.0128)	0.0729*** (0.0080)	0.0949*** (0.0282)	0.0785** (0.0281)	0.1239*** (0.0237)	0.0531*** (0.0070)	0.0648*** (0.0059)	0.0478** (0.0185)
$\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$	0.9944*** (0.0072)	1.0047*** (0.0029)	0.9956*** (0.0045)	1.0064*** (0.0043)	0.9972*** (0.0070)	1.0005*** (0.0039)	1.0092*** (0.0044)	1.0087*** (0.0049)
loc. stability	stable	unstable	stable	unstable	stable	unstable	unstable	unstable
$R_{crit.}$	1.0047*** (0.0072)	1.0150*** (0.0029)	1.0023*** (0.0045)	1.0227*** (0.0044)	1.0047*** (0.0070)	1.0137*** (0.0039)	1.0282*** (0.0045)	1.0256*** (0.0050)

Likewise, the estimates of $\phi_2 - (R + \bar{\alpha})$ are still positive, but no longer significant for some countries. The asynchronous updating parameter δ is no longer significant for JP, CH and SE when we take $\bar{\alpha} = 0.005$ instead of 0, and less significant for NL. The values of the Bayesian information criterion (BIC-values) indicate that the model with $\bar{\alpha} = 0$, presented in the main text, is more suitable than the model with $\bar{\alpha} = 0.005$.

As a second robustness check we performed a diagnostic check of the estimated baseline model (with $\bar{\alpha} = 0$) by investigating the autocorrelation of the residuals \hat{u}_t and of their absolute values $|\hat{u}_t|$. For all countries we observed mild (0.2–0.4) autocorrelation in the residuals, significant up to 5 lags for most countries, and substantial autocorrelation (0.3–0.7) in the absolute residuals, significant up to 10 lags. The latter clearly indicates the presence of (conditional) heteroskedasticity. To accommodate for heteroskedasticity we estimated the conditional variance in the residuals using a GARCH(1,1) model.¹⁶ These estimated conditional variances were subsequently used to perform a second stage weighted nonlinear least squares fit of the model. The standardized residuals of this second stage estimation step no longer had any visible heteroskedasticity as judged from the autocorrelation of their absolute values. The resulting parameter estimates and standard errors are shown in Table 5. A comparison with the unweighted NLS estimates in Table 3 shows that the estimation results are not very sensitive to the use of heteroskedastic errors. Taking into account heteroskedasticity when present should be expected to lead to more efficient estimation. Indeed, allowing for GARCH(1,1) heteroskedastic errors appears to lead to a small reduction in the standard errors of most of the estimated parameters. In particular it can be observed that the intensity of choice parameter β benefits from this, becoming more significant for US, JP and CH. This is consistent with the fact that the BIC-values reported in Table 5 are smaller than those in Table 3, suggesting that the model GARCH(1,1) errors fits the data better.

The standardized residuals of the second stage estimation still contained some significant but mild autocorrelation up to lag 5. Although this is not optimal from an econometric model specification perspective, we have made no attempts to correct this for three reasons. Firstly, the data provided by the OECD are seasonally adjusted, which could account for a substantial part of the autocorrelation present up to lag 4. Secondly, although one could econometrically easily take into account a, say, AR(1) term in u_t in the NLS fit to reduce residual autocorrelation, this would not do justice to our aim to estimate a behavioral model rather than an econometric model; no direct behavioral interpretation would be available for the coefficient of such an estimated AR(1) noise term. Thirdly, a behavioral modeling alternative to adding an AR(1) term in u_t would be to add another expectation rule (type) to the behavioral model. However, this would lead to an increase of the model complexity, while the aim here is to come up with a parsimonious behavioral model that captures the most important consequence of introducing heterogeneous agents in the model, being that this gives rise to endogenously induced periods of dynamical instability and stability.

3.3. Temporary bubbles in house prices

Figs. 4 and 5 show for all eight countries the log-difference between house prices and fundamentals (upper panels), the estimated proportion n_1 of agents forming expectations of type 1 associated with $\phi_1 < 1$, in other words, those agents

¹⁶ We also tried specifying the conditional variance as being proportional to $n_{1,t}(1 - n_{1,t})$ reflecting the binomial nature of the choice each agent makes, but this led to a poorly specified conditional variance.

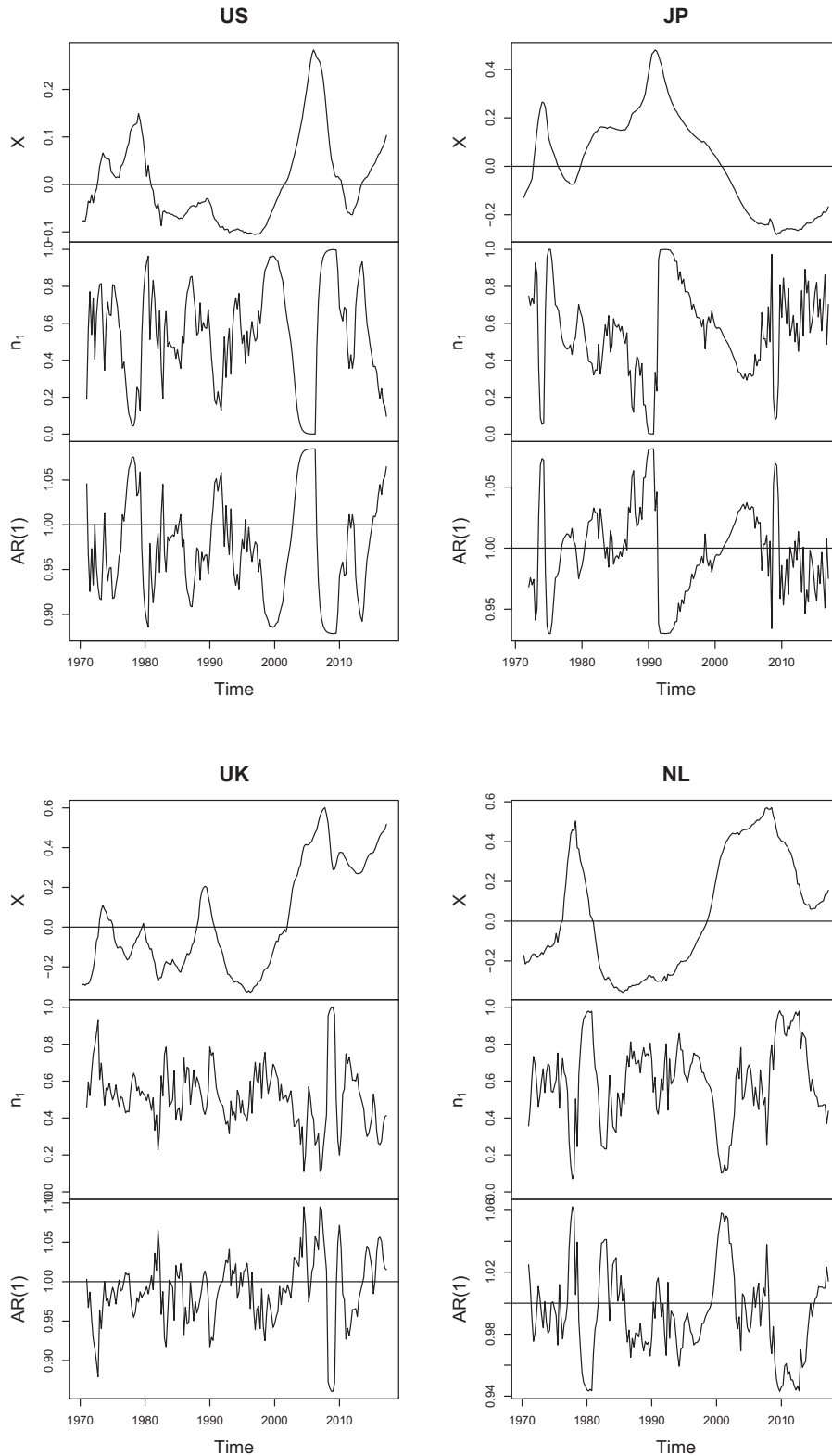


Fig. 4. Relative price deviations X_t from fundamentals (top panels), estimated fractions of agents of type 1, i.e. fundamental mean-reverting agents (middle panels) and implied AR(1) coefficients, i.e. market sentiment (bottom panels) for the US, JP, UK and NL. Temporary bubbles arise when the market sentiment exceeds 1.

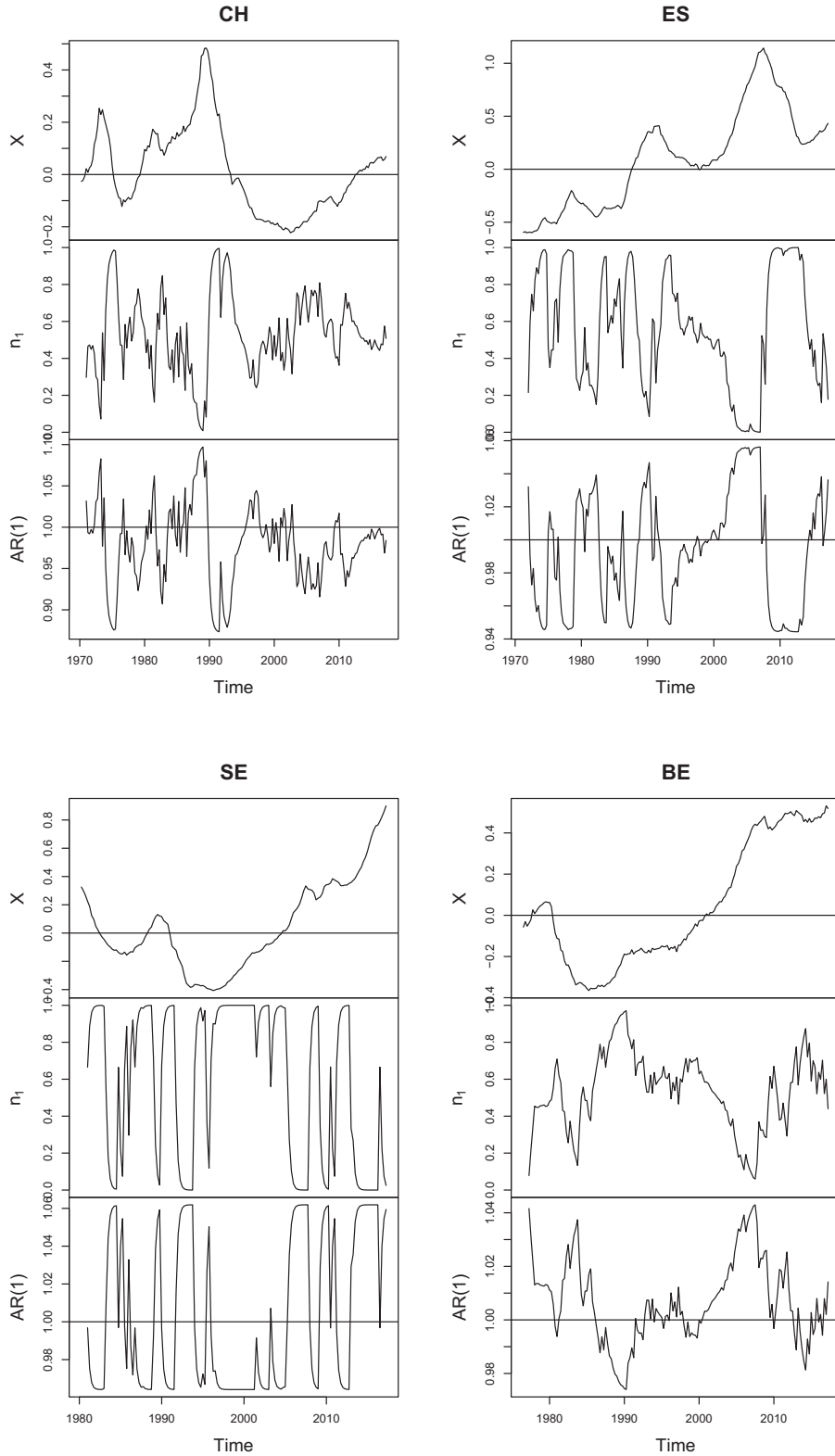


Fig. 5. Relative price deviations X_t from fundamentals (top panels), estimated fractions of agents of type 1, i.e. fundamental mean-reverting agents (middle panels) and implied AR(1) coefficients, i.e. market sentiment (bottom panels) for CH, ES, SE and BE. Temporary bubbles arise when the market sentiment exceeds 1.

who expect mean-reversion towards the fundamental value (middle panels) and, finally, the estimated time-varying AR(1) coefficients in (15) showing the time variation of the market sentiment (lower panels).

There is considerable time-variation in the distribution across agents of the fundamental mean-reverting and trend-following rules and in the average market sentiment. An immediate observation is that in all countries temporary housing market bubbles occur, that is, there are prolonged periods of several years during which the market sentiment is explosive (i.e. exceeds 1). For the US, for example, four episodes where the AR(1) coefficient is explosive can be identified: in the late 1970s, early 1990s, during 2004–2007 and in the last few years of the sample. The first, third and last of these coincide with increasing prices above the fundamental, while the second period is a ‘negative’ bubble with prices decreasing below the fundamental. The UK, NL, SE and ES also exhibit housing bubbles with explosive market sentiment in the years 2004–2007 or even for a longer period (BE). Exceptions are CH and JP for which positive bubbles arose much earlier, between 1970 and 1990, later followed by a negative bubble. In particular JP experienced a strong negative bubble with explosive market sentiment during most of 2000–2010 with house prices continuously declining. Our simple stylized model thus provides an easy to use tool for identifying housing market bubbles and the market sentiment may serve as an early warning signal of housing bubbles signaling when the market becomes explosive and enters a (temporary) bubble regime. When a bubble bursts after a few years a majority of agents typically switches to the type 1 fundamental forecasting rule and strong mean-reverting market sentiment brings house prices back closer to fundamentals. In particular, after 2007 in the US and somewhat later also in the NL and ES, the housing market was dominated by a strong mean-reverting market sentiment bringing prices back closer to fundamentals.

Similar heterogeneous expectations models have been estimated on various data in recent years including stock market data, exchange rates, survey data and macroeconomic data (see the references in the introduction). An interesting and characteristic feature of the housing market data are the long-lasting bubbles, over many years, detected in all eight countries. For example, ES has a long-lasting bubble characterized by temporary exploding market sentiment of about 10 years from the late 1990s to the financial crisis, while Japan has a declining bubble over roughly the same period. For UK, SE and BE, the three countries in the sample that have not gone through a major correction yet, the duration is arguably longer. In other data sets such long-lasting bubbles are rare, with the only exception being the dot-com bubble in the stock market. Using yearly data of the S&P500 (Boswijk et al., 2007) estimate the dot-com bubble to last for 6 years, 1995–2000, while Hommes and in’t Veld (2017) find the same for quarterly data. Hence, based on the estimation results of similar HAMs, we conclude that house prices exhibit the longest temporary bubbles compared to other markets and macro data¹⁷.

4. Policy implications

Blanchard (2014) recently stressed that “The main lesson of the crisis is that we were much closer to “dark corners” – situations in which the economy could badly malfunction – than we thought. Now that we are more aware of nonlinearities and the dangers they pose, we should explore them further theoretically and empirically.” Blanchard argued for the coexistence of non-linear models along the standard, general equilibrium models: “If macroeconomic policy and financial regulation are set in such a way as to maintain a healthy distance from dark corners, then our models that portray normal times may still be largely appropriate. Another class of economic models, aimed at measuring systemic risk, can be used to give warning signals that we are getting too close to dark corners, and that steps must be taken to reduce risk and increase distance. Trying to create a model that integrates normal times and systemic risks may be beyond the profession’s conceptual and technical reach at this stage.”

What are the policy implications of our nonlinear HAM? Based on the analysis of the house price dynamics in our stylized nonlinear heterogeneous expectations switching model for the range of empirically relevant parameter values we can draw some general policy conclusions. In particular, the multiplicity of steady states and the global instability of the system has implications for how policy can avoid the “dark corners” of the economy. We focus on the role of R as the policy parameter, as this is an important parameter that can be influenced by policy makers.

Recall from Section 2.3 that the local stability condition (14) of the fundamental steady state equilibrium is

$$\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})} = \frac{(1 + g)(\phi_1 + \phi_2)}{2(1 + r + \alpha)} < 1.$$

where we have dropped the absolute value, since all estimated parameters are positive. Note that the fundamental equilibrium is locally unstable when the average of ϕ_1 and ϕ_2 is larger than $R + \bar{\alpha}$, which means that the local stability is affected directly by behavioral parameters as well as structural parameters. Notice also that for sufficiently large R the fundamental steady state will be locally stable. Recall that $R = \frac{1+r^f+\omega}{1+g}$, so that increasing R is equivalent to increasing the risk-free mortgage interest rate r^f , increasing the tax rate or maintenance costs ω or decreasing the growth rate g of the rent.

For all eight countries, the estimated parameter values of the model were found to be close to the border of (in)stability of the fundamental steady state. The bifurcation diagrams in Fig. 6 show numerically the transitions that occur for each country as R decreases, with all other parameters fixed at the country specific estimates.¹⁸ The fundamental steady state

¹⁷ See Ellen et al. (2018) for a general discussion of empirical HAMs comparing bubbles for different asset classes.

¹⁸ In a related paper (Diks and Wang, 2016) estimate a stochastic cusp catastrophe model to house prices and interest rates in different countries and also find multiple non-fundamental steady states for several countries.

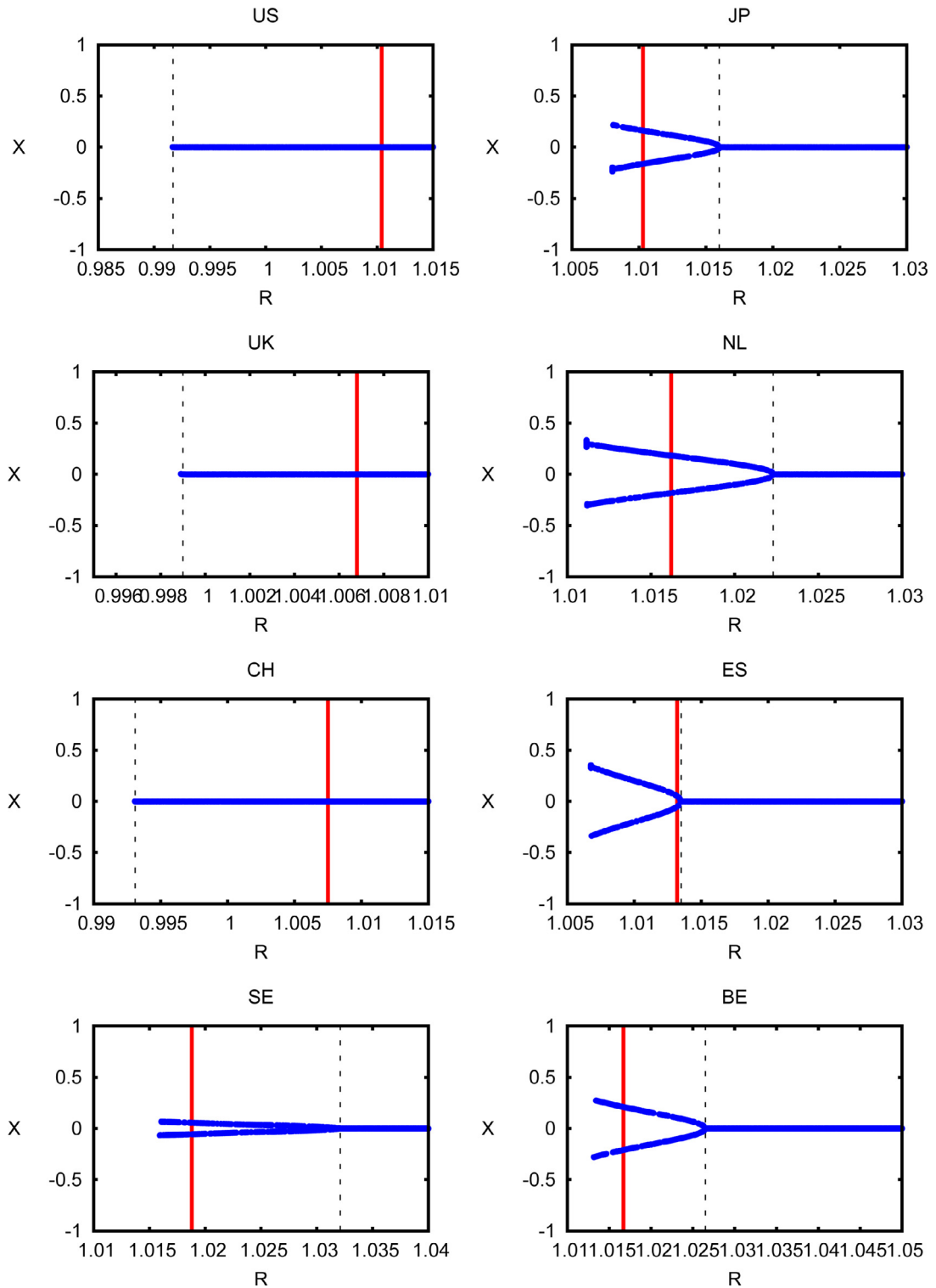


Fig. 6. Bifurcation diagrams showing the long run behavior for each country as a function of the bifurcation parameter R , with other parameters fixed at the estimated values. The solid (red) vertical lines indicate the estimated values of R and the dashed vertical lines the implied critical value R_{crit} of R at which the primary pitchfork bifurcation occurs (transition from local stability to (local) instability of the fundamental steady state). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

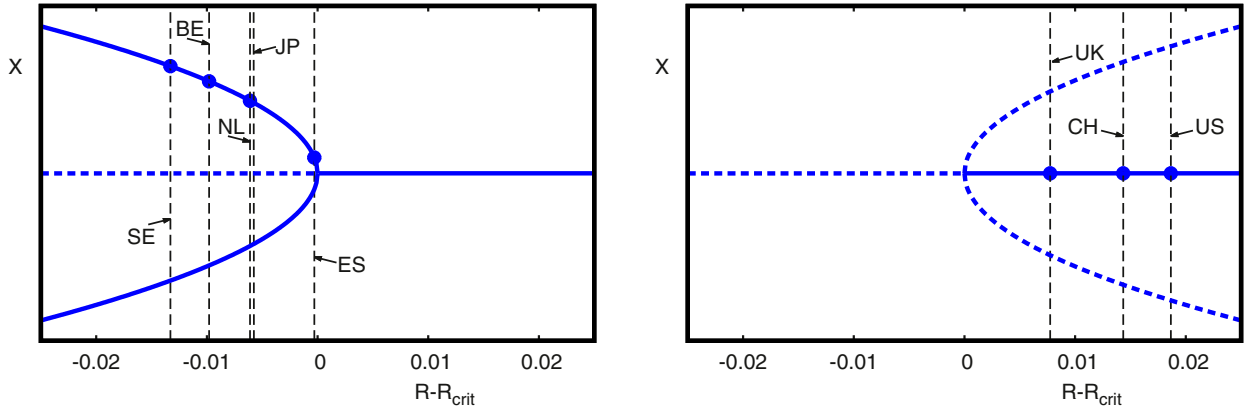


Fig. 7. Pitchfork bifurcation w.r.t. the policy parameter R . Bold curves denote stable steady states; dotted curves are unstable steady states. **Left Panel:** supercritical pitchfork bifurcation with stable fundamental steady state for $R > R_{crit}$ and unstable fundamental steady state surrounded by two stable non-fundamental steady states for $R < R_{crit}$. **Right Panel:** subcritical pitchfork bifurcation with stable fundamental steady state surrounded by a corridor of stability bounded by two unstable non-fundamental steady states for $R > R_{crit}$ and (globally) unstable fundamental steady state with exploding dynamics for $R < R_{crit}$. The dots represent the estimated values $R - R_{crit}$ for the eight countries.

destabilizes when R decreases and hits the critical threshold

$$R_{crit} = \frac{\phi_1 + \phi_2}{2} - \bar{\alpha},$$

the estimated values of which are shown in the last row of Table 3.

The bifurcation diagrams in Fig. 6 are constructed by plotting for each parameter value R (from a finite grid) 100 subsequent states visited by the model after a transient of 100 iterations. A small amount of dynamic noise was added, $NID(0, \sigma^2)$ with $\sigma = 10^{-4}$, to enable the system to move away from the equilibrium once it becomes unstable.

There are two different bifurcation scenarios observed for these countries. For JP, NL, ES, SE and BE two co-existing stable non-fundamental steady states arise, one above and one below the unstable fundamental steady state, for values of R below the critical threshold. Hence, for these countries the nonlinear model exhibits multiple stable steady state equilibria for low values of R . The diagram also illustrates that as R decreases further, the model becomes globally unstable with exploding house prices. In contrast, the primary bifurcations to instability for US, UK and CH are different: the fundamental steady state becomes globally unstable, with exploding house prices for R values immediately below the critical threshold. A careful reader may observe that the critical thresholds R_{crit} for the US and, UK and CH are below 1 (see Table 3) and therefore might think that a bifurcation towards instability is unlikely to occur in practice as $R > 1$. However, the fundamental steady state is only locally stable for this nonlinear system, and only within the “corridor of stability” formed by the two unstable non-fundamental steady states will house prices remain bounded. For values of R above but close to the critical value R_{crit} global instability may arise due to small exogenous shocks driving the state outside of the “corridor of stability” and onto unbounded bubble solutions.

These empirical results in the housing model for the eight countries in Fig. 6 are in line with the theoretical analysis in Section 2.3, Proposition 2, where we showed that the model exhibits a pitchfork bifurcation. There are two different cases for the pitchfork bifurcation, a supercritical and a subcritical, and both theoretical cases, together with the empirical estimates for the eight countries, are illustrated in Fig. 7. For JP, NL, ES, SE and BE we find a supercritical pitchfork bifurcation, while for US, UK and CH we find a subcritical pitchfork bifurcation.

What then can a policy maker do to stabilize house price bubbles and prevent market instability? Our model is a partial equilibrium approach that aims to classify house price changes into categories of acceptable and unacceptable (i.e. dangerous) movements. To fully understand the welfare implications of such instability one would need to incorporate them into general equilibrium models.

Nevertheless, what this model can do is provide early warnings when the system may be approaching what (Blanchard, 2014) called the ‘dark corners’. The general lesson for policy makers to be drawn from our analysis of the nonlinear housing model is that structural knowledge of the system may yield important insights in policies that can prevent local or even global instability. In general terms, the policy maker should prevent the system from getting too close to bifurcation points that may destabilize the system. An interesting new methodological contribution of our analysis of the stylized housing model is that the policy maker should in particular be aware of preventing the so-called “hard bifurcations” of the system, such as the subcritical pitchfork, which may cause a sudden critical transition of the system leading to an exploding bubble or market collapse.

In terms of our stylized housing model and the bifurcation diagrams in Figs. 6 and 7 the policy maker should keep the parameter R sufficiently large, so that the system stays away from the locally or even globally unstable fundamental steady state (the dark corners). Recall that $R = \frac{1+r}{1+g}$, where g is the growth rate of housing rents and $r = r^{rf} + \omega$ is the sum of the risk-free mortgage rate and the tax/maintenance cost rate. Hence, stabilizing policies include an increase of the

mortgage interest rate,¹⁹ a decrease of mortgage tax deduction rates, an increase of tax rates for home owners, and/or a decrease of (the growth of) housing rents. Notice also an important difference in policy implications compared to the RE fundamental benchmark model. Under RE these policies *only* affect the level to fundamental house prices. In our behavioral 2-type switching model these policies affect the level of the fundamental steady state as well as the market volatility, that is, the (*local*) stability of the fundamental steady state and the mean-reversion towards the fundamental steady state are directly affected by these policy parameters. For example, in an environment with higher interest rates the stable mean-reverting rule performs relatively better than the trend-extrapolating rule and, therefore, agents will learn and adapt their behavior after the policy change by switching to the better performing mean-reverting rule making the house price dynamics more stable under evolutionary selection in an environment with higher interest rates. Note that, although they are stabilizing the effectiveness of these policies may be limited however, because changes in the interest rate, the tax rate or the growth rates of housing rents only lead to small changes in the parameter R and therefore only to small changes of the mean-reversion of the system.²⁰ Other policies putting restrictions on housing demand, such as loan-to-value restriction, may be a more effective way to stabilizing housing markets (cf. [Baptista et al., 2016](#)).

5. Concluding remarks

We have developed a nonlinear empirical housing market model with heterogeneous beliefs. The standard user cost of capital model with house price fundamentals based on imputed rents is nested as a special case within the general heterogeneous setup. Agents however disagree in their expectations about future house prices and switch between a fundamental mean-reverting and a mean-diverting, trend-following forecasting rule based upon their relative performance. The heterogeneous beliefs housing model with endogenous switching displays nonlinear aggregate price fluctuations with booms and busts around the fundamental price triggered by stochastic shocks and strongly amplified by self-fulfilling expectations.

Our goal was to develop a general structural model that can be estimated on house prices of different countries. Using quarterly data on rents and house prices, we estimate the model parameters for eight different countries, US, UK, NL, JP, CH, ES, SE and BE. In all countries the data support heterogeneity in expectations, with temporary switching between fundamental mean-reverting and trend-following beliefs. For all countries we identify long-lasting temporary house price bubbles amplified by trend extrapolation. For three countries, US, NL, and ES we identify strong housing bubbles in the period 2004–2007, while for JP and CH housing bubbles in the 1980–1990s are identified, in all cases strongly amplified by trend-following behavior. When these bubbles burst, the majority of agents switches to a fundamental mean-reverting strategy reinforcing a strong correction of house prices. Similar HAMs have been estimated on various data sets, including stock prices, commodity prices, exchange rates and macro data. Comparing estimation results of similar 2-type HAMs we conclude that housing markets exhibit the longest temporary bubbles with housing bubbles up to 6–10 years being the rule rather than the exception.

These results have important policy implications. The underlying nonlinear switching model exhibits multiple steady states and/or global instability for parameter-values close to the estimated values for all countries. We have argued that a decrease of the (mortgage) interest rate, a decrease of the tax rate for home owners, an increase of mortgage tax deduction rates and/or an increase of housing rents all shift the nonlinear system closer to multiple equilibria and global instability. Policy should prevent the system getting too close to bifurcation in order to avoid critical transitions to global instability. The market sentiment, that is the average extrapolation factor of our model, may serve as an early warning indicator for policy intervention when the system is approaching the border of instability.

Our housing model with heterogeneous beliefs is very stylized and the results should be viewed as a ‘proof of principle’ to show that a nonlinear model with switching can lead to very different behavior than a benchmark linear model. Nonlinear systems may easily turn unstable, but structural nonlinear modeling can also provide new insights for policy makers on how to prevent critical transitions towards instability and collapse. Our stylized nonlinear housing market model is just a simple but empirically relevant example illustrating potential policy tools for taming instability. Building more realistic nonlinear economic models based upon country specific institutional details can give important new insights for policy makers, particularly in extreme times of crises.

In order to study realistic policy scenarios in more detail future research should focus on a number of extensions. First of all, our benchmark fundamental value (a simple dynamic Gordon model) is a very stylized and general benchmark that applies to different countries. But our HAM is formulated in terms of relative deviations some benchmark fundamental. For policy analysis it would be important to include country specific institutional details of the housing market, such as tax rules and mortgage requirements, in the fundamental value. One could then easily re-estimate the model and study the

¹⁹ The stabilizing effect of the interest rate is also discussed in the recent housing market model of [Martin et al. \(2019\)](#), who, inspired by [Taylor \(2009\)](#), [Agnello et al. \(2018\)](#) and particularly by [Lambertini et al. \(2013\)](#), consider a dynamic interest rate rule. Interestingly, [Martin et al. \(2019\)](#) show that the central bank has the ability to suppress a pitchfork and a Neimark-Sacker bifurcation via the interest rate. However, we do not argue that the interest rate should be used to stabilize financial imbalances. There are arguably other macro prudential tools to consider that might be better suited ([Agur and Demertzis, 2019](#)). What the paper does argue is that policy intervention can stabilize bounded rational expectations of the way we described.

²⁰ [Bao and Hommes \(2015\)](#) design an experimental housing market to study bubble formation. They consider three different treatments with different discount factors $1/R$ and find that explosive bubbles emerge for $1/R = 0.95$, boom and bust cycles emerge for $1/R = 0.85$ and a globally stable fundamental steady state emerges for $1/R = 0.71$.

amplification mechanism of endogenous belief switching around an improved and more realistic fundamental benchmark. Second, our two forecasting rules are in fact the simplest linear examples with only one lag. It would be of interest to have further guidance on which types of forecasting rules to use in these models, for example through laboratory experiments on expectations with human subjects and/or using surveys of forecasts. In general, building more realistic behavioral models for policy analysis should be high on the research agenda of academics and policy makers.

Acknowledgments

Views expressed are our own and do not reflect those of the institutions with which we are affiliated. Earlier versions of the paper have been presented at the 14th DNB Annual Conference 2011 on “Complex Systems: Towards a better understanding of financial stability and crises”, the Latsis Symposium 2012, Zürich, Sept. 11-14, 2012, the seminar series at Massey University, Albany, New Zealand, 2012, the Computation in Economics and Finance (CEF) Conference, Vancouver, July 10-12, 2013, the Society for Nonlinear Dynamics and Econometrics (SNDE) Symposium, New York, April 17-18, 2014, the European Economic Association (EEA) Conference, Toulouse, August 25-29, 2014, the Banque de France and ESSEC Business School Workshop “Expectations and Forecasting in International Macroeconomics”, Dec. 10, 2014 the European Central Bank conference on “Non-linearities in macroeconomics and finance in the light of crises”, Dec. 15-16, 2014, and at seminars at UNSW, Sydney, December 10, 2014, Technical University Delft, the Netherlands, Feb. 5, 2015, Statistics Netherlands, March 20, 2017, Leiden University, the Netherlands, May 22, 2017 and UvA Institute for Advanced Study, Amsterdam, the Netherlands, Jan. 12, 2018. We are grateful to Vincent Sterk and Simon van Norden, as well as seminar participants at the Bank of England and the European Commission for comments and suggestions. This research has been supported by the Netherlands Organisation for Scientific Research (NWO) under the Complexity program and by the Netherlands Institute for Advanced Study in the Humanities and Social Sciences (NIAS). The usual disclaimer applies.

Appendix A

A1. Fundamental price models

Static Gordon model

Assuming a constant growth rate g , interest plus maintenance rate $r = r^{rf} + \omega$ and risk aversion parameter α , we can write (imposing a transversality condition)

$$\begin{aligned} P_t^{*,static} &= E_t \left(\frac{(1 + r^{rf})Q_t}{1 + r + \alpha} \right) + E_t \left(\frac{(1 + r^{rf})Q_t(1 + g)}{(1 + r + \alpha)^2} \right) + E_t \left(\frac{(1 + r^{rf})Q_t(1 + g)^2}{(1 + r + \alpha)^3} \right) + \dots \\ &= \frac{(1 + r^{rf})Q_t}{1 + r + \alpha} \sum_{i=1}^{\infty} \left(\frac{1 + g}{1 + r + \alpha} \right)^i \\ &= \frac{1 + r^{rf}}{1 + r + \alpha} \frac{Q_t}{1 - \frac{1+g}{1+r+\alpha}} = \frac{1 + r^{rf}}{r + \alpha - g} Q_t. \end{aligned}$$

Dynamic Gordon model

In case we allow r_t^{rf} and g_t to be time-varying with mean r^{rf} and g , respectively, using the law of iterated expectations and imposing a transversality condition, one obtains

$$\begin{aligned} P_t^* &= \mathbb{E}_t \left(\frac{(1 + r_t^{rf})Q_t}{1 + r_t^{rf} + \omega + \alpha} \right) + \mathbb{E}_t \left(\frac{(1 + r_t^{rf})Q_t(1 + g_{t+1})}{(1 + r_t^{rf} + \omega + \alpha)(1 + r_{t+1}^{rf} + \omega + \alpha)} \right) + \\ &+ \mathbb{E}_t \left(\frac{(1 + r_t^{rf})Q_t(1 + g_{t+1})(1 + g_{t+2})}{(1 + r_t^{rf} + \omega + \alpha)(1 + r_{t+1}^{rf} + \omega + \alpha)(1 + r_{t+2}^{rf} + \omega + \alpha)} \right) + \dots \\ &= \mathbb{E}_t \left(1 + \sum_{j=1}^{\infty} \left(\prod_{i=1}^j \frac{1 + g_{t+i}}{1 + r_{t+i}^{rf} + \omega + \alpha} \right) \right) Y_t, \end{aligned}$$

where

$$Y_t \equiv \frac{1 + r_t^{rf}}{1 + r_t^{rf} + \omega + \alpha} Q_t,$$

is known at time t . This expression for the fundamental price allows straightforward evaluation of the partial derivatives required for the time-varying Gordon model, which are found to be

$$\begin{aligned} \left. \frac{\partial P_t^*}{\partial r_{t+j}^{rf}} \right|_{(r^{rf}, g)} &= E_t \left(\sum_{k=j}^{\infty} \frac{-1}{1+r^{rf}+\omega+\alpha} \left(\frac{1+g}{1+r^{rf}+\omega+\alpha} \right)^k \right) Y_t \\ &= \frac{-(1+g)^j}{(1+r^{rf}+\omega+\alpha)^{j+1}} \frac{1}{1-\frac{1+g}{1+r^{rf}+\omega+\alpha}} Y_t \\ &= -\frac{1}{r^{rf}+\omega+\alpha-g} \left(\frac{1+g}{1+r^{rf}+\omega+\alpha} \right)^j Y_t, \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial P_t^*}{\partial g_{t+j}} \right|_{(r^{rf}, g)} &= E_t \left(\sum_{k=j}^{\infty} \frac{1}{1+g} \left(\frac{1+g}{1+r^{rf}+\omega+\alpha} \right)^k \right) Y_t \\ &= \frac{1}{1+g} \left(\frac{1+g}{1+r^{rf}+\omega+\alpha} \right)^j \frac{1}{1-\frac{1+g}{1+r^{rf}+\omega+\alpha}} Y_t \\ &= \frac{1+r^{rf}+\omega+\alpha}{(1+g)(r^{rf}+\omega+\alpha-g)} \left(\frac{1+g}{1+r^{rf}+\omega+\alpha} \right)^j Y_t. \end{aligned}$$

If we let $\gamma \equiv \frac{1+g}{1+r^{rf}+\omega+\alpha}$ ($= \frac{1}{R+\alpha}$), the first order Taylor approximation (based on the Taylor expansion of (r_t^{rf}, g_t) around (r^{rf}, g)) of the fundamental price is given by

$$\begin{aligned} P_t^* &\approx \left[\frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} - \frac{1}{r^{rf}+\omega+\alpha-g} \mathbb{E}_t \left(\sum_{j=1}^{\infty} \gamma^j (r_{t+j}^{rf} - r^{rf}) \right) \right. \\ &\quad \left. + \frac{1+r^{rf}+\omega+\alpha}{(1+g)(r^{rf}+\omega+\alpha-g)} \mathbb{E}_t \left(\sum_{j=1}^{\infty} \gamma^j (g_{t+j} - g) \right) \right] Y_t. \end{aligned}$$

Assuming AR(1) expectations on future values r_{t+j}^{rf} and g_{t+j} , with AR(1) coefficient ρ and ϕ , respectively, this becomes

$$\begin{aligned} P_t^* &\approx \left[\frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} - \frac{1}{r^{rf}+\omega+\alpha-g} \sum_{j=1}^{\infty} \gamma^j \rho^j (r_t^{rf} - r^{rf}) \right. \\ &\quad \left. + \frac{1+r^{rf}+\omega+\alpha}{(1+g)(r^{rf}+\omega+\alpha-g)} \sum_{j=1}^{\infty} \gamma^j \phi^j (g_t - g) \right] Y_t \\ &= \left[\frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} - \frac{1}{r^{rf}+\omega+\alpha-g} \frac{\gamma \rho}{1-\gamma \rho} (r_t^{rf} - r^{rf}) \right. \\ &\quad \left. + \frac{1+r^{rf}+\omega+\alpha}{(1+g)(r^{rf}+\omega+\alpha-g)} \frac{\gamma \phi}{1-\gamma \phi} (g_t - g) \right] Y_t \\ &= \left[1 - \frac{r^{rf}+\alpha-g}{(1+r^{rf}+\omega+\alpha)(r^{rf}+\omega+\alpha-g)} \frac{\gamma \rho}{1-\gamma \rho} (r_t^{rf} - r^{rf}) \right. \\ &\quad \left. + \frac{r^{rf}+\alpha-g}{(1+g)(r^{rf}+\omega+\alpha-g)} \frac{\gamma \phi}{1-\gamma \phi} (g_t - g) \right] \frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} Y_t. \end{aligned}$$

Using

$$\begin{aligned} \frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} Y_t &= \frac{1+r^{rf}+\omega+\alpha}{r^{rf}+\alpha-g} \frac{1+r_t^{rf}}{1+r_t^{rf}+\omega+\alpha} Q_t \\ &= \frac{1+r^{rf}+\omega+\alpha}{1+r^{rf}} \frac{1+r_t^{rf}}{1+r_t^{rf}+\omega+\alpha} P_t^{*,static} \\ &= \left(1 + \left(\frac{1}{1+r^{rf}} - \frac{1}{1+r^{rf}+\omega+\alpha} \right) (r_t^{rf} - r^{rf}) + \mathcal{O}((r_t^{rf} - r^{rf})^2) \right) P_t^{*,static}, \end{aligned}$$

it is straightforward to show that, up to first order in $(r_t^{rf} - r^{rf}, g_t - g)$, we have

$$P_t^* \approx (1 + c_r(r_t^{rf} - r^{rf}) + c_g(g_t - g)) P_t^{*,static}$$

with

$$c_r = -\frac{r^{rf} + \alpha - g}{(1 + r^{rf} + \omega + \alpha)(r^{rf} + \omega + \alpha - g)} \frac{\gamma \rho}{1 - \gamma \rho} + \frac{1}{1 + r^{rf}} - \frac{1}{1 + r^{rf} + \omega + \alpha}$$

and

$$c_g = \frac{r^{rf} + \alpha - g}{(1 + g)(r^{rf} + \omega + \alpha - g)} \frac{\gamma \phi}{1 - \gamma \phi}.$$

A2. Expressing R_{t-1} in terms of X_{t-1} and X_{t-2}

Since according to the static Gordon model

$$X_t = \frac{P_t}{P_t^*} - 1 = \frac{P_t}{\left(\frac{1+r^{rf}}{r+\alpha-g} Q_t\right)} - 1,$$

we can write $P_t = (X_t + 1) \frac{1+r^{rf}}{r+\alpha-g} Q_t$, and express R_{t-1} as

$$\begin{aligned} R_{t-1} &= \frac{P_{t-1} + (1 + r^{rf})Q_{t-2} - (1 + r)P_{t-2}}{P_{t-2}} \\ &= (X_{t-1} + 1) \frac{1 + r^{rf}}{r + \alpha - g} \frac{Q_{t-1}}{P_{t-2}} + \left[(1 + r^{rf}) - (1 + r) \frac{1 + r^{rf}}{r + \alpha - g} (X_{t-2} + 1) \right] \frac{Q_{t-2}}{P_{t-2}} \\ &\approx (X_{t-1} + 1) \frac{1 + r^{rf}}{r + \alpha - g} (1 + g) \frac{Q_{t-2}}{P_{t-2}} + \left[(1 + r^{rf}) - (1 + r) \frac{1 + r^{rf}}{r + \alpha - g} (X_{t-2} + 1) \right] \frac{Q_{t-2}}{P_{t-2}} \\ &= \frac{1 + r^{rf}}{r + \alpha - g} \frac{Q_{t-2}}{P_{t-2}} ((X_{t-1} + 1)(1 + g) + [(r + \alpha - g) - (1 + r)(X_{t-2} + 1)]) \\ &= (1 + g)Y_{t-2}(X_{t-1} + \bar{\alpha} - RX_{t-2}), \end{aligned}$$

where $Y_{t-2} = \frac{1+r^{rf}}{r+\alpha-g} \frac{Q_{t-2}}{P_{t-2}} = \frac{P_{t-2}^*}{P_{t-2}}$.

A3. Proof of Proposition 1

The price is at its fundamental value when the deviation $X = 0$. Furthermore, using (12) and $X_{t-1} = X_{t-2} = X_{t-3} = 0$ at the fundamental steady state, it follows immediately that the corresponding fundamental steady state fractions are $n_1^* = n_2^* = \frac{1}{2}$.

For the dynamical systems (12) and (13) we observe that $n_{1,t} \equiv n_{1,t}(X_{t-1}, X_{t-2}, X_{t-3}, n_{1,t-1})$. Hence, the system can be written as a 4-D dynamical system. A straightforward computation shows that at the fundamental steady state $X^* = 0$ and $n_1^* = n_2^* = \frac{1}{2}$ the eigenvalues of the Jacobian matrix are $\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}$, δ and a double eigenvalue 0. Since $0 < \delta < 1$ it follows that the fundamental steady state is locally stable if $|\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})}| < 1$.

A4. Proof of Proposition 2

Using the dynamical system (13) steady states x must satisfy the equation

$$x = \frac{n_1 \phi_1 + n_2 \phi_2}{R + \bar{\alpha}} x.$$

Solutions either satisfy $x = 0$ (the fundamental steady state) or they must satisfy

$$\frac{n_1 \phi_1 + n_2 \phi_2}{R + \bar{\alpha}} = 1.$$

Since $n_1 = 1 - n_2$, this is equivalent to

$$n_1 = \frac{\phi_2 - \bar{\alpha} - R}{\phi_2 - \phi_1}.$$

At a steady state x , using (12) the corresponding steady state fraction n_1 is given by

$$n_1 = \frac{1}{1 + e^{\beta(U_2 - U_1)}} = \frac{1}{1 + e^{\beta[\bar{\alpha}(\phi_2 - \phi_1)x + (R-1)(\phi_1 - \phi_2)x^2]}.$$

Assume now $\bar{\alpha} = 0$, then the function $n_1 \equiv n_1(x)$ is symmetric²¹ around $x = 0$ (i.e. $n_1(-x) = n_1(x)$) and $n_1(0) = \frac{1}{2}$.

²¹ In the non-symmetric case $\bar{\alpha} \neq 0$ the two non-fundamental steady states are created in a saddle-node bifurcation slightly before or after the fundamental steady state becomes unstable.

First consider the case (i) $R = R_{crit} > 1$. We have $n_1(\pm\infty) = 1$ and n_1 has a global minimum $\frac{1}{2}$ at $x = 0$. Hence, there exist two additional non-fundamental steady states $\pm x$ when $n_1 = \frac{\phi_2 - R}{\phi_2 - \phi_1} > \frac{1}{2}$ or, equivalently, when $\frac{\phi_1 + \phi_2}{2R} > 1$, i.e. for $R < R_{crit}$ when the fundamental steady state is unstable. This corresponds to a supercritical pitchfork bifurcation at $R = R_{crit} = \frac{\phi_1 + \phi_2}{2}$ (see Fig. 3, left panel).

Second, consider the case (ii) $R = R_{crit} < 1$. We have $n_1(\pm\infty) = 0$ and n_1 has a global maximum $\frac{1}{2}$ at $x = 0$. Hence, there exist two additional non-fundamental steady states $\pm x$, when $\frac{\phi_2 - R}{\phi_2 - \phi_1} < \frac{1}{2}$, or equivalently when $\frac{\phi_1 + \phi_2}{2R} < 1$, i.e. for $R > R_{crit}$ when the fundamental steady state is locally stable. This corresponds to a subcritical pitchfork bifurcation at $R = R_{crit} = \frac{\phi_1 + \phi_2}{2}$ (see Fig. 3 right panel).

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