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DOI
10.1016/j.mathsocsci.2019.06.002
Publication date
2019
Document Version
Author accepted manuscript
Published in
Mathematical Social Sciences

Constrained Stochastic Cost Allocation*

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June 20, 2019

Abstract

This paper presents a model of a multi-divisional firm to share the joint yet uncertain and fixed cost of running a central operational unit. A firm aims at allocating this cost *ex ante*, subject to constraints imposed by the asymmetric and limited liabilities of the different divisions. We study solutions that are made up of a vector of *ex ante* payments which are allocated in absence of costs, and a remaining solution that is contingent on the cost. Under a mild continuity condition we find different classes of *egalitarian* solutions. The class of egalitarian proportional solutions is characterized by dependency on the disutility of the total cost instead of details of the distribution. In this class, there is a unique proportional solution which systematically minimizes the maximal transfer. A fundamentally different egalitarian solution is the *stochastic egalitarian constrained equal costs* solution. It is characterized using a local symmetry property which states that incremental costs should be distributed equally among those divisions with sufficient liability. This egalitarian solution has a smaller largest transfer than any egalitarian proportional solution. We conclude by showing how our results generalize when egalitarianism is replaced by a more general fairness property.

Keywords: stochastic cost allocation, egalitarian solution, rationing, constrained equal awards rule, proportional rule.
JEL Classification: C79, D31, D81, M41.

1 Introduction

Consider a multi-divisional firm with a central service unit - to which each of n divisions have equal access. Running this shared facility is costly and the divisions are charged for the full and uncertain cost. We will focus on cost allocation, and study for *ex ante* contracts that the firm may use to share the *ex post* realized cost. The firm puts upper bounds on the liability of a division, just as long as the total of maximal liabilities is enough to cover the costs arising in the worst-case scenario. The maximal liabilities of the divisions may be the result from exogenous risk capital allocations within the firm, and are limiting the divisions' capacity to bear risk (see, e.g., Myers and Read, 2001).

We assume that it is up to a benevolent manager to allocate the random cost, which is considered a social bad, among the divisions. We propagate an allocation of the total cost that

*The authors thank Itzhak Gilboa, Hervé Moulin, Hans Schumacher, Juan D. Moreno-Ternero, and two anonymous reviewers for their comments on an earlier version of this paper.

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reconciles the possible asymmetric way in which the divisions can be ultimately be held responsible for upon realization of the cost. In this paper we take egalitarianism as the fundamental principle that should govern the allocation, which means that ideally the realizations of the cost are shared equally by the divisions. However, this is not necessarily feasible in case liabilities are different and the realized cost is in this respect high enough. In such scenarios the divisions with high liabilities may *have to* contribute more than those with low liabilities. Despite the fact that the divisions cannot be treated equally in those cases, we will cherish the idea of cost allocations that are symmetric functions of the liability profiles.

We pose the question whether symmetric *ex ante* solutions exist according to which all agents face the same *ex ante* disutility level, even if this requires asymmetric solutions to realized instances of the constrained cost allocation problem. In order to make sense of intercomparison of the agents, we will assume that all have the same beliefs regarding the underlying probability distribution, and assume that all try to minimize an expected cost with possible subjective but homogeneous probabilities. We interpret this expected cost as disutility. Using the same disutility function for the collective of agents makes the concept of egalitarianism straightforward.

In case the liabilities are not high enough to be able to bear the share of $1/n$ of the realized cost, solutions may demand higher contributions from those agents with the higher liabilities. When such solutions are egalitarian, these agents will be compensated by those with low liabilities in case of a low realized cost. This compensation scheme via *transfers* is such that all agents are assured to be exposed, *ex ante*, to the same disutility.

One particularly interesting class of solutions is when allocations for these *constrained cost allocation problems* are obtained by solving a *rationing problem* by a *rationing rule*. A rationing problem describes the situation in which we allocate a given amount (often referred to as estate) among a group of agents when the available amount is not enough to satisfy all their claims. A rationing rule calculates shares for agents such that 1) no agent gets more than his/her claim, and 2) all get a non-negative share.[1] With the realized cost as estate and the profile of liabilities as claims, each constrained cost allocation problem is the natural counterpart of a rationing problem. In fact, the constrained cost allocation problem generalizes the rationing problem to allow for a stochastic cost. Then, each rationing rule can be taken to define a cost allocation solution. We show that each rationing rule that is continuous in the "claims" component can be used to define an egalitarian solution. In particular, included are many solutions that are symmetric as function of the profile of liabilities, meaning that agents are regarded equal and possible asymmetries between the proposed allocations should be motivated by the differences in liability. We discuss two special subclasses of solutions therein, one that is generated by proportional rules and the other is based on the constrained equal awards rationing rule.

Characteristic of the proportional solutions is that all satisfy an invariance property regarding the underlying distribution of the total cost; as long as the disutility of the total cost is the same, the cost allocation solution is the same. We prove that egalitarian solutions with this *Invariance to total Disutility Preserving Preferences* (IDPP) property are in fact proportional solutions. The egalitarian solution that we get using the constrained equal awards rationing rule as generator is denoted as the *stochastic egalitarian constrained equal costs* (SEEC) solution. The solution is characterized as the unique egalitarian solution with the *Local Symmetry* (LS) property, according to which marginal increases of the realized cost affect the agent's marginal contribution in the same way – for those agents whose liabilities are still not met.

Our aim is to allocate equal cost shares whenever this is feasible. In case the liabilities are high enough so that we can allocate the share of $1/n$ of the realized cost, we propose this as the final settlement of the cost allocation problem. We will refer to this property as *Symmetry un-*

[1] For rationing problems in practice and rationing rules see, e.g., O'Neill (1982), Aumann and Maschler (1985), or the overviews of Moulin (2002), and Thomson (2003, 2015).

der Sufficient Liability (SSL). A particular proportional solution is introduced as an egalitarian proportional solution with the SSL property. Additionally, we show that this proportional solution systematically minimizes the maximal transfers within the class of egalitarian proportional solutions. More precisely, the vector of transfers generated by this solution Lorenz-dominates all others used by the egalitarian proportional solutions. Nevertheless, we show that SEEC also satisfies SSL and also that it uses a (weakly) lower maximal transfer than does any egalitarian proportional solution. In particular this means that the vector of transfers according to SEEC is not dominated by the vector of transfers corresponding to any proportional solution. We discuss the egalitarian solution underpinned by the constrained equal losses rationing rule as an example of a solution that does not even satisfy SSL, despite the fact that the underlying rationing rule is credited egalitarian properties in the rationing framework.

Whereas egalitarian solutions play a central role in this paper, this does not exclude meaningful deviations from the egalitarian allocation. We finalize this paper by showing that the strict interpretation of egalitarianism may be relaxed, and that much of the reasoning throughout the paper can be used to allocate the total cost such that the disutility of agents are desired proportions of the total disutility. Then this more general set-up bridges the gap between the theory of allocating a random cost and a social norm. It is the social norm that governs the fixed proportions of the total disutility *ex ante*, subject to *ex post* feasibility defined by the individual liabilities. And those social norms exogenous to the model may require other proportions in which the disutility of the total cost is shared *ex ante*, leading to alternative concepts of *fairness*. Deviations from pure egalitarianism may be motivated by non-symmetric initially received contributions to bare the total cost. Our fairness constraint is akin to the financial fairness condition in risk-sharing (Pazdera et al., 2017). This more general idea of extending a pure egalitarian setup to fairness only requires a slight adaptation of our original model formulation. In addition, the *fair* solutions inherit the very same structure of the pure egalitarianism that is elemental to the standard cost allocation solutions like SEEC; basically, the solutions only make a correction of the transfers at zero cost. The concept of fairness was originally introduced for risk exchanges by Bühlmann and Jewell (1979). Bühlmann and Jewell (1979) and also Pazdera et al. (2017) consider the case of risk-sharing, where the agents have initial stochastic endowments to be shared, and the focus is on the induced incentives for agents to participate or not. In this paper we concentrate on the fundamentally different problem of allocating the stochastic cost of a public good.

Habis and Herings (2013) and Ertemel and Kumar (2018) also study stochastic rationing problems, but where both the estate and the claims are considered stochastic. Like us, the authors concentrate on *ex ante* solutions, but with a focus on notions of stability and enhancing cooperation. Kıbrıs and Kıbrıs (2013), Karagözoğlu (2014) and Boonen (2017) study investment problems with an endogenous and stochastic estate, which are seen as bankruptcy problems in case of default and that are solved as such. Instead of investing with risk of a defaulting counterparty, we concentrate in this paper on a public good of which the cost needs to be allocated under liability constraints. Hougaard and Moulin (2018) study a problem of sharing the cost of a stochastic network, where the focus is on determining cost shares *ex ante* such that these equal the expectation over the random realization of the network of the shares *ex post*. Xue (2018) and Long et al. (2019) study cost-sharing without liability constraints, but with uncertain claims on a divisible commodity. Their objectives focus on maximizing social welfare functions based on notions of waste and deficit.

The rest of this paper is organized as follows. Section 2 specifies the model, and Section 3 provides our construction of egalitarian solutions and transfers. The ordering of transfers is studied in Section 4. Sections 5 and 6 are devoted to defining and characterizing of our solutions based on the proportional and constrained equal awards rules. Section 7 compares these solutions based on a ranking of the transfers. Section 8 provides a generalization of the concept of egalitarianism to the case where we first exogenously allocate the disutility of the total cost in the desired ratio to the agents. Section 9 provides a remark in which we explain how our solution concept can be generalized to situations where egalitarian solutions do not exist, i.e., where some of the agents have too low liabilities to take a fair share of the risk. Finally, Section 10 concludes. All proofs are delegated to the appendix.

2 The constrained cost allocation model and solutions

2.1 Mathematical notation

In this paper we will focus on a fixed and finite set of *agents* $N = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Two special vectors will be used frequently: $\mathbf{e} \in \mathbb{R}^N$ is the vector with $\mathbf{e}_i = 1$ for all $i \in N$, and $\mathbf{0} \in \mathbb{R}^N$ is the zero vector with $\mathbf{0}_i = 0$ for all $i \in N$. By $\mathbb{R}^N_+$ we mean the set of all vectors in $\mathbb{R}^N$ with non-negative coordinates, $\mathbb{R}^N_{++}$ is the subset of the vectors with positive coordinates. For two vectors $x, y \in \mathbb{R}^N$ we will understand $x \leq y$ element-wise, which means $x_i \leq y_i$ for all $i \in N$. For $x \in \mathbb{R}^N$ and $S \subseteq N$ we define $x(S) := \sum_{i \in S} x_i$ and $x_S := \{x_i\}_{i \in S} \in \mathbb{R}^S$.

2.2 The constrained cost allocation problem

Agents aim to share the cost of a risky project. The cost of the project, denoted by C, is a bounded, non-negative random variable on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the state space, $\mathcal{F}$ is the σ-algebra, and $\mathbb{P}$ is a probability measure. We assume that the image of C equals $[0, \overline{c}]$, where $\overline{c} > 0$ is a fixed constant and thus the highest possible realization of C. We denoted by $\mathcal{L}^\infty$ the class of all bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

We consider an ordered pair $(C, L) \in \mathcal{L}^\infty \times \mathbb{R}^N_{++}$, where C is the random cost that has to be allocated to the agents in N under liability constraints, and L is a vector of liabilities. We assume that the allocation for an agent i is at most equal to its liability $L_i > 0$. This liability of an agent is the maximal obligation to contribute. Moreover, (C, L) satisfies:

Admissibility: $L(N) \geq \overline{c}$.

Admissibility implies that whatever the realization of the project will turn out to be, the collective of agents can afford it. Throughout this paper, we assume without loss of generality that L is an ordered vector, i.e.,

$$L_1 \leq \cdots \leq L_n.$$

Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. Below we interpret $\mathbb{Q}$ as a *subjective* probability measure by which agents value *ex ante* the *disutility* of a random variable. Random variables in $\mathcal{L}^\infty$ represent random costs that are allocated to an agent, from which the agent incurs disutility. Throughout this paper, we will only consider random variables $X \in \mathcal{L}^\infty$ such that the ordered pair $(C, X) \in (\mathcal{L}^\infty)^2$ is *comonotonic*.[2] Each agent wants to minimize his disutility $V(X)$, where V is such that

$$V(X) = \mathbb{E}_{\mathbb{Q}}[X], \text{ for all } X \in \mathcal{L}^\infty \text{ such that } (C, X) \text{ is comonotonic.} \tag{1}$$

So, V summarizes the *ex ante* preferences of an agent over random variables. We assume that $\mathbb{Q}(C \leq c)$ is strictly increasing in c on $[0, \overline{c}]$, so that C has full support on $[0, \overline{c}]$. The set of all

[2]The ordered pair $(Y, Z) \in (\mathcal{L}^\infty)^2$ is comonotonic if there exists a non-decreasing function h such that $(Y, Z) \stackrel{d}{=} (Y, h(Y))$.

such preferences V is denoted $\mathcal{V}$. For instance, the function V can be a representation of the expectation or of *dual utility* (Yaari, 1987).[3] If V is represented by dual utility, then

$$V(X) = \int_0^\infty g^V(1 - F_X(x))\, dx + \int_{-\infty}^0 [1 - g^V(1 - F_X(x))]\, dx, \text{ for all } X \in \mathcal{L}^\infty,$$

for a left-continuous and strictly increasing function $g^V : [0,1] \to [0,1]$ with $g^V(0) = 0$ and $g^V(1) = 1$, where F_X is the cumulative density function (CDF) of X. Then, $\mathbb{Q}(X \leq x) = 1 - g^V(1 - F_X(x))$ for all $x \in \mathbb{R}$. Also, V can represent the preferences of De Giorgi and Post (2008), that are given by

$$V(X) = (1-\alpha)\mathbb{E}_{\mathbb{P}}[X] + \alpha\hat{V}(X), \tag{2}$$

with $\alpha \in [0,1]$ and $\hat{V}$ a representation of dual utility. Then, the subjective probability measure of V is given by $\mathbb{Q}(X \leq x) = (1-\alpha)\mathbb{P}(X \leq x) + \alpha\hat{\mathbb{Q}}(X \leq x)$ for all $x \in \mathbb{R}$, where $\hat{\mathbb{Q}}$ is the subjective probability measure of $\hat{V}$. If $\alpha = 0$, all agents are risk-neutral and aim to minimize the expected value of X.

We emphasize that we do not assume heterogeneous beliefs or heterogeneous preferences for the agents; the same $V \in \mathcal{V}$ is used by the collective of agents. In case of heterogeneous preferences we would need to compare interpersonal risk aversion, which can be considered problematic in several aspects. First of all, there is a complicating factor that there is no way to assure that agents will reveal their true preferences. For instance, Anthropelos and Karatzas (2017) show that agents would have an incentive to misrepresent their true risk aversion if it is self-reported. But even if we would know the true preferences, the problem of ambiguity involved in making such intercomparison is intrinsically hard to overcome, see Young (1990). This motivates the use of a "representative agent" model where the agents are assumed to have the same preferences.

We will call the ordered tuple $(C, L, V) \in \mathcal{L}^\infty \times \mathbb{R}^N_{++} \times \mathcal{V}$ a *constrained cost allocation problem* if (C, L) is admissible and (C, L, V) satisfies the following condition:

V-sufficiency: $L \geq \frac{1}{n}V(C)\,\mathbf{e}$.

This means that the liabilities are such that in principle each agent can take a fair share of the random cost, as measured by V. The set of all such constrained cost allocation problems is denoted by $\mathcal{P}$.

2.3 Solutions

For each *realized cost allocation problem* (c, L) that is a combination of a realization $c \in [0, \overline{c}]$ of C and liability vector $L \in \mathbb{R}^N_{++}$, the set of feasible cost allocations is given by:

$$\mathcal{A}(c, L) := \left\{x \in \mathbb{R}^N : x(N) = c, x \leq L\right\}.$$

This set $\mathcal{A}(c, L)$ is a finite dimensional bounded space, and as the intersection of a finite number of closed half-spaces it is a (non-empty) convex polytope. In the sequel we will see that there is no lack of feasible cost allocations, which justifies our focus on *cost allocation solutions* to be defined below.

For each $(C, L, V) \in \mathcal{P}$, a cost allocation solution ψ maps every $c \in [0, \overline{c}]$ into $\mathcal{A}(c, L)$. So ψ defines a cost allocation for each realization of the total cost, and $\psi(C, L, V)(c) \in \mathcal{A}(c, L)$ for

[3] Disutility that is represented as dual utility is alternatively defined as a distortion risk measure (Wang, 1996; Wang et al, 1997).

$c \in [0, \overline{c}]$ stands for the cost allocation solution for the realization c of random variable C. Note that this implies for all $(C, L, V) \in \mathcal{P}$ that

$$\psi\left(C,L,V\right)(c) \leq L \text{ and } \sum_{i \in N} \psi_i\left(C,L,V\right)(c) = c, \text{ for all } c \in [0, \overline{c}]. \tag{3}$$

Denote by $\mathcal{Z}$ the set of all cost allocation solutions (or, in short, solutions). At this point it is essential to realize that the only restriction on $\psi(C, L, V)(0)$ is that it is contained in $\mathcal{A}(0, L)$, and it is not necessarily equal to the zero vector. If non-zero, the cost allocation when $C = 0$ is negative for one or more agents and positive for one or more of the others. Negative cost allocations are possible in order to allow for compensations between agents. The standard literature on distributive justice is more restrictive in this sense, and allows for non-negative shares only.

Below we will discuss several classes of solutions, which all satisfy the following weak monotonicity property – which we think is compelling for any solution:

Comonotonicity: $c \mapsto \psi_i(C, L, V)(c)$ is non-decreasing on $[0, \overline{c}]$ for all $i \in N$.

Comonotonicity expresses the idea that if the total cost C increases, no agent should benefit by paying less. So, it can be considered a weak property assuring that no agent benefits if the total cost increases. Note that this assumption implies that $X = \psi_i(C, L, V)$ is comonotonic with C, and is therefore considered admissible as a random variable in (1). The property comonotonicity is closely related to the property resource monotonicity in the literature on (deterministic) rationing rules (Chun, 1999).

Consider the set $\mathcal{Z}_0 \subset \mathcal{Z}$ of all comonotonic solutions ψ that are *zero-normalized* in the sense that $\psi(C, L, V)(0) = 0$ for all $(C, L, V) \in \mathcal{P}$. It is trivial but instructive to see that any solution ψ can be written as the sum of the solution at 0 cost and a zero-normalized solution $\psi_0 \in \mathcal{Z}_0$, as follows

$$\begin{aligned} \psi(C,L,V)(c) &= \psi(C,L,V)(0) + (\psi(C,L,V)(c) - \psi(C,L,V)(0)) \\ &= \psi(C,L,V)(0) + \psi_0(C,L,V)(c), \end{aligned}$$

for all $c \in [0, \overline{c}]$.

This paper focuses foremost on the class of solutions in $\mathcal{Z}$ that we will refer to as *standard*, i.e., those $\psi \in \mathcal{Z}$ for which the corresponding $\psi_0 \in \mathcal{Z}_0$ can be written as

$$\psi_0(C,L,V)(c) = f(c, L, t, V(C)), \text{ for all } c \in [0, \overline{c}], \tag{4}$$

where $t = \psi(C, L, V)(0) \in \mathcal{A}(0, L)$ and $f(c, L, t, V(C)) \leq L - t$ for all problems $(C, L, V) \in \mathcal{P}$. Let $\mathcal{Z}^{\mathrm{s}} \subset \mathcal{Z}$ be the set of all standard solutions. Then solutions in $\mathcal{Z}^{\mathrm{s}}$ capture the idea that asymmetries derived from the liabilities can be neutralized with respect to the preferences using a profile of *ex ante* transfers t; after realization of the total cost the *ex post* cost allocation may still be influenced by V but only through t and $V(C)$. Note that the upper bound on f is meant to keep feasibility.

Example 1 The class of *proportional solutions* $\mathcal{Z}^{\mathrm{p}} \subset \mathcal{Z}^{\mathrm{s}}$ consists of all solutions ψ^{α} given by

$$\psi^{\alpha}(C,L,V)(c) = t + \alpha(L, t, V(C)) \cdot c, \text{ for all } c \in [0, \overline{c}], \tag{5}$$

where $t \in \mathcal{A}(0, L)$ and $\alpha(L, t, V(C)) \in \mathbb{R}^N_+$ is a vector such that $\sum_{i \in N} \alpha_i(L, t, V(C)) = 1$ and $\alpha(L, t, V(C)) \leq (L - t)/\overline{c}$. So, a proportional solution conveys the idea that *within* each problem

$(C, L, V) \in \mathcal{P}$ the cost is allocated to the agents in a fixed ratio. For the sake of exposition we will slightly abuse our notation and simplify $\alpha(L, t, V(C))$ to α, when no confusion arises. $\triangledown$

Mathematically speaking, each realized cost sharing problem (c, L) is a rationing problem in the sense of O'Neill (1982), Moulin (2002) and Thomson (2003, 2015) – only the interpretation is different. Formally, a rationing problem is an ordered pair $(c, L) \in \mathcal{R} := \mathbb{R}_+ \times \mathbb{R}^N_{++}$ such that $L(N) \geq c$, so that L_i is interpreted as the justified claim of agent i on the amount of (a now desirable) good c. A *rationing rule* is a mapping $r : \mathcal{R} \to \mathbb{R}^N_+$ such that for each $(c, L) \in \mathcal{R}$ we have

$$\mathbf{0} \leq r\,(c, L) \leq L \text{ and } \sum_{i \in N} r_i\,(c, L) = c.$$

While the cost sharing and rationing model have diametrically different interpretations, the mathematical formulations of a rationing model and a realized cost allocation problem are similar. The only differences between both formulations are the non-negativity constraints in rationing problems. The set of feasible cost allocations $\mathcal{A}(c, L)$ is a superset of the set of allocations in a rationing problem. Thus, we may take any rationing rule as a tool to select a cost allocation in our model. In this way any rationing rule *induces* a cost allocation solution. Moreover, each zero-normalized solution ψ_0 can be seen as a rationing rule by which for each realization of cost c the ordered pair (c, L) is assigned a cost allocation $\psi_0(C, L, V)(c) = \psi(C, L, V)(c) - \psi(C, L, V)(0) \leq L - t$, where $t = \psi(C, L, V)(0)$. It then can be argued that this rationing rule should take $L - t$ as the vector of liabilities, instead of L. The liabilities in the model serve the goal of specifying the extent to which agents may be exposed to the total cost, and after having made the *ex ante* payment of t the room left for cost allocations is set at the vector $L - t$.

Let $\mathcal{Z}^{\mathrm{R}} \subset \mathcal{Z}^{\mathrm{S}}$ be the set of solutions ψ that agree with this idea so that it can be written as

$$\psi(C, L, V)(c) = t + \varphi^{V(C)}(c, L - t), \text{ for all } c \in [0, \overline{c}], \tag{6}$$

where $t = \psi(C, L, V)(0) \in \mathcal{A}(0, L)$ and $\varphi^{V(C)}$ is rationing rule – and that rationing rule may depend on $V(C)$. Note that while $\varphi^{V(C)}$ is a rationing rule, the interpretation of $\varphi^{V(C)}$ is that it constitutes a cost allocation solution.

Example 2 The class of *proportional rationing solutions* $\mathcal{Z}^{\mathrm{PR}} \subset \mathcal{Z}^{\mathrm{P}} \cap \mathcal{Z}^{\mathrm{R}}$ consists of all solutions $\psi^\alpha \in \mathcal{Z}^{\mathrm{P}}$ based on φ^{P} related to the proportional rationing rule $r^p(c, L) := L/L(N) \cdot c$, so that[4]

$$\psi^\alpha(C, L, V)(c) = t + \alpha\, c = t + \varphi^{\mathrm{P}}(c, L - t) = t + \frac{L - t}{L(N)}\, c, \text{ for all } c \in [0, \overline{c}],$$

where $t \in \mathcal{A}(0, L)$. Note that any such solution is feasible due to $\alpha = (L - t)/L(N) \leq (L - t)/\bar{c}$. There is only one proportional solution when $L(N) = \bar{c}$, which is the proportional rationing solution. Since in that situation, for fixed $t \in \mathcal{A}(0, L)$, the only proportion α in (5) that we can chose is $\alpha = (L - t)/L(N)$. $\triangledown$

Example 3 Define cost allocations by φ^{EC} such that $\varphi^{\mathrm{EC}}(c, L) = \min\{\lambda(c), L_i\}$ for $i \in N$, where $\lambda(c)$ solves

$$\sum_{i \in N} \min\{\lambda(c), L_i\} = c.$$

Then φ^{EC} is based on the *constrained equal awards* rationing rule (see Aumann and Maschler, 1985). The class of *constrained equal costs solutions* $\mathcal{Z}^{\mathrm{CEC}} \subset \mathcal{Z}^{\mathrm{R}}$ consists of solutions ψ such that

$$\psi(C, L, V)(c) = t + \varphi^{\mathrm{EC}}(c, L - t), \text{ for all } c \in [0, \overline{c}],$$

where $t \in \mathcal{A}(0, L)$. $\triangledown$

[4]Here, dividing a vector by a positive scalar is understood element-wise.

Example 4 Define cost allocations by $\varphi^{\text{DEC}}(c, L) := L - \varphi^{\text{EC}}(L(N) - c, L)$. Then φ^{DEC} is based on the *constrained equal losses* rationing rule (see Aumann and Maschler, 1985) which is the *dual* of the constrained equal awards rationing rule. Then a *dual constrained equal costs* solution $\psi \in \mathcal{Z}^{\text{R}}$ can be written as

$$\psi(C, L, V)(c) = t + \varphi^{\text{DEC}}(c, L - t), \text{ for all } c \in [0, \bar{c}],$$

where $t \in \mathcal{A}(0, L)$. $\bigtriangledown$

3 Egalitarian solutions and transfers

The central issue addressed in this paper is the question how to deal with situations where individual liabilities are too skewed to be able to see to a pure egalitarian solution for all realized costs *ex post*, but that at least *ex ante* the asymmetric random cost allocation is equally preferred by all the agents. The leading property is the following. We will call a solution *egalitarian* if the agents with homogeneous preferences in $\mathcal{V}$ all are subjected to the same disutility, for all problems in $\mathcal{P}$. Formally:

Egalitarianism: ψ is called *egalitarian* if $V(\psi_i(C, L, V)) = V(\psi_j(C, L, V))$ for all $i, j \in N$ and all $(C, L, V) \in \mathcal{P}$.

This means that the advocated notion of egalitarianism agrees with the Dutta and Ray (1989) interpretation. We will see how the vectors of transfers $t = \psi(C, L, V)(0)$ play a key role in fine-tuning basic solutions to egalitarian solutions. And, particularly, we will need the possibility of negative cost allocations to accomplish this. Note, however, that comonotonicity implies that if $\psi(C, L, V)(0)$ is the zero vector, then allocations are all non-negative (and so in line with the standard literature). In Section 8, the concept of egalitarianism is generalized to the case where we asymmetrically allocate proportions of the disutility of the total cost to the agents. From (1) we get that if ψ satisfies comonotonicity, then for $i \in N$

$$V(\psi_i(C, L, V)) = \mathbb{E}_{\mathbb{Q}}[\psi_i(C, L, V)].$$

Therefore, any egalitarian and comonotonic solution satisfies

$$V(\psi_i(C, L, V)) = \tfrac{1}{n} V(C), i \in N. \tag{7}$$

This can be taken as additional argument to restrict the attention to V-sufficient constrained cost allocation problems. Note that admissibility and V-sufficiency are obviously necessary for an egalitarian and comonotonic solution to exist.

The subclasses of $\mathcal{Z}^{\text{R}}$ introduced in Examples 2-4 are parameterized by the transfers t when the realized total cost is zero. The following theorem shows that a mild condition on the rationing method underpinning a standard solution assures existence of a vector t so that the corresponding solution is egalitarian.

Theorem 1 *Consider $\psi \in \mathcal{Z}^{\text{R}}$ and corresponding $\varphi^{V(C)}$ such that $\psi(C, L, V)(c) := t + \varphi^{V(C)}(c, L - t)$ for some $t \in \mathcal{A}(0, L)$ and all $(C, L, V) \in \mathcal{P}$. If $x \mapsto \varphi^{V(C)}(c, x)$ is continuous for all $c \in [0, \bar{c}]$, then $t = t(C, L, V) \in \mathcal{A}(0, L)$ may be chosen such that ψ is egalitarian.*

Theorem 1 provides a condition on $\varphi^{V(C)}$ under which there exist vectors of transfers such that the disutility of agents is the same. However, there may be multiple vectors of transfers. For each of the members of the previously discussed classes of proportional solutions there exist unique vectors of transfers that make the solutions egalitarian, as we will show now.

Theorem 2 *The solution $\psi^{\alpha} \in \mathcal{Z}^{\mathrm{P}}$ given by $\psi^{\alpha}(C,L,V)(c) = t + \alpha c$ for all $c \in [0,\bar{c}]$ and $t \in \mathcal{A}(0,L)$ is egalitarian if and only if*

$$t = V(C)(\tfrac{1}{n}\mathbf{e} - \alpha) \text{ and } \alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

Corollary 1 *The unique egalitarian solution in $\mathcal{Z}^{\mathrm{PR}}$ is given by the egalitarian proportional rule $\psi^{\mathrm{EP}}(C,L,V) := t + \varphi^{\mathrm{P}}(C, L-t)$ with*

$$t = \frac{(\frac{1}{n}L(N)\,\mathbf{e} - L)V(C)}{L(N) - V(C)}.$$

Where for the proportional solutions the transfers can be explicitly found, this is not true for the egalitarian solutions based on either φ^{EC} or φ^{DEC}. Still, we have uniqueness of the transfers for the egalitarian solution based on φ^{EC}.

Theorem 3 *There is a unique egalitarian solution in $\mathcal{Z}^{\mathrm{CEC}}$.*

We define the *stochastic egalitarian constrained equal costs* (SEEC) solution by $\psi^{\mathrm{SEEC}}(C,L,V) := t + \varphi^{\mathrm{EC}}(C, L-t)$, where $t \in \mathcal{A}(0,L)$ is chosen such that ψ^{SEEC} is egalitarian.

Example 5 In this example, we describe two solutions ψ that we will characterize in this paper. Let $V(C) = \mathbb{E}_{\mathbb{P}}[C]$, $N = \{1,2,3\}$, $C \sim Un(0,10)$, and $L = (2,3,8)$. So, $V(C) = 5$. It is easily verified that the problem (C,L,V) satisfies admissibility and V-sufficiency.

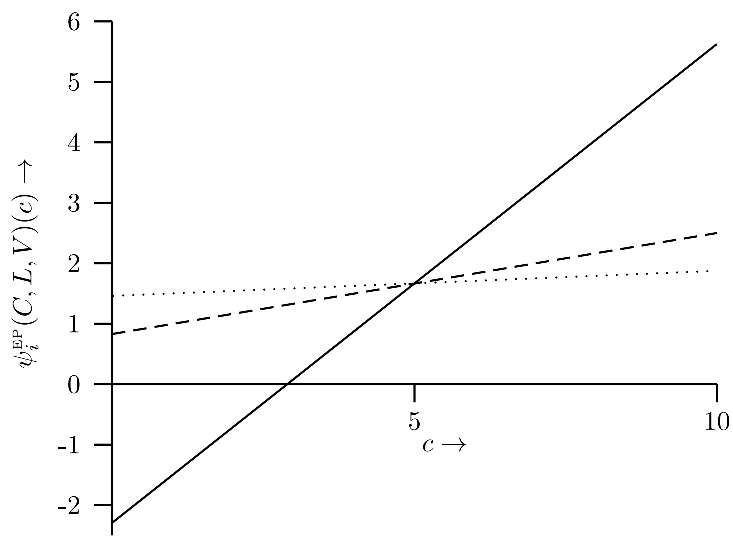

Figure 1: Graphical illustration of the egalitarian proportional solution $\psi^{\mathrm{EP}}(C,L,V)(c) = t + \varphi^{\mathrm{P}}(c, L-t)$ corresponding to Example 5. The dotted line represents $\psi_1^{\mathrm{EP}}(C,L,V)(\cdot)$, the dashed line $\psi_2^{\mathrm{EP}}(C,L,V)(\cdot)$ and the solid line $\psi_3^{\mathrm{EP}}(C,L,V)(\cdot)$.

From Corollary 1, we compute a unique vector of transfers $t \approx (1.46, 0.83, -2.29)$ for the egalitarian solution $\psi^{\mathrm{EP}}(C,L,V)$. Moreover, we derive a unique vector of transfers $t \approx (0.51, -0.13, -0.38)$

for the egalitarian solution ψ^{SEEC}, defined in Theorem 3. The solutions $\psi^{\text{EP}}(C, L, V)$ and $\psi^{\text{SEEC}}(C, L, V)$ are displayed in Figures 1 and 2. The solution $\psi^{\text{EP}}(C, L, V)$ is linear, and the solution $\psi^{\text{SEEC}}(C, L, V)$ is piecewise linear such that marginal contributions due to cost increase are equally shared among the agents whose liabilities are not fully attained. We see that the transfers of $\psi^{\text{SEEC}}(C, L, V)$ are closer to $\mathbf{0}$ than the transfers of $\psi^{\text{EP}}(C, L, V)$. This is the topic of that we study further in Section 7.

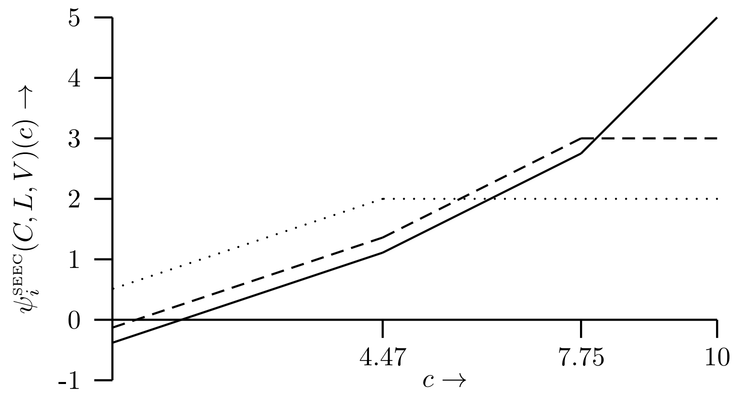

Figure 2: Graphical illustration of the egalitarian solution $\psi^{\text{SEEC}}(C, L, V)(c) = t + \varphi^{\text{EC}}(c, L - t)$ corresponding to Example 5. The dotted line represents $\psi_1^{\text{SEEC}}(C, L, V)(\cdot)$, the dashed line $\psi_2^{\text{SEEC}}(C, L, V)(\cdot)$ and the solid line $\psi_3^{\text{SEEC}}(C, L, V)(\cdot)$. Here, the "cut-off" points where the consecutive agents become tight are given by approximately $(4.47, 7.75, 10)$.

$\triangledown$

4 Ranking of transfers

Basically, Theorem 1 says that problems arising in *ex post* cost allocation due to heterogeneity in liabilities can be repaired using fixed *ex ante* payments, so that the resulting solution is still egalitarian. The idea is that agents who have low liabilities could reimburse the others when they can. This may be accomplished by letting them contributing more at low realized cost levels, and in particular by assigning to these agents higher transfers when the cost is zero. Below it is shown that this reversion of the ordering of transfers relative to liabilities holds for standard solutions in $\mathcal{Z}^{\text{R}}$ if the standard solution uses an *order preserving* component $\varphi^{V(C)}$ in (6):

Order preserving (OP): $\varphi_i^{V(C)}(\cdot, L) \leq \varphi_j^{V(C)}(\cdot, L)$ whenever $L_i \leq L_j$.[5]

This property is tantamount to the property discussed by Aumann and Maschler (1985) for rationing methods – or, here in the framework with costs rather than awards, for mappings

[5]Note that if $\varphi^{V(C)}$ is order preserving, then it also satisfies the weaker and well-known property of *equal treatment*: $L_i = L_j \implies \varphi_i^{V(C)}(\cdot, L) = \varphi_j^{V(C)}(\cdot, L)$.

$\varphi^{V(C)}$ that are based on a rationing rule. The following result shows a reversion of the ordering of transfers relative to liabilities if $\varphi^{V(C)}$ satisfies OP.

Theorem 4 *Consider $\psi \in \mathcal{Z}^{\mathrm{R}}$ as in (6) with order preserving $\varphi^{V(C)}$, and assume that $x \mapsto \varphi^{V(C)}(c,x)$ is continuous for all c. Then for all $(C,L,V) \in \mathcal{P}$ we may choose a vector of transfers $t = \psi(C,L,V)(0)$ such that ψ is egalitarian and $L_i \leq L_j \implies t_i \geq t_j$ for all $i,j \in N$.*

Note that the proportional solutions with $\varphi^{V(C)}(c,L) = \alpha c$ preserve ordering if $\alpha_i \leq \alpha_j \Leftrightarrow L_i \leq L_j$. Also, both φ^{EC} and φ^{DEC} are order preserving.

5 Characterization of proportional solutions $\mathcal{Z}^{\mathrm{P}}$

In this section, we characterize proportional solutions. The characterization is based on the following property:

Invariance to total Disutility Preserving Preferences (IDPP): for all $(C,L,V),(C,L,V^*) \in \mathcal{P}$, if $V^*(C) = V(C)$ then $\psi(C,L,V^*) = \psi(C,L,V)$.

The property IDPP states that for all $V \in \mathcal{V}$ with the same value $V(C)$, the corresponding solution is the same. This property implies that solutions depend on V only via $V(C)$. Our next result shows that this property characterizes the class of proportional solutions.

Theorem 5 *An egalitarian $\psi \in \mathcal{Z}^{\mathrm{S}}$ satisfies IDPP if and only if $\psi = \psi^{\alpha} \in \mathcal{Z}^{\mathrm{P}}$ with*

$$\psi(C,L,V)(0) = V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha), \text{ and } \alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

6 Characterization of constrained egalitarian solutions $\mathcal{Z}^{\mathrm{CEC}}$

We impose the following condition on solutions ψ:

Local Symmetry (LS): for all $(C,L,V) \in \mathcal{P}$, $i,j \in N$ and $c \in [0,\bar{c}]$, we have

$$\psi_i\,(C,L,V)\,(c) < L_i, \psi_j\,(C,L,V)\,(c) < L_j \implies \tfrac{\partial}{\partial c}\psi_i\,(C,L,V)\,(c) = \tfrac{\partial}{\partial c}\psi_j\,(C,L,V)\,(c).$$

A solution is locally symmetric if the marginal increases of the cost are accounted for by agents in a similar fashion – as long as the liability constraints are not met. The property LS states that, *ex post*, an additional unit in the cost is shared equally among the agents who can afford it. So, LS is a property that ensures *ex post* egalitarianism among marginal cost changes.[6]

Clearly, a solution $\psi \in \mathcal{Z}$ satisfying LS is comonotonic. In fact, we will show next that such ψ belongs to the class of constrained equal costs solutions.

Theorem 6 *Solution $\psi \in \mathcal{Z}$ has the property LS if and only if $\psi \in \mathcal{Z}^{\mathrm{CEC}}$, i.e., we have*

$$\psi\,(C,L,V)\,(c) = t + \varphi^{\mathrm{EC}}\,(c, L - t)\,, \text{ for all } c \in [0,\bar{c}],$$

for all $(C,L,V) \in \mathcal{P}$ and $t = \psi(C,L,V)(0) \in \mathcal{A}(0,L)$.

[6] Instead of considering marginal costs, it is easy to see that all results will still remain valid with a notion of local symmetry that takes care of non-infinitesimal increases of the cost. Rather than working with difference quotients, we prefer to stick to the present formulation for the sake of the exposition.

The fact that we may decompose any solution with the property LS in this fashion pins down a solution if we select transfers. The following is a direct consequence of Theorem 3 and Theorem 6.

Theorem 7 *There is a unique egalitarian $\psi \in \mathcal{Z}$ which satisfies LS, and that is $\psi = \psi^{\text{SEEC}}$.*

Remark: In Koster and Boonen (2014), we provide an alternative characterization of ψ^{SEEC} based on properties stemming from the rationing literature. It relies on properties mimicking homonymous properties for rationing rules as there are the ideas of consistency, a secured lower bound, and the composition up property. These properties for rationing rules are translated to the stochastic context, where the transfers are already paid up-front. *Consistency* is a general idea that tells us how a solution behaves over problems with different agent sets (see, e.g., Moulin, 2000; Thomson, 2003). The *secured lower bound* provides a minimal allocation, and is introduced by Moreno-Ternero and Villar (2004) and popularized by, e.g., Dominguez and Thomson (2006). The property *composition up* states that, given an increase of cost, new cost shares can be allocated from the information of the earlier cost shares alone. This property (and its dual) were introduced by Moulin (1987) and Young (1988). The characterization in Koster and Boonen (2014) relies then on the result of Yeh (2008) for rationing problems.

7 Minimizing the use of transfers and symmetry under sufficient liability

This paper focuses on solutions of allocating a stochastic cost when asymmetries between liabilities cause problems. Our intention is to share the stochastic cost equally if this is feasible. Under sufficient liability, we propose the symmetric solution:

Symmetry under Sufficient Liability (SSL): $L \geq \frac{1}{n}\bar{c}\,\mathbf{e} \implies \psi(C,L,V)(c) = \frac{1}{n}\,c\,\mathbf{e}$ for all $c \in [0,\bar{c}]$.

If every element of the liability vector L is large, then the property SSL requires that the allocated stochastic cost is always the same for all agents. In other words, if $\frac{1}{n}\,c\,\mathbf{e} \in \mathcal{A}(c,L)$ holds for all $c \in [0,\bar{c}]$, then $\psi(C,L,V)(c) = \frac{1}{n}\,c\,\mathbf{e}$. This condition is satisfied by ψ^{SEEC}. However, SSL does not need to hold for proportional solutions, as one may show that SSL is not satisfied by the solutions in $\mathcal{Z}^{\text{PR}}$.

Example 6 Consider the proportional solution ψ^{P} which is the egalitarian solution in $\mathcal{Z}^{\text{P}}$ such that

$$\alpha = \frac{\varphi^{\text{EC}}(\bar{c},L) - \frac{1}{n}V(C)\mathbf{e}}{\bar{c} - V(C)}. \tag{8}$$

Since $\varphi^{\text{EC}}(\bar{c},L) \leq L$, it holds that α satisfies the condition in Theorem 5, so that egalitarianism implies

$$t = \psi^{\text{P}}(C,L,V)(0) = V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha).$$

Also we have $\alpha \geq \mathbf{0}$. To see this, suppose that $\varphi_i^{\text{EC}}(\bar{c},L) = \min\{L_i, \lambda(\bar{c})\} < \frac{1}{n}V(C)$ for some i. Then by V-sufficiency we must have $\lambda(\bar{c}) < \frac{1}{n}V(C)$, so that

$$\sum_{j\in N} \varphi_j^{\text{EC}}(\bar{c},L) \leq n\lambda(\bar{c}) < V(C) \leq \bar{c},$$

which provides the desired contradiction. Obviously ψ^{P} satisfies SSL, since $L \geq \frac{1}{n}\,\bar{c}\,\mathbf{e}$ implies $\varphi^{\mathrm{EC}}(\bar{c}, L) = \frac{1}{n}\,\bar{c}\,\mathbf{e}$, so that $\alpha = \frac{1}{n}\,\mathbf{e}$ and $t = \mathbf{0}$. The proportional rationing solution does not satisfy SSL. $\triangledown$

Consider two transfer vectors $x, y \in \mathbb{R}^N$ that are ordered in descending order. Then we say that x Lorenz-dominates y if $\sum_{k=1}^{i} x_k \leq \sum_{k=1}^{i} y_k$ for all $i = 1, \ldots, n$, and we will write $x \succeq_L y$ in that case. For any $t \in \mathcal{A}(0, L)$, it holds that $\mathbf{0} \succeq_L t$. So, this criterion aims to select transfers that are "closest" to the zero vector.

Theorem 8 *Consider an egalitarian $\psi^{\alpha} \in \mathcal{Z}^{\mathrm{P}}$ and $(C, L, V) \in \mathcal{P}$. Then if $\alpha_i \leq \alpha_j \Leftrightarrow L_i \leq L_j$, we have $\psi^{\mathrm{P}}(C, L, V)(0) \succeq_L \psi^{\alpha}(C, L, V)(0)$ for all $(C, L, V) \in \mathcal{P}$.*

This means that among the set of transfers generated by the egalitarian proportional solutions, the one corresponding to ψ^{P} has the minimal largest transfer, and given the set of all minimizing this largest transfers it minimizes the second largest transfer and so on. However, compared to the proportional solution ψ^{P}, the standard egalitarian solution ψ^{SEEC} calculates smaller largest transfers.

Theorem 9 *For all $(C, L, V) \in \mathcal{P}$, it holds $\psi_1^{\mathrm{SEEC}}(C, L, V)(0) \leq \psi_1^{\mathrm{P}}(C, L, V)(0)$.*

Note that by Theorem 4 and the fact that φ^{EC} satisfies OP, the transfer of Agent 1 is the largest among the elements of the transfer vector $\psi^{\mathrm{SEEC}}(C, L, V)(0)$. So, the intuition is that ψ^{SEEC} is better equipped for minimization of the largest transfer than ψ^{P}, and thus than any egalitarian proportional solution due to Theorem 8. Whether this comparison is systematic in the sense that $\psi^{\mathrm{SEEC}}(C, L, V)(0) \succeq_L \psi^{\mathrm{P}}(C, L, V)(0)$ will be left as an open problem.

Example 7 We return to the problem (C, L, V) of Example 5, but now we study a solution as in Example 4. Our aim of this example is to show that the use of egalitarian dual constrained equal costs solutions may yield a very dispersed transfer vector. We consider the unique vector of transfers such that $\psi^{\mathrm{SEERC}}(C, L, V)(c) := t + \varphi^{\mathrm{DEC}}(c, L - t)$ is egalitarian. We call this solution the *stochastic egalitarian constrained equal remaining costs* solution. The unique vector of transfers is given by $t \approx (1.67, 1.67, -3.33)$. Then, we get $\psi_1^{\mathrm{SEERC}}(C, L, V)(c) = \psi_2^{\mathrm{SEERC}}(C, L, V)(c) = 1.67$ and $\psi_3^{\mathrm{SEERC}}(C, L, V)(c) = -3.33 + c$ for all $c \in [0, 10]$. This solution is such that Agents 1 and 2 pay only their deterministic transfers, and do not bear any risk. All risk due to the total cost C is borne by Agent 3. This solution is displayed in Figure 3. Note that the transfers are not close to the zero vector $\mathbf{0}$. In general, the solution $\psi^{\mathrm{SEERC}}(C, L, V)$ is piecewise linear.

It is straightforward to check that the solution $\psi^{\mathrm{SEERC}}(C, L, V)$ does not satisfy SSL. Moreover, note that the solution $\psi^{\mathrm{SEEC}}(C, L, V)$ is be insensitive to choices of L_3, as long as $L_3 \geq 5$. The solution $\psi^{\mathrm{SEERC}}(C, L, V)$ does not satisfy this property. In fact, it is very sensitive to choices of L_3 such that $5 \leq L_3 \leq 8$. For these reasons, we do not discuss this solution any further. $\triangledown$

8 Non-egalitarian solutions

In this section, we study an adaptation of the egalitarianism. We define the following property for a vector $a \in \mathbb{R}_+^N$ such that $a(N) = 1$ and $L \geq aV(C)$:

a-fairness: $V\left(\psi_i\left(C, L, V\right)\right) = a_i V(C)$ for all $i \in N$.

The property a-fairness is inspired by a desire to allocate the total cost C in a non-egalitarian manner. Here, the vector a is exogenously given, and assigns a proportion of the disutility of

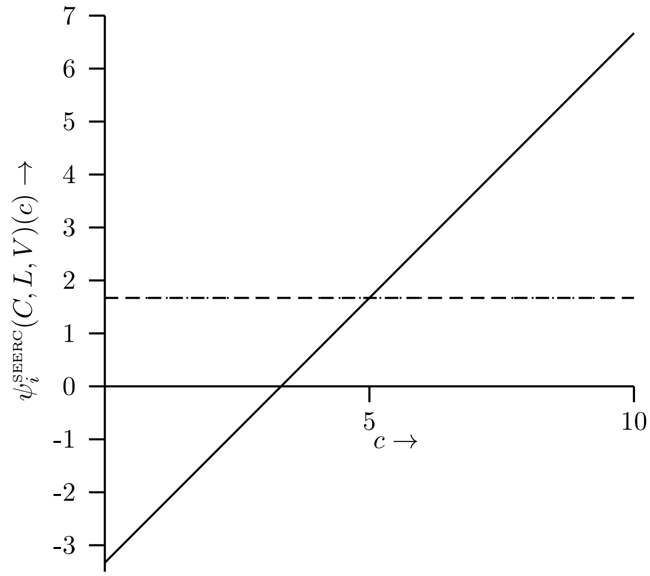

Figure 3: Graphical illustration of the egalitarian solution $\psi^{\text{SEERC}}(C, L, V)(c) = t + \varphi^{\text{DEC}}(c, L - t)$ corresponding to Example 7. The dash-dotted line represents $\psi_1^{\text{SEERC}}(C, L, V)(\cdot) = \psi_2^{\text{SEERC}}(C, L, V)(\cdot)$, and the solid line $\psi_3^{\text{SEERC}}(C, L, V)(\cdot)$.

the total cost $V(C)$ to the agents. For instance, a may be inspired by non-symmetric up-front contributions that the agents received to bear the stochastic cost C.

The property a-fairness is inspired by Pazdera et al. (2017), who study the related concept of financial fairness in risk-sharing. For this financial fairness approach, there is a given pricing measure and agents aim to share risk such that the price of the allocated risk is equal to the price of initial wealth. In the setting of this paper, the disutility function V is an expectation, and can alternatively be interpreted as a pricing functional. Moreover, the value of a_i can be interpreted as the price of initial wealth of agent i.

Suppose we first allocate the amount $\delta_i := (a_i - \frac{1}{n})V(C)$ to every agent $i \in N$. The remaining constrained cost allocation problem is given by $\left(\hat{C}, \hat{L}, V\right) \in \mathcal{P}$, where

$$\begin{aligned} \hat{C} &= C - \sum_{i \in N} \delta_i = C, \\ \hat{L} &= L - \delta. \end{aligned}$$

Then, a-fairness of the solution $\psi(C,L,V)$ is equivalent to V-egalitarianism of the solution $\hat{\psi}(\hat{C},\hat{L},V)$, where $\psi(C,L,V)=\delta+\hat{\psi}(\hat{C},\hat{L},V)$. Moreover, V-sufficiency of $(\hat{C},\hat{L},V)$ is equivalent to $L \geq aV(C)$ for the original problem (C,L,V). Hence, all results from Sections 3, 5, and 6 can be readily modified so that solutions are a-fair instead of V-egalitarian. For instance, Theorem 7 yields directly the following result.

Corollary 2 *On the class of all cost allocation problems $(C,L,V) \in \mathcal{L}^{\infty} \times \mathbb{R}^N_{++} \times \mathcal{V}$ and $a \in \mathbb{R}^N_{+}$ such that $a(N)=1$, (C,L) is admissible and $L \geq aV(C)$, there is a unique a-fair solution ψ that satisfies LS, and that is $\psi=\psi^a$ which is defined by*

$$\psi^{a}\left(C,L,V\right)(c) := t+\varphi^{\mathrm{EC}}\left(c,L-t\right), \text{ for all } c \in [0,\overline{c}], \tag{9}$$

where $t \in \mathcal{A}(0,L)$ is a unique vector of transfers such that (9) is a-fair.

9 Generalization to liability profiles that are not V-sufficient

If the problem (C,L,V) is admissible but not V-sufficient, then it holds that

$$L_1 < \tfrac{1}{n}V(C).$$

In particular we have to conclude that egalitarian solutions do not exist. As the project might be beneficial for all parties, and all parties are needed to support the project, we consider an alternative solution. We want the solution to be as egalitarian as possible. For instance, we can lexicographic minimize the vector $V(\psi_i(C,L,V)), i \in N$. Then, we erase Agent 1 that is not V-sufficient from the constrained cost allocation problem. Agent 1 pays the transfer $t=L_1$ regardless of the realization of C. Consider the reduced constrained cost allocation problem $\left(\tilde{C},\tilde{L},V\right)$, where

$$\begin{aligned}\tilde{C} &= C-L_1,\\ \tilde{L} &= L_{\{2,3,\ldots,n\}}.\end{aligned}$$

Clearly, this problem is again admissible, but is it V-sufficient? If not, i.e., if $L_2 < \frac{1}{n-1}V(C-L_1)$, remove Agent 2 in the same way for the new problem and continue. If the reduced problem becomes V-sufficient, then apply the solution to the reduced constrained cost allocation problem. Note that the cost $\tilde{C}$ may have negative realizations, but are bounded from below by $-L_1$. Then, it is easy to show that our results of this paper still hold true.

Hence, the idea is that where we are limited in our choices, we propose to adopt the idea of egalitarianism under participation constraints as in Dutta and Ray (1989).

10 Conclusion

This paper studies optimal cost allocation under liability constraints. The problem is a generalization of a rationing problem, where there is a stochastic estate that we here interpret as a cost. We aim to share a stochastic cost in an *ex ante* egalitarian way, which is defined in terms of an expectation or dual utility. We show two conditions which are necessary and sufficient to have existence of egalitarian solutions. First, all cost levels can be accounted for. Second, the individual agent's liability equals at least a fair share of the disutility of the total cost. Since there is typically not a unique egalitarian solution, we characterize specific egalitarian solutions by means of properties. The solutions that we propose are analogous to the proportional and constrained equal awards rules within a rationing context. Within these classes of solutions, we introduce zero-sum transfers that are paid/received before the realization of the stochastic cost is known; this vector is the solution if the realized cost is zero. There is a unique vector of transfers to guarantee egalitarianism for a class of solutions that is intrinsically connected to continuous rationing methods.

The class of standard proportional solutions is characterized by the property that the solutions may depend only on the disutility of the total cost. Within this class we identify a particular proportion vector that is consistent with the idea that when transfers are not necessary, the solution will do without these. A stronger result is that this solution determines a Lorenz-dominant vector of transfers within the class of egalitarian standard proportional solutions. Then we considered another class of solutions, based on the egalitarian rationing solution known as the constrained equal awards rule. Especially we characterize the SEEC solution which is the unique egalitarian solution in this class that can be considered locally egalitarian as well by using any possibility to share marginal costs equally among the agents. In particular, this solution uses transfers such that the largest transfer is smaller than the largest transfer used with any proportional solution.

We generalize our results with the concept of fairness, which is a condition that generalizes the concept of egalitarianism to allow for, *a priori*, asymmetries between the agents. Finally, if egalitarianism is not possible, we conclude this paper by providing a method that adopts the idea of egalitarianism under participation constraints of Dutta and Ray (1989).

We conclude this section with a suggestion for further research. We would like to extend the study of this paper to the setting where the preferences are given by a maximization of expected utility. If individuals minimize an expected cost, it can be shown that any cost allocation is Pareto-optimal. For expected utilities, however, a notion of Pareto-optimality is relevant. For strictly concave utility functions, we do not expect multiple cost allocations that are both egalitarian and Pareto-optimal, and so further characterizations based on the properties Invariance to total Disutility Preserving Preferences (IDPP) or Local Symmetry (LS) are redundant.

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A Proofs

Proof of Theorem 1: First of all, we will show existence of a vector t with the desired property, by application of egalitarianism:

$$\begin{aligned} V(\psi(C,L,V)) = \tfrac{1}{n}V(C)\,\mathbf{e} \quad &\Leftrightarrow \quad t + V(\varphi^{V(C)}(C, L-t)) = \tfrac{1}{n}V(C)\,\mathbf{e} \\ &\Leftrightarrow \quad t = \tfrac{1}{n}V(C)\mathbf{e} - V(\varphi^{V(C)}(C, L-t)) \\ &\Leftrightarrow \quad L - t = L - \tfrac{1}{n}V(C)\,\mathbf{e} + V(\varphi^{V(C)}(C, L-t)). \end{aligned}$$

Define $F : \Delta \to \Delta$ by $F(x) = L - \frac{1}{n}V(C)\,\mathbf{e} + V(\varphi^{V(C)}(C,x))$, where $\Delta := \{y \in \mathbb{R}^N_+ : y(N) = L(N), y \geq L - \frac{1}{n}V(C)\mathbf{e}\}$ is compact and convex. Note that $V(\psi(C,L,V)) = \frac{1}{n}V(C)\,\mathbf{e}$ if and only if $F(L-t) = L-t$, so that for the given combination (C,L,V) we find a desired corresponding vector of transfers through fixed points of F. The function F is continuous on Δ by continuity of $\varphi^{V(C)}$. Then the existence of a fixed point x^* of F follows by Brouwer's Fixed Point Theorem. So the vector of transfers we are looking for is given by $t^* = L - x^*$. Notice that $L - t^* \geq 0$, since we have

$$t^* = L - x^* \leq L - (L - \tfrac{1}{n}V(C)\,\mathbf{e}) = \tfrac{1}{n}V(C)\,\mathbf{e},$$

and $(C,L,V) \in \mathcal{P}$ satisfies V-sufficiency. Note that indeed t^* depends on (C,L,V). □

Proof of Theorem 2: The vector of transfers t can explicitly be found using egalitarianism, since

$$V(t + \alpha C) = \tfrac{1}{n}V(C)\,\mathbf{e} \quad \Leftrightarrow \quad t + \alpha V(C) = \tfrac{1}{n}V(C) \Leftrightarrow t = V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha)$$

Since $\alpha \geq \mathbf{0}, \alpha(N) = 1$ and V-sufficiency, it holds that $t \in \mathcal{A}(0,L)$. Only we need to have that $\alpha\, c \leq L - t$ for all $c \in [0,\bar{c}]$, so that a sufficient and necessary condition on α is that

$$\alpha\,\bar{c} \leq L - V(C)(\tfrac{1}{n}\,\mathbf{e} - \alpha) \Leftrightarrow \alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}.$$

□

Proof of Theorem 3: By Theorem 1, it suffices to show that the fixed point of the mapping F is unique. Recall $x = L - t$, where $t = \psi^{\text{SEEC}}(C,L,V)(0)$. Suppose that there are two fixed points, $\tilde{x}$ and $\bar{x}$, so that $\tilde{x} \neq \bar{x}$. Take the smallest index i such that $\tilde{x}_i \neq \bar{x}_i$. Since $\tilde{x}(N) = \bar{x}(N)$, we have without loss of generality that $\tilde{x}_i < \bar{x}_i$ for $i < n$. Since $\tilde{x}$ and $\bar{x}$ are ordered as a result of Theorem 4, we have, with $\tilde{x}_i^+ = \sum_{j<i}\tilde{x}_j + (n-i+1)\tilde{x}_i$,

$$\varphi_i^{\text{EC}}(c,\tilde{x}) = \begin{cases} \varphi_i^{\text{EC}}(c,\bar{x}) & \text{if } c \leq \tilde{x}_i^+, \\ \tilde{x}_i & \text{if } c > \tilde{x}_i^+, \end{cases}$$

and therefore

$$\varphi_i^{\text{EC}}(c,\bar{x}) - \varphi_i^{\text{EC}}(c,\tilde{x}) = \begin{cases} 0 & \text{if } c \leq \tilde{x}_i^+, \\ \min\left\{\frac{c - \tilde{x}_i^+}{n-i+1}, \bar{x}_i - \tilde{x}_i\right\} & \text{if } c > \tilde{x}_i^+. \end{cases}$$

Thus, $\mathbb{Q}(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x}) < \bar{x}_i - \tilde{x}_i) \geq \mathbb{Q}(C \leq \tilde{x}_i^+)$. Moreover, $\mathbb{Q}(C \leq \tilde{x}_i^+) = 0$ holds only if $\tilde{x}_i^+ = 0$, since $\mathbb{Q}$ has a positive density on $[0,\bar{c}]$. In that case also $\tilde{x}_i = 0$ so that $\tilde{t}_i = L_i$, $V(\tilde{t}_i + \varphi_i^{\text{EC}}(C,\tilde{x})) = L_i$, and $\mathbb{Q}(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x}) < \bar{x}_i) = \mathbb{Q}(C < (n-i-1)\bar{x}_i) > 0$.

Thus, $\mathbb{Q}(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x}) < \bar{x}_i - \tilde{x}_i) > 0$ and so

$$V(\varphi_i^{\text{EC}}(C,\bar{x}) - \varphi_i^{\text{EC}}(C,\tilde{x})) < \bar{x}_i - \tilde{x}_i = (L_i - \bar{t}_i) - (L_i - \tilde{t}_i) = \tilde{t}_i - \bar{t}_i.$$

But then this leads to contradiction with egalitarianism as

$$\begin{aligned} V(\bar{t}_i + \varphi_i^{\mathrm{EC}}(C, \bar{x})) &= \bar{t}_i + V(\varphi_i^{\mathrm{EC}}(C, \bar{x})) \\ &< \bar{t}_i + (\tilde{t}_i - \bar{t}_i) + V(\varphi_i^{\mathrm{EC}}(C, \tilde{x})) = \tilde{t}_i + V(\varphi_i^{\mathrm{EC}}(C, \tilde{x})). \end{aligned}$$

□

Proof of Theorem 4: Consider again the mapping F that we used in the proof of Theorem 1, but now restricted to $\Delta^* := \{y \in \mathbb{R}_+^N : y_1 \leq y_2 \leq \cdots \leq y_n, y(N) = L(N), y \geq L - \frac{1}{n}V(C)\mathbf{e}\}$, which is compact and convex, and we have $F(\Delta^*) \subseteq \Delta^*$. Then – again by using Brouwer's Fixed Point Theorem – there is at least one fixed point for F, say x^*. If $L_i \leq L_j$ (or $i < j$), then $(L - t^*)_i = x_i^* \leq x_j^* = (L - t^*)_j$ so that by OP we have $\varphi_i^{V(C)}(c, L - t^*) \leq \varphi_j^{V(C)}(c, L - t^*)$. Consequently

$$t_i^* = \tfrac{1}{n}V(C) - V(\varphi_i^{V(C)}(c, L - t^*)) \geq \tfrac{1}{n}V(C) - V(\varphi_j^{V(C)}(c, L - t^*)) = t_j^*.$$

□

We continue with an intermediary result that we need in the proofs of Theorems 5 and 6.

Lemma 1 *If $\psi \in \mathcal{Z}$ satisfies comonotonicity, then the mapping $c \to \psi_i (C, L, V) (c)$ is Lipschitz-continuous.*

Proof of Lemma 1: We will show that $\|\psi (C, L, V) (c_1) - \psi (C, L, V) (c_2)\| \leq \sqrt{n}\, |c_1 - c_2|$ for all $c_1, c_2 \in [0, \bar{c}]$. First recall that for fixed L and $c_1, c_2 \in [0, \bar{c}], c_1 < c_2$, we have $\psi (C, L, V) (c_1) \leq \psi (C, L, V) (c_2)$. Moreover, since $\psi \in \mathcal{Z}$, it holds that $\sum_{j \in N} \psi_j (C, L, V) (c_1) = c_1$ and $\sum_{j \in N} \psi_j (C, L, V) (c_2) = c_2$, so that for any choice of $i \in N$ we have

$$\psi_i (C, L, V) (c_2) - \psi_i (C, L, V) (c_1) + \sum_{j \neq i} (\psi_j (C, L, V) (c_2) - \psi_j (C, L, V)) (c_1) = c_2 - c_1.$$

Now, suppose $\psi_i (C, L, V) (c_2) - \psi_i (C, L, V) (c_1) > c_2 - c_1$. Then, $\sum_{j \neq i} (\psi_j (C, L, V) (c_2) - \psi_j (C, L, V) (c_1)) < 0$, and there must be $k \in N$ with $\psi_k (C, L, V) (c_2) - \psi_k (C, L, V) (c_1) < 0$, which contradicts with the assumption that ψ satisfies comonotonicity. So we have $\psi_i (C, L, V) (c_2) - \psi_i (C, L, V) (c_1) \leq c_2 - c_1$ for all $i \in N$.

Then,

$$\begin{aligned} \|\psi (C, L, V) (c_1) - \psi (C, L, V) (c_2)\|^2 &= \sum_{i \in N} (\psi_i (C, L, V) (c_2) - \psi_i (C, L, V) (c_1))^2 \\ &\leq \sum_{i \in N} (c_2 - c_1)^2 = n (c_2 - c_1)^2, \end{aligned}$$

which proves our claim by taking the square-root on both sides. □

Proof of Theorem 5: The "if" part is easy, as it basically follows from Theorem 2. So now we will turn to the "only if" part. Since ψ is standard, we may write $\psi(C, L, V)(c) = t + f(c, L, t, V(C))$ with $t = \psi(C, L, V)(0)$. Suppose ψ is egalitarian and take a V as in (1) with a cumulative distribution function $F(c) = \mathbb{Q}(C \leq c)$. Since $\mathbb{Q}$ has a positive density on $[0, \bar{c}]$, F is strictly increasing on $[0, \bar{c}]$. Below we will construct $V^* \in \mathcal{V}$ with $V^*(C) = V(C)$, so that in turn IDPP implies $\psi(C, L, V) = \psi(C, L, V^*)$. Especially we have $t = \psi(C, L, V)(0) = \psi(C, L, V^*)(0)$ so that for all i:

$$V^*(f_i(C, L, t, V^*(C))) = \tfrac{1}{n} V^*(C) - t_i = \tfrac{1}{n} V(C) - t_i = V(f_i(C, L, t, V(C))). \tag{10}$$

We will now show that an agent should then pay a proportion of the realized cost, that is fixed for all realizations. Assume an agent i who does not pay a fixed proportion of the realized cost after transfers. We will point out that in that case we may construct V^* with $V^*(C) = V(C)$ but such that (10) is not satisfied, and we have the desired contradiction. For the sake of the exposition we will write $f_i(c, L, t, V(C))$ as $z(c)$. Since $c \mapsto z(c)$ is non-decreasing, it is Lipschitz-continuous due to Lemma 1, and hence differentiable almost everywhere. Since Agent i does not pay a fixed share of the realized cost after transfers, there must be c_1 and c_2 such that $z'(c_1)$ exists, and

- $z'(c_1) > \dfrac{z(c_2) - z(c_1)}{c_2 - c_1}$, or
- $z'(c_1) < \dfrac{z(c_2) - z(c_1)}{c_2 - c_1}$.

We will show a proof for the second case only, as a proof for the first case follows the same lines. So, assume that for small $\varepsilon > 0$ and $c \in (c_1, c_1 + \varepsilon)$ we have

$$\frac{z(c) - z(c_1)}{c - c_1} < \frac{z(c_2) - z(c_1)}{c_2 - c_1}. \tag{11}$$

We will show that for those cases we may shift some probability mass from $\mathbb{Q}$ to $\mathbb{Q}^\Delta$, and obtain in this way a new distribution under which the disutility of the total cost for i increases whilst $V^\Delta(C) := \mathbb{E}_{\mathbb{Q}^\Delta}[C] = \mathbb{E}_{\mathbb{Q}}[C] = V(C)$. We will point out how to construct V^* from here, yielding the desired conflict with the assumption of an egalitarian solution.
The distribution $\mathbb{Q}^\Delta$ is constructed by removing mass $\Delta_F(\varepsilon) := F(c_1 + \varepsilon) - F(c_1)$ on the interval $[c_1, c_1 + \varepsilon]$ and replace it by point masses on c_1 and c_2. In order to keep the disutility as measured by (1) of the total cost under the redistribution constant, determine $\delta(\varepsilon) \in (0, 1)$ such that

$$\delta(\varepsilon)\Delta_F(\varepsilon)c_1 + (1 - \delta(\varepsilon))\Delta_F(\varepsilon)c_2 + \int_{c_1+\varepsilon}^{c_2} c\, dF(c) = \int_{c_1}^{c_2} c\, dF(c). \tag{12}$$

Then the cumulative distribution function F^Δ for $\mathbb{Q}^\Delta$ is specified by

$$F^\Delta(c) = \mathbb{Q}^\Delta(C \le c) = \begin{cases} F(c) & c < c_1 \text{ or } c \ge c_2, \\ F(c_1) + \delta(\varepsilon)\Delta_F(\varepsilon) & c \in [c_1, c_1 + \varepsilon], \\ F(c) - (1 - \delta(\varepsilon))\Delta_F(\varepsilon) & c \in (c_1 + \varepsilon, c_2), \end{cases}$$

and define $V^\Delta(X) = \mathbb{E}_{\mathbb{Q}^\Delta}[X]$ for any $X \in \mathcal{L}^\infty$ that is comonotonic with C. By construction in (12), we obtain $V^\Delta(C) = V(C)$ and

$$\begin{aligned} V(z(C)) &= \int_0^{\bar{c}} z(c)\, dF(c), \\ V^\Delta(z(C)) &= \int_0^{c_1} z(c)\, dF(c) + \int_{c_2}^{\bar{c}} z(c)\, dF(c) \\ &\quad + \delta(\varepsilon)\Delta_F(\varepsilon)z(c_1) + \int_{c_1+\varepsilon}^{c_2} z(c)\, dF(c) + (1 - \delta(\varepsilon))\Delta_F(\varepsilon)z(c_2). \end{aligned}$$

It holds

$$\begin{aligned} V^{\Delta}(z(C)) - V(z(C)) &= -\delta(\varepsilon)\Delta_F(\varepsilon)(z(c_2) - z(c_1)) + \Delta_F(\varepsilon)z(c_2) - \int_{c_1}^{c_1+\varepsilon} z(c)\, dF(c) \\ &= -\delta(\varepsilon)\Delta_F(\varepsilon)(z(c_2) - z(c_1)) + \int_{c_1}^{c_1+\varepsilon} (z(c_2) - z(c))\, dF(c). \end{aligned}$$

Note that from equation (12) we get

$$\delta(\varepsilon)\Delta_F(\varepsilon) = \frac{1}{c_2 - c_1} \int_{c_1}^{c_1+\varepsilon} (c_2 - c)\, dF(c)$$

and thus

$$\begin{aligned} V^{\Delta}(z(C)) - V(z(C)) &= -\int_{c_1}^{c_1+\varepsilon} (c_2 - c)\, dF(c) \cdot \frac{z(c_2) - z(c_1)}{c_2 - c_1} + \int_{c_1}^{c_1+\varepsilon} (z(c_2) - z(c))\, dF(c) \\ &= -\int_{c_1}^{c_1+\varepsilon} ((c_2 - c_1) + (c_1 - c))\, dF(c) \cdot \frac{z(c_2) - z(c_1)}{c_2 - c_1} \\ &\quad + \int_{c_1}^{c_1+\varepsilon} ((z(c_2) - z(c_1)) + (z(c_1) - z(c)))\, dF(c) \\ &= -\Delta_F(\varepsilon)(c_2 - c_1)\frac{z(c_2) - z(c_1)}{c_2 - c_1} + \int_{c_1}^{c_1+\varepsilon} (c - c_1)\, dF(c) \cdot \frac{z(c_2) - z(c_1)}{c_2 - c_1} \\ &\quad + \Delta_F(\varepsilon)(z(c_2) - z(c_1)) - \int_{c_1}^{c_1+\varepsilon} (z(c) - z(c_1))\, dF(c) \\ &= \int_{c_1}^{c_1+\varepsilon} (c - c_1) \cdot \frac{z(c_2) - z(c_1)}{c_2 - c_1}\, dF(c) - \int_{c_1}^{c_1+\varepsilon} (z(c) - z(c_1))\, dF(c) \\ &> \int_{c_1}^{c_1+\varepsilon} (c - c_1) \cdot \frac{z(c) - z(c_1)}{c - c_1}\, dF(c) - \int_{c_1}^{c_1+\varepsilon} (z(c) - z(c_1))\, dF(c) = 0. \end{aligned}$$

Note that $\mathbb{Q}^{\Delta}$ does not have a positive density on $[0, \bar{c}]$, so that $V^{\Delta} \notin \mathcal{V}$. Similar as in (2), we now mix the two densities of $\mathbb{Q}^{\Delta}$ and $\mathbb{Q}$. For small enough $\lambda \in (0, 1)$, we define $\mathbb{Q}^*(C \leq c) := (1 - \lambda)\mathbb{Q}^{\Delta}(C \leq c) + \lambda\mathbb{Q}(C \leq c)$ for all $c \in [0, \bar{c}]$. Then, $\mathbb{Q}^*$ has a positive density on $[0, \bar{c}]$. Thus, it holds for the corresponding admissible preference $V^*(\cdot) = (1-\lambda)V^{\Delta}(\cdot) + \lambda V(\cdot)$ that $V^* \in \mathcal{V}$. Moreover, we have $V^*(C) = V(C)$ and $V^*(z(C)) > V(z(C))$. This is the desired contradiction with (10). □

Proof of Theorem 6: Let $(C, L, V) \in \mathcal{P}$. Since solution $\psi \in \mathcal{Z}$ is comonotonic, the mapping $c \mapsto \psi(C, L, V)(c)$ is Lipschitz-continuous due to Lemma 1, and this mapping is therefore absolutely continuous. This implies

$$\psi(C, L, V)(c) - \psi(C, L, V)(0) = \int_0^c \frac{\partial}{\partial s}\psi(C, L, V)(s) ds \text{ for all } c \in [0, \bar{c}]. \qquad (13)$$

Fix $\psi(C, L, V)(0)$, and let $k : N \to N$ be a bijection such that

$$k(i) \leq k(j) \Leftrightarrow L_{k(i)} - \psi_{k(i)}(C, L, V)(0) \leq L_{k(j)} - \psi_{k(j)}(C, L, V)(0).$$

Define iteratively constants $c_0, c_1, c_2, \ldots, c_n \in [0, \bar{c}\,]$ by $c_0 := 0$,

$$c_1 := n\left(L_{k(1)} - \psi_{k(1)}(C, L, V)(0)\right),$$

$$\ell \geq 2 : c_\ell := c_{\ell-1} + (n-\ell+1)\left(L_{k(\ell)} - \psi_{k(\ell)}(C,L,V)(0) - \sum_{r=1}^{\ell-1} \frac{c_r - c_{r-1}}{n-r+1}\right).$$

By construction, it holds $c_0 \leq c_1 \leq \cdots \leq c_n$. Then, by LS, we can write (13) for $c \in [c_{\ell-1}, c_\ell]$ as

$$\psi_{k(i)}(C,L,V)(c) = \begin{cases} \psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{i} \frac{c_r - c_{r-1}}{n-r+1} & \text{if } k(i) < \ell, \\ \psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{\ell-1} \frac{c_r - c_{r-1}}{n-r+1} + \frac{c - c_{\ell-1}}{n-\ell+1} & \text{if } k(i) \geq \ell, \end{cases}$$

where $\ell = 1, \ldots, n$. Note that for $r \in \{1, 2, \ldots, n\}$ we have

$$c_r \wedge c - c_{r-1} \wedge c = \begin{cases} 0 & \text{if } c \leq c_{r-1}, \\ c - c_{r-1} & \text{if } c \in (c_{r-1}, c_r), \\ c_r - c_{r-1} & \text{if } c \geq c_r, \end{cases}$$

where $c_r \wedge c := \min\{c_r, c\}$. So, for the random cost C with realization $c \in [0, \overline{c}]$, we get

$$\begin{aligned} \psi_{k(i)}(C,L,V)(c) &= \left(\psi_{k(i)}(C,L,V)(0) + \sum_{r=1}^{i} \frac{c_r \wedge c - c_{r-1} \wedge c}{n-r+1}\right) \wedge L_{k(i)} \\ &= \psi_{k(i)}(C,L,V)(0) + \left(\sum_{r=1}^{i} \frac{c_r \wedge c - c_{r-1} \wedge c}{n-r+1}\right) \wedge \left(L_{k(i)} - \psi_{k(i)}(C,L,V)(0)\right) \\ &= \psi_{k(i)}(C,L,V)(0) + \varphi^{\text{EC}}_{k(i)}(c, L - \psi(C,L,V)(0)). \end{aligned}$$

Hence, $\psi(C,L,V)(c) = t + \varphi^{\text{EC}}(c, L-t)$ for $t = \psi(C,L,V)(0)$. This completes the proof. $\square$

Proof of Theorem 8: If $L \geq \frac{1}{n}\bar{c}\,\mathbf{e}$, then $\varphi^{\text{EC}}(\bar{c}, L) = \frac{1}{n}\bar{c}\,\mathbf{e}$ so that $\alpha = \frac{1}{n}\,\mathbf{e}$ and $\psi^{\text{P}}(C,L,V)(0) = \mathbf{0}$. Clearly, in $\mathcal{A}(0,L)$ the Lorenz dominant element is $\mathbf{0}$. Now assume that not $L \geq \frac{1}{n}\bar{c}\,\mathbf{e}$. Then pick $i \in N$ so that

$$\sum_{k<i} L_k + (n-i+1)L_i \leq \bar{c} \leq \sum_{k \leq i} L_k + (n-i)L_{i+1},$$

which exists due to admissibility. Then

$$\varphi^{\text{EC}}_j(\bar{c}, L) = \begin{cases} L_j & \text{if } j \leq i, \\ \frac{1}{n-i}(\bar{c} - \sum_{k \leq i} L_k) & \text{if } j > i. \end{cases}$$

Since ψ^α is egalitarian, by Theorem 5 it must hold that

$$\alpha \leq \frac{L - \frac{1}{n}V(C)\,\mathbf{e}}{\bar{c} - V(C)}$$

which means that for $j \leq i$

$$\alpha_j \leq \frac{\varphi^{\text{EC}}_j(\bar{c}, L) - \frac{1}{n}V(C)}{\bar{c} - V(C)}$$

so that

$$\psi_j^{\mathrm{P}}(C,L,V)(0) = V(C)\left(\tfrac{1}{n} - \frac{\varphi_j^{\mathrm{EC}}(\bar{c},L) - \frac{1}{n}V(C)}{\bar{c} - V(C)}\right) \leq V(C)\left(\tfrac{1}{n} - \alpha_j\right) = \psi_j^{\alpha}(C,L,V)(0),$$

and thus we have for all $k = 1,\ldots,i$ that

$$\sum_{j=1}^{k} \psi_j^{\mathrm{P}}(C,L,V)(0) \leq \sum_{j=1}^{k} \psi_j^{\alpha}(C,L,V)(0).$$

Now suppose that there exists an $\ell > i$ such that

$$\sum_{j=1}^{\ell} \psi_j^{\mathrm{P}}(C,L,V)(0) > \sum_{j=1}^{\ell} \psi_j^{\alpha}(C,L,V)(0).$$

Assume without loss of generality that ℓ is the smallest number with this property, so that $1 < \ell < n$ and $\psi_\ell^{\alpha}(C,L,V)(0) < \psi_\ell^{\mathrm{P}}(C,L,V)(0)$. Since we assumed that α is ordered in the same way as L, it holds $\alpha_1 c \leq \alpha_2 c \leq \cdots \leq \alpha_n c$ for all $c \in [0,\bar{c}]$. Suppose $\psi_j^{\alpha}(C,L,V)(0) < \psi_\ell^{\alpha}(C,L,V)(0)$ for $j \geq \ell$, then $\psi_j^{\alpha}(C,L,V)(0) + \alpha_j c < \psi_\ell^{\alpha}(C,L,V)(0) + \alpha_\ell c$ for all $c \in [0,\bar{c}]$, and so $V(\psi_j^{\alpha}(C,L,V)(0) + \alpha_j C) < V(\psi_\ell^{\alpha}(C,L,V)(0) + \alpha_\ell C)$, which is a contradiction with egalitarianism. Thus, it holds for $j \geq \ell$ that

$$\psi_j^{\alpha}(C,L,V)(0) \leq \psi_\ell^{\alpha}(C,L,V)(0) < \psi_\ell^{\mathrm{P}}(C,L,V)(0) = \psi_j^{\mathrm{P}}(C,L,V)(0),$$

where the equality follows from the fact that φ^{EC} satisfies OP. But then this leads to a contradiction, since

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \psi_j^{\alpha}(C,L,V)(0) = \sum_{j\leq\ell} \psi_j^{\alpha}(C,L,V)(0) + \sum_{j>\ell} \psi_j^{\alpha}(C,L,V)(0) \\ &< \sum_{j\leq\ell} \psi_j^{\mathrm{P}}(C,L,V)(0) + \sum_{j>\ell} \psi_\ell^{\mathrm{P}}(C,L,V)(0) = 0. \end{aligned}$$

This means that ℓ cannot exist, and that for all $k = 1,\ldots,n$ we have

$$\sum_{j=1}^{k} \psi_j^{\mathrm{P}}(C,L,V)(0) \leq \sum_{j=1}^{k} \psi_j^{\alpha}(C,L,V)(0),$$

which concludes the proof. □

Proof of Theorem 9: Let $(C,L,V) \in \mathcal{P}$. Suppose that $\psi_1^{\mathrm{SEEC}}(C,L,V)(0) > \psi_1^{\mathrm{P}}(C,L,V)(0)$. If $L_1 \geq \frac{1}{n}\bar{c}$, then by SSL that $\psi_1^{\mathrm{SEEC}}(C,L,V)(0) = \psi_1^{\mathrm{P}}(C,L,V)(0) = 0$, which is a contradiction. Assume $L_1 < \frac{1}{n}\bar{c}$. By LS, we have that $\frac{\partial}{\partial c}\psi_1^{\mathrm{SEEC}}(C,L,V)(c) \geq \frac{1}{n}$ whenever $\psi_1^{\mathrm{SEEC}}(C,L,V)(c) < L_1$. Moreover,

$$\begin{aligned} \frac{\partial}{\partial c}\psi_1^{\mathrm{P}}(C,L,V)(c) = \alpha_1 &= \frac{\varphi_1^{\mathrm{EC}}(\bar{c},L) - \frac{1}{n}V(C)}{\bar{c} - V(C)} \\ &= \frac{L_1 - \frac{1}{n}V(C)}{\bar{c} - V(C)} < \frac{\frac{1}{n}\bar{c} - \frac{1}{n}V(C)}{\bar{c} - V(C)} = \frac{1}{n}, \end{aligned}$$

for all $c \in [0,\bar{c}]$, and $\psi_1^{\mathrm{P}}(C,L,V)(c) < L_1$ for all $c \in [0,\bar{c})$. Thus, $\psi_1^{\mathrm{SEEC}}(C,L,V)(c) > \psi_1^{\mathrm{P}}(C,L,V)(c)$ for all $c \in [0,\bar{c})$. Thus, since $V(C) < \bar{c}$, we have $V(\psi_1^{\mathrm{SEEC}}(C,L,V)) > V(\psi_1^{\mathrm{P}}(C,L,V))$, which is a contradiction with egalitarianism of ψ^{SEEC} and ψ^{P}. □