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Hadron structure in the description of electromagnetic reactions

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The description of electromagnetic reactions at intermediate energies, such as pion electroproduction or (virtual) Compton scattering, traditionally starts from covariant tree-level Feynman diagrams (Born or pole terms). Internal hadron structure is included by means of (on-shell) form factors in the vertices while free propagators are used. To overcome problems with gauge invariance, simple prescriptions, such as, choosing $F_1^V(q^2) = F_\pi(q^2)$ in pion electroproduction or the “minimal substitution,” are used. We discuss the inherent assumptions of such approaches and study the general structure of electromagnetic vertices and propagators for pions and nucleons. We show which part of the vertex is entirely determined through the Ward-Takahashi identity and point out that detailed features of the q^2 dependence of the form factors cannot be derived from this condition. Recipes to enforce gauge invariance, including the minimal substitution, are critically examined in the light of the exact treatment. The interplay between irreducible contact terms and “off-shell effects” in vertices and propagators is demonstrated for real Compton scattering on a pion. The need for a consistent microscopic treatment of reaction amplitudes is stressed and illustrated by an example in the framework of chiral perturbation theory. Shortcomings of the minimal substitution method when applied to effective Lagrangians are pointed out.

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I. INTRODUCTION

High-precision experiments with electromagnetic probes are presently being carried out at intermediate energies to investigate the internal structure of the participating hadrons and to probe details of the reaction dynamics. Recent examples are virtual Compton scattering on the nucleon to extract the (generalized) polarizabilities [1], or pion electroproduction on the nucleon to determine the axial and electromagnetic form factors of the nucleon and the pion, respectively [2,3]. Most theoretical descriptions of these intermediate-energy reactions are based on covariant tree-level Feynman diagrams (Born or pole terms), where the free hadron vertices are the building blocks. One limitation of the Born-term approach is immediately obvious from a simple observation: the two-step reactions under study necessarily involve intermediate particles that are “off mass shell” [4–6]. Nevertheless, encouraged by theorems that express reaction amplitudes in certain low-energy limits in terms of on-shell properties of the participating particles, it has been customary to use “Born amplitudes,” which only involve on-shell information in vertices and propagators. Interestingly enough, the papers deriving these low-energy theorems [7,8] were the first to point out the more complex structure of vertices and propagators in the general case at higher energies. What they also emphasized was that the internal structure of the hadrons necessarily leads to new classes of amplitudes (see Fig. 1). Both features are absent in the Born-term description.

An objective criterion whether some important physics is missing is provided by gauge invariance. When form factors are included in the Born terms, one typically obtains amplitudes where the electromagnetic current to which the photon

couples, is not conserved. Even though the cause of this problem can be understood and remedied by consistently dealing with the origin of the internal structure, most authors reverted to a variety of *ad hoc* recipes: terms are added by hand or form factors are adjusted with the sole purpose to produce a conserved current. In pion electroproduction, for example, a commonly made assumption for this purpose is that the pion and nucleon isovector form factors are identical [9–11]. Similarly, both the pion-pole term and the nucleon-pole term, which is needed for gauge invariance, are proportional to the pion form factor in the model [12,13] used to analyze the data in Ref. [3]. These assumptions may very well limit the use of this model for extracting the pion form factor from experiment.

Another way to preserve gauge invariance in phenomenological models is the minimal substitution. It generates a restricted or “minimal” set of terms that guarantees a gauge-invariant total amplitude. This recipe also has serious shortcomings since the nonminimal terms, which it fails to produce, can be very important [14]. It has mainly been applied in pion photoproduction [15], where the use of a strong pion-nucleon form factor leads, in general, to a photoproduc-

FIG. 1. Examples of related one-particle-irreducible contributions to the electromagnetic vertex of a nucleon (a) and to the Compton-scattering amplitude (b).

tion amplitude that is not gauge invariant.

In view of the shortcomings of the standard theoretical approaches and the goals of the modern experiments, we examine here the description of the structure of hadrons in electromagnetic reactions. Instead of just enforcing current conservation in a phenomenological description, we focus on the constraints imposed by the Ward-Takahashi (WT) identity [16,17] on the general form of vertices, propagators, and entire reaction amplitudes. Such a consistent theoretical description allows for an entirely different q^2 dependence than that assumed in the *ad hoc* recipes. We use the pion as the main example, since the formalism remains simple in comparison to particles with higher spin while the relevant aspects can still be studied.

We start in Sec. II with a discussion of the features of the general electromagnetic vertex of the pion. While known for decades [18–20], they have largely been ignored in the theoretical descriptions. We discuss the restrictions due to the WT identity for the electromagnetic vertex and show that its q^2 dependence arises from terms not constrained by this condition. We examine to what extent this vertex can be obtained through minimal substitution into the pion self-energy and then extend the discussion to the nucleon. Finally, we show how to correctly obtain a gauge-invariant amplitude for pion electroproduction on the nucleon without inter-relating the pion and nucleon form factors as is often done.

In Sec. III we investigate Compton scattering off a pion as a prototype for a two-step process. Special attention is paid to the question how “off-shell effects” can contribute to the amplitude and polarizabilities; it is shown that they cannot be entirely eliminated by use of the WT identity as has been suggested. We then contrast the general Compton scattering amplitude for a pion with the amplitude generated through minimal substitution.

Clearly the most sensible way to address the physics of hadron structure and reaction mechanism in a consistent fashion, while avoiding the complications and recipes mentioned above, is to work in the framework of a well-defined (effective) field theory, such as chiral perturbation theory (ChPT). In Sec. IV we illustrate this by looking at the pion electromagnetic form factor to one loop and demonstrate shortcomings of the minimal recipe, both for generating a vertex for a given self-energy and when applied to the Lagrangian itself. General conclusions are presented in Sec. V.

II. ELECTROMAGNETIC VERTEX OF COMPOSITE PARTICLES: GENERAL FEATURES, THE MINIMAL SUBSTITUTION, AND OTHER RECIPES

A. The electromagnetic vertex of the pion and its q^2 dependence

For a spin-0 particle (we will for simplicity assume a charged pion), Lorentz invariance restricts the form of the electromagnetic vertex to

$$\Gamma^\mu(p', p) = e[(p' + p)^\mu F(q^2, p'^2, p^2) + q^\mu G(q^2, p'^2, p^2)], \quad (1)$$

where p and p' are the initial and final momenta of the meson, respectively, and $q = p' - p$ is the photon momentum.

The functions F and G are form factors that, in general, depend on three scalar variables. We suppress an index to show if a form factor refers to a positive or negative pion; the neutral pion is its own antiparticle and, because of C invariance, has no electromagnetic form factor even off-shell. Time-reversal invariance imposes further constraints on the functions F and G

$$\begin{aligned} F(q^2, p'^2, p^2) &= F(q^2, p^2, p'^2), \\ G(q^2, p'^2, p^2) &= -G(q^2, p^2, p'^2), \end{aligned} \quad (2)$$

whereas from Hermiticity follows, for $q^2 \leq 0$,

$$F(q^2, m^2, m^2) = F^*(q^2, m^2, m^2). \quad (3)$$

Assuming that we are dealing with the irreducible electromagnetic vertex, the requirement of gauge invariance yields the WT identity [16,17]

$$q_\mu \Gamma^\mu(p', p) = e_\pi [\Delta^{-1}(p') - \Delta^{-1}(p)], \quad (4)$$

where $\Delta(p)$ denotes the dressed, renormalized propagator of the meson and $e_\pi = e \hat{e}_\pi$ ($\hat{e}_{\pi^\pm} = \pm 1$). Using the general form in Eq. (1), the WT identity becomes

$$\begin{aligned} (p'^2 - p^2)F(q^2, p'^2, p^2) + q^2 G(q^2, p'^2, p^2) \\ = \hat{e}_\pi [\Delta^{-1}(p') - \Delta^{-1}(p)], \end{aligned} \quad (5)$$

providing a relation between the functions F and G and the propagator Δ of the meson.

From Eq. (2) or the WT identity it can be easily seen that

$$G(q^2, p^2, p^2) = 0, \quad (6)$$

i.e., this function vanishes when the invariant masses of the initial and final meson are equal. Therefore, we write G as

$$G(q^2, p'^2, p^2) = (p'^2 - p^2)g(q^2, p'^2, p^2), \quad (7)$$

where the function g is nonsingular at $p'^2 = p^2$. We now use Eq. (7) to rewrite the general electromagnetic vertex of the scalar particle as

$$\begin{aligned} \Gamma^\mu(p', p) &= e[(p' + p)^\mu f(p'^2, p^2) \\ &\quad + (q^\mu q^\nu - g^{\mu\nu} q^2)(p' + p)_\nu g(q^2, p'^2, p^2)], \end{aligned} \quad (8)$$

where f is defined as

$$f(p'^2, p^2) = F(q^2, p'^2, p^2) + q^2 g(q^2, p'^2, p^2). \quad (9)$$

The parametrization we introduced in Eq. (8) has not been used in the literature before and is particularly useful to demonstrate two important new points concerning the vertex. First, by contracting Eq. (8) with q_μ , we see that the function f is entirely determined through the WT identity and depends only on *two* scalar variables, the invariant masses p'^2 and p^2 , and not on q^2 ,

$$f(p'^2, p^2) = \hat{e}_\pi \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2}. \quad (10)$$

Second, the term in Eq. (8) involving the function $g(q^2, p'^2, p^2)$ is separately gauge invariant. This leads to the important conclusion that the condition of gauge invariance, the WT identity, clearly cannot help to determine the q^2 dependence. The general fact that the function $g(q^2, p'^2, p^2)$ —or $G(q^2, p'^2, p^2)$ —is crucial for the vertex to have a q^2 dependence at all can already be seen by looking at the WT identity. The right-hand side of Eq. (5) depends only on p^2 and p'^2 , but not on q^2 . Hence without the term involving G , the form factor F (and with it the entire vertex) could not depend on q^2 [19].

We stress that the vanishing of $G(q^2, p'^2, p^2)$ for $p'^2 = p^2$ according to Eq. (6) does not rule out a q^2 dependence of the vertex, since in that limit

$$\lim_{p'^2 \rightarrow p^2} \Gamma^\mu(p', p) = e(p + p')^\mu F(q^2, p^2, p^2), \quad (11)$$

and we have from the WT identity, Eq. (5), and Eq. (7)

$$F(q^2, p^2, p^2) = \hat{e}_\pi \frac{\partial \Delta^{-1}(p)}{\partial p^2} - q^2 g(q^2, p^2, p^2). \quad (12)$$

This includes of course the on-shell case, $p^2 = p'^2 = m^2$, which is characterized by a single on-shell form factor

$$F_\pi(q^2) \equiv F(q^2, p^2, p^2)|_{p^2=m^2}. \quad (13)$$

The significance of the second term in Eq. (1) is not clearly recognized in much of the literature, even though it is crucial for maintaining gauge invariance. Such gauge terms proportional to q_μ are often dropped right from the start with the argument that they will not contribute when the vertex is contracted with the conserved electron current, an argument that only applies in certain gauges [21,22].

Actually, for the interaction of a pion with a *real* photon this term can indeed be ignored when calculating an amplitude. Since $\epsilon \cdot q = 0$ and $q^2 = 0$, one only needs

$$\Gamma_{q^2=0}^\mu(p', p) = e(p' + p)^\mu f(p'^2, p^2) \quad (14)$$

$$= e(p' + p)^\mu F(0, p'^2, p^2) \quad (15)$$

$$= e_\pi(p' + p)^\mu \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2}. \quad (16)$$

Thus by using Lorentz symmetry and gauge invariance, the electromagnetic vertex needed for real photons is entirely determined in terms of the inverse meson propagator. This includes the dependence of the vertex on the off-shell variables p^2 and p'^2 . The situation is entirely different for virtual photons: knowledge of Δ is not sufficient, the function F depends also on q^2 and the term involving G must, in general, be retained.

B. The minimal-substitution vertex

We now compare the general pion vertex discussed above with the vertex Γ_{MS}^μ , which one obtains by minimal substitution (MS), an often used prescription. An extensive review of the minimal substitution can be found in Ref. [23]. The vertex is obtained by making the “substitution” $p \rightarrow p - e_\pi A$ in $-\Delta^{-1}(p)$ and then identifying—in general, via a functional derivative—the term linear in A_μ . The result can be written in terms of the finite-difference derivative of the inverse propagator

$$\begin{aligned} \Gamma_{\text{MS}}^\mu(p', p) &= e_\pi(p' + p)^\mu \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2} \\ &= e_\pi(p' + p)^\mu f(p'^2, p^2). \end{aligned} \quad (17)$$

One sees that the minimal-substitution vertex differs from the general vertex in Eq. (8) in two respects: operator structure and dependence on scalar variables. Minimal substitution generates the first term $\sim (p' + p)^\mu$, which depends only on p'^2 and p^2 . It does not produce the second, separately gauge-invariant term, which is crucial for having a vertex that depends on the scalar variable q^2 —clearly an unsatisfactory result for virtual photons. In the following we will refer to this second term as “nonminimal” in distinction to the “minimal” term of Eq. (17).

However, we stress that minimal substitution into the self-energy yields precisely the off-shell vertex for a *real* photon, in Eq. (16),

$$\Gamma_{\text{MS}}^\mu(p', p) = \Gamma_{q^2=0}^\mu(p', p). \quad (18)$$

Thus, for the interaction of a real photon with a pion, the result of the minimal substitution into the self-energy and of a microscopic Lorentz- and gauge-invariant calculation are identical, as long as the underlying dynamics is dealt with consistently. However, this finding at $q^2 = 0$ is an exception as we will see in the following section.

C. Electromagnetic vertex of the nucleon

In this section, we derive similarities and differences for the general electromagnetic vertex of a spin-1/2 particle, such as the nucleon.

In contrast to earlier work [4,5,18], we choose for our purpose the form [24]

$$\begin{aligned} \Gamma^\mu(p', p) &= e \left(1 - \frac{q^2}{4M^2} \right)^{-1} \sum_{i,j=+,-} \Lambda_i(p') \\ &\quad \times \left(\frac{P^\mu}{M} G_{ij}^E + N^\mu G_{ij}^M + \frac{q^\mu}{2M} G_{ij}^q \right) \Lambda_j(p), \end{aligned} \quad (19)$$

where we have used

$$N^\mu = \frac{\gamma^\mu \not{P} \not{q} - \not{q} \not{P} \gamma^\mu}{4M^2}, \quad P = \frac{p' + p}{2}, \quad (20)$$

with $q \cdot N = 0$, and the projection operators

$$\Lambda_{\pm}(p) = \frac{W \pm \not{p}}{2W}, \quad W = \sqrt{p^2}. \quad (21)$$

The $G_{ij}^{E,M,q}$ —the 12 form factors that characterize the off-shell vertex—are scalar functions of q^2 , p'^2 , and p^2 . For simplicity we suppress charge (or isospin) indices.

Similar to the pion vertex, the functions multiplying the term proportional to q^μ must vanish for $p^2 = p'^2$ and it is convenient to introduce [cf. Eq. (7)]

$$G_{ij}^q(q^2, p'^2, p^2) = (p'^2 - p^2) g_{ij}^q(q^2, p'^2, p^2). \quad (22)$$

With this property, it is easily seen that the on-shell matrix element of Eq. (19) yields the well-known parametrization of the free current involving the Sachs form factors G_E and G_M .

Further restrictions for the vertex arise from the WT identity

$$q_\mu \Gamma^\mu(p', p) = e_N [S^{-1}(p') - S^{-1}(p)], \quad (23)$$

where $e_N = e \hat{e}_N$ is the charge of the nucleon ($\hat{e}_p = 1$, $\hat{e}_n = 0$) and S^{-1} is the inverse propagator of the nucleon, which we parametrize as

$$S^{-1}(p) = A(p^2) + \frac{\not{p}}{W} B(p^2). \quad (24)$$

Projecting out terms in the WT identity by using the operators $\Lambda_{\pm}$, we obtain [21]

$$\begin{aligned} & G_{ij}^E(q^2, p'^2, p^2) + q^2 g_{ij}^q(q^2, p'^2, p^2) \\ &= \frac{2M}{p'^2 - p^2} \left(1 - \frac{q^2}{4M^2} \right) \hat{e}_N \\ & \quad \times [A(p'^2) - A(p^2) + iB(p'^2) - jB(p^2)]. \end{aligned} \quad (25)$$

The WT identity thus makes it possible to eliminate G_{ij}^E , yielding a general covariant nucleon vertex in a form similar to Eq. (8),

$$\begin{aligned} \Gamma^\mu(p', p) &= e_N (p' + p)^\mu \frac{S^{-1}(p') - S^{-1}(p)}{p'^2 - p^2} \\ &+ e \left(1 - \frac{q^2}{4M^2} \right)^{-1} \sum_{i,j=+,-} \Lambda_i(p') \\ &\times \left[N^\mu G_{ij}^M(q^2, p'^2, p^2) + (q^\mu q^\nu - g^{\mu\nu} q^2) \right. \\ &\quad \left. \times \frac{P_\nu}{M} g_{ij}^q(q^2, p'^2, p^2) \right] \Lambda_j(p). \end{aligned} \quad (26)$$

Just as for the pion vertex in Eq. (8), the form that we have chosen for the nucleon vertex allows us to draw two general conclusions: the first part is again independent of q^2 and through the WT identity completely determined in terms

of the inverse propagator, including its off-shell dependence. The q^2 dependence of the vertex resides entirely in the second part, which is separately gauge invariant, and contains an off-shell dependence not determined by the dressed propagator S . The WT identity again cannot be used to further constrain this q^2 dependence. In contrast to the pion, however, we see that the WT identity is not sufficient to determine the vertex for a real photon. While $G_{\pm\pm}^E(0, p'^2, p^2)$, can be expressed through properties of the nucleon propagator through Eq. (25), the unconstrained, separately gauge-invariant term proportional to N_μ in Eq. (19) does contribute for a real photon.

Minimal substitution generates again only an off-shell dependence related to the nucleon propagator. As was pointed out in Ref. [23], due to the noncommutativity of the γ matrices the minimal-substitution procedure is not unique; different results differ by gauge-invariant terms. None of the recipes leads to a q^2 dependence, an obvious shortcoming when applying minimal substitution to virtual photons.

We finally note in passing that at the tree level with free propagators, $S^{-1} = \not{p} - m$, the first term yields

$$e_N \frac{(p' + p)^\mu \not{q}}{(p' + p) \cdot q} \quad (27)$$

rather than the usual $e_N \gamma^\mu$. The two expressions can be seen to differ by a separately gauge-invariant term.

D. The pion form factor in pion electroproduction

The standard calculations of pion electroproduction in the literature are based on pole terms or Born diagrams. All claim to be “gauge invariant,” meaning that the virtual photon couples to a conserved total hadronic current, which in most cases is achieved by using *ad hoc* recipes. The strict use of the WT identity leads to a completely different, physically more meaningful implementation of “gauge invariance” than in these calculations.

Among the Born terms, the pion-pole term is the only one where the electromagnetic form factor of the pion appears. Since this term is not separately gauge invariant, gauge invariance of the total amplitude must be achieved through cancellation with other terms. To make this cancellation with other diagrams, which involve *nucleon* form factors, possible, one is forced to assume a relation between these form factors, namely,

$$F_1^V(q^2) = F_\pi(q^2), \quad (28)$$

where F_1^V is the electromagnetic isovector form factor of the nucleon. Of course, this assumption about the form factors for the sake of gauge invariance is in contrast with our findings about the implications of the WT identity for the q^2 dependence of pion and nucleon vertices above. In a similar vein, both the Reggeized pion-pole term and the nucleon-pole term were described through the same form factor F_π to ensure gauge invariance in Refs. [12,13], which was the basis for the form factor determination of Ref. [3]. Clearly such assumptions, which are commonly made (see, e.g., Refs.

FIG. 2. Pion-pole term contributing to charged pion electroproduction.

[10,11])—unless they can be put on a firm physical basis and assessed quantitatively—should be avoided when extracting detailed information such as a pion form factor from electroproduction experiments.

In contrast, a description that consistently uses a dressed pion propagator and vertex, in agreement with the WT identity, does not have to assume that the pion and nucleon form factors are equal. The intermediate pion in the pion-pole term is half off-shell (see Fig. 2). When one tests gauge invariance and contracts the pion-pole term with the photon momentum, it is easily seen from Eq. (4) that, with $p'^2 = m^2$ and $\tilde{p}$ the momentum of the intermediate pion, one obtains

$$q_\mu \Gamma^\mu(p', \tilde{p}) \Delta(\tilde{p}) = e_\pi [\Delta^{-1}(p') - \Delta^{-1}(\tilde{p})] \Delta(\tilde{p}) = -e_\pi. \quad (29)$$

The pion-pole term, in general, depends on $\tilde{p}^2$ and q^2 . However, the above observation proves that its divergence, when internal structure is consistently included in pion vertex and propagator, is independent of q^2 and of any detailed electromagnetic properties of the pion. Analogous statements hold for the nucleon-pole terms and the mutual cancellation of pion- and nucleon-pole terms is obtained without any assumptions about the different form factors. What the WT identity obviously tells us is that we must simultaneously and consistently deal with both the electromagnetic vertex and the propagator of the same hadron to arrive at a gauge-invariant description. (Of course, the pion-pole term is also the only term involving the strong πNN form factor for an off-shell pion. We refer to, e.g., Refs. [18,25] for a discussion how to deal consistently with this aspect.)

III. REAL COMPTON SCATTERING ON A MESON

We now consider real Compton scattering (RCS) on a meson, e.g., a π^+ meson,

$$\gamma(q, \epsilon) + \pi^+(p) \rightarrow \gamma(q', \epsilon') + \pi^+(p'). \quad (30)$$

The invariant amplitude can be written as

$$\mathcal{M} = -i \epsilon_\mu \epsilon'_\nu {}^* M^{\mu\nu} = -i \epsilon_\mu \epsilon'_\nu {}^* (M_A^{\mu\nu} + M_B^{\mu\nu}), \quad (31)$$

where we have divided the total amplitude into the most general s - and u -channel meson-pole terms (class A) and the rest (class B) [8].¹ Class B contains all terms that cannot be reduced to a form where only a pion propagator connects the two electromagnetic vertices of the pion. It thus contains irreducible diagrams involving the pion as well as the contributions from intermediate states other than the pion.

The “off-shell effects” we discussed in the preceding section are contained in the class-A terms. Here we discuss how they enter into the general Compton amplitude. Our discussion will mainly be based on Lorentz and gauge invariance, but uses also the discrete symmetries P , T , and C as well as crossing symmetries. We have to include the irreducible meson terms contained in the class B into this discussion. For these terms a separate WT identity holds, which relates it to the electromagnetic meson vertex. As we will see below, the off-shell dependence in one of the two vertices is canceled directly by the off-shell meson propagator. This was the starting point of the work by Kaloshin [26], who arrived at the conclusion that in the end *all* off-shell effects in class A necessarily cancel. We will show that this claim is not true. Finally, we will also compare the general form of the RCS amplitude with the amplitude obtained from the minimal-substitution prescription.

A. The general structure of the RCS tensor

The class-A part of the Compton tensor $M^{\mu\nu}$ has the general form [8,18,27,28]

$$M_A^{\mu\nu} = \Gamma^\nu(p', p' + q') \Delta(p' + q') \Gamma^\mu(p + q, p) + \Gamma^\mu(p', p' - q) \Delta(p' - q) \Gamma^\nu(p - q', p). \quad (32)$$

The building blocks of this part of the tensor are Γ^μ , the renormalized one-particle-irreducible vertex, and the renormalized propagator Δ . In other words, this part involves quantities that take into account that the intermediate meson is not on its mass shell through the dressed propagator Δ and the form factor F . We now want to examine to what extent these off-shell effects contribute to RCS.

For RCS, the external particles are on their mass shell, $p^2 = p'^2 = m^2$, and for real photons we have $q^2 = q'^2 = 0$. The class-A contribution, therefore, reduces to ($e^2/4\pi \approx 1/137$)

$$M_A^{\mu\nu} = e^2 [(2P^\nu + q^\nu) F(0, m^2, s) \Delta(p' + q') (2P^\mu + q'^\mu) \times F(0, s, m^2) + (2P^\mu - q'^\mu) F(0, m^2, u) \Delta(p' - q) \times (2P^\nu - q^\nu) F(0, u, m^2)], \quad (33)$$

with $2P = p + p'$ and the Mandelstam variables

¹The neutral pion is its own antiparticle and class A vanishes in this case.

$$s = (p + q)^2, \quad t = (p - p')^2, \quad u = (p - q')^2, \\ s + t + u = 2m^2. \quad (34)$$

We have dropped all terms proportional to q^μ and q'^ν , since they do not contribute to the RCS amplitude when contracting the Compton tensor with the polarization vectors since $q \cdot \epsilon = q' \cdot \epsilon' = 0$.

The WT identity yields for the form factors in half-off-shell vertices [29]

$$F(0, s, m^2) = \Delta^{-1}(p' + q')(s - m^2)^{-1}, \\ F(0, u, m^2) = \Delta^{-1}(p' - q)(u - m^2)^{-1}. \quad (35)$$

As a result the off-shell dependence of one of the two vertices gets canceled by the presence of the dressed propagator and the class-A tensor becomes

$$M_A^{\mu\nu} = e^2 \left\{ (2P^\nu + q^\nu)(2P^\mu + q'^\mu) \right. \\ \times \frac{F(0, m^2, m^2) + [F(0, m^2, s) - F(0, m^2, m^2)]}{s - m^2} \\ + (2P^\mu - q'^\mu)(2P^\nu - q^\nu) \\ \times \left. \frac{F(0, m^2, m^2) + [F(0, m^2, u) - F(0, m^2, m^2)]}{u - m^2} \right\} \\ \equiv M_{\text{pole}}^{\mu\nu} + \Delta M^{\mu\nu}. \quad (36)$$

The total class-A tensor has been split into two parts: The first part involves on-shell properties only; this part is what one commonly refers to as the “pole terms.” Because of $F(0, m^2, m^2) = 1$, we have

$$M_{\text{pole}}^{\mu\nu} = e^2 \left[\frac{(2P^\nu + q^\nu)(2P^\mu + q'^\mu)}{s - m^2} \right. \\ \left. + \frac{(2P^\mu - q'^\mu)(2P^\nu - q^\nu)}{u - m^2} \right]. \quad (37)$$

The term $\Delta M^{\mu\nu}$, which is of interest to us, contains all off-shell contributions of class A. It is proportional to the terms between square brackets in Eq. (36)

$$\Delta M^{\mu\nu} = e^2 [(2P^\nu + q^\nu)(2P^\mu + q'^\mu)h(s) \\ + (2P^\mu - q'^\mu)(2P^\nu - q^\nu)h(u)] \\ = e^2 \{ 4[h(s) + h(u)]P^\mu P^\nu + 2[h(s) - h(u)] \\ \times (P^\mu q^\nu + q'^\mu P^\nu) + [h(s) + h(u)]q'^\mu q^\nu \}, \quad (38)$$

where we have introduced the function

$$h(z) = \frac{F(0, m^2, z) - F(0, m^2, m^2)}{z - m^2}.$$

Note that $h(z)$ is analytical for $z \rightarrow m^2$, because $F(0, m^2, m^2) = 1$.

In order to address the question to what extent the off-shell contribution $\Delta M^{\mu\nu}$ survives in the total tensor, we first rewrite $\Delta M^{\mu\nu}$ and $M_B^{\mu\nu}$ in a tensor basis that is convenient for this purpose. As shown in the Appendix, both tensors can be expanded as

$$\sum_{i=1}^4 \mathcal{T}_i^{\mu\nu} c_i = g^{\mu\nu} c_1 + P^\mu P^\nu c_2 + [4x^2 g^{\mu\nu} - 4x(P^\mu q^\nu + q'^\mu P^\nu) \\ + 4y P^\mu P^\nu] c_3 + \frac{1}{2}(y g^{\mu\nu} - q'^\mu q^\nu) c_4,$$

where the coefficients c_i are functions of two kinematical variables. The last two tensors in this expansion are separately gauge invariant, while the first two are not. Expressing $\Delta M^{\mu\nu}$ of Eq. (38) in terms of this tensor basis

$$\Delta M^{\mu\nu} = \sum_{i=1}^4 \mathcal{T}_i^{\mu\nu} \Delta a_i, \quad (39)$$

we find

$$\Delta a_1 = e^2 [F(0, m^2, s) + F(0, m^2, u) - 2], \\ \Delta a_2 = -\frac{2e^2}{x} [F(0, u, m^2) - F(0, m^2, s)], \quad (40)$$

where $x = \frac{1}{4}(s - u)$. For the coefficients of the separately gauge-invariant terms we obtain

$$\Delta a_3 = -\frac{e^2}{2x} [h(s) - h(u)], \\ \Delta a_4 = -2e^2 [h(s) + h(u)]. \quad (41)$$

Let us now turn to class B. Using the Ward-Takahashi identity Eq. (4), one can show that gauge invariance of the total tensor $M^{\mu\nu}$ leads to constraints for the class-B part [18, 27, 28]

$$q_\mu M_B^{\mu\nu} = e [\Gamma^\nu(p' - q, p) - \Gamma^\nu(p', p + q)],$$

$$q'_\nu M_B^{\mu\nu} = e [-\Gamma^\mu(p' + q', p) + \Gamma^\mu(p', p - q')], \quad (42)$$

which relates this part of the tensor to the vertex. Expanding the class-B tensor in the same basis

$$M_B^{\mu\nu} = \sum_{i=1}^4 \mathcal{T}_i^{\mu\nu} b_i, \quad (43)$$

and contracting with q_μ , we obtain

$$q_\mu M_B^{\mu\nu} = b_1 q^\nu + x b_2 P^\nu. \quad (44)$$

Using the real photon vertex Eq. (15), we have from Eq. (42)

$$q_\mu M_B^{\mu\nu} = e^2 \{ 2P^\nu [F(0, u, m^2) - F(0, m^2, s)] \\ - q^\nu [F(0, u, m^2) + F(0, m^2, s)] \}, \quad (45)$$

where we have dropped terms proportional to q'^ν to be consistent with the steps for the class- B parametrization which led to Eq. (44). By comparing Eqs. (44) and (45), we obtain

$$\begin{aligned} b_1 &= -e^2[F(0,u,m^2) + F(0,m^2,s)], \\ b_2 &= \frac{2e^2}{x}[F(0,u,m^2) - F(0,m^2,s)]. \end{aligned} \quad (46)$$

Since $F(0,u,m^2) - F(0,m^2,s)$ is an odd function of x , there is no singularity for $x \rightarrow 0$ in Eq. (46). Using the second condition of Eq. (42) leads to the same results for b_1 and b_2 [27]. Gauge invariance, of course, yields no constraints for b_3 and b_4 .

When we now combine the contributions from the off-shell tensor $\Delta M^{\mu\nu}$ and the class- B part $M_B^{\mu\nu}$, we obtain for the coefficient of the first tensor

$$\Delta a_1 + b_1 = -2e^2, \quad (47)$$

i.e., all off-shell dependence has canceled and we are left with a “seagull term” that is required by gauge invariance. Analogously, we find

$$\Delta a_2 + b_2 = 0, \quad (48)$$

where again the off-shell dependence introduced through the class- A term has been canceled. No such statement can be derived for the coefficients of the two separately gauge-invariant tensor structures, $T_3^{\mu\nu}$ and $T_4^{\mu\nu}$, which are not constrained by the WT identity.

In summary, the above procedure has yielded the total result

$$\begin{aligned} M^{\mu\nu} &= M_{\text{pole}}^{\mu\nu} + \Delta M^{\mu\nu} + M_B^{\mu\nu} \\ &= M_{\text{pole}}^{\mu\nu} - 2e^2 g^{\mu\nu} + T_3^{\mu\nu}(\Delta a_3 + b_3) + T_4^{\mu\nu}(\Delta a_4 + b_4) \\ &\equiv M_{\text{Born}}^{\mu\nu} + T_3^{\mu\nu}(\Delta a_3 + b_3) + T_4^{\mu\nu}(\Delta a_4 + b_4). \end{aligned} \quad (49)$$

Here,

$$M_{\text{Born}}^{\mu\nu} = M_{\text{pole}}^{\mu\nu} - 2e^2 g^{\mu\nu}$$

is the gauge-invariant tensor for a “point particle,” commonly denoted as the Born terms. It contains the tree-level diagrams of scalar QED without any off-shell effects in vertices or propagators. The remainder

$$(\Delta a_3 + b_3)T_3^{\mu\nu} + (\Delta a_4 + b_4)T_4^{\mu\nu}$$

are the separately gauge-invariant parts. It is in these gauge-invariant terms where the off-shell effects Δa_3 and Δa_4 enter, together with the class- B contributions. From the point of view of gauge invariance and the symmetries we have used above, there is clearly no requirement that all “off-shell” contributions of the most general class- A terms are canceled by class B . It is only true for the $g^{\mu\nu}$ and $P^\mu P^\nu$ structures, where a cancellation of off-shell effects *must* occur [see Eqs. (47) and (48)]. This type of cancellation has been observed

in other reactions. Examples are pion photoproduction on the nucleon [30] and NN bremsstrahlung [31].

As has been discussed by Fearing and Scherer [32–34], there is no absolute and unique meaning to the “off-shell effects”; they are not observables. They depend not only on the microscopic model, but also on the representation one chooses for the interpolating fields of the (intermediate) particles [32–35]. By changing the representation of the meson field, off-shell effects can be transformed from class- A into class- B terms and vice versa. When carrying out such a transformation to another representation, the combined values of $(\Delta a_3 + b_3)$ and $(\Delta a_4 + b_4)$ will not change, but the splitting into “off-shell” and class- B contributions will change. In particular, this means that, e.g., the electromagnetic polarizabilities one defines from separately gauge-invariant terms will, in a general covariant framework, receive contributions from both the off-shell behavior of the meson *and* irreducible class- B terms involving the meson or other intermediate particles.

In principle, there is a “canonical representation,” where it is possible to work with free electromagnetic form factors and propagators [36]. The price one pays for this is, of course, the presence of a number of irreducible reaction-specific “contact” terms. This representation is particularly convenient for general purposes, such as the derivation of low-energy theorems for (virtual) Compton scattering [27,36]. However, the possibility to move off-shell effects into contact terms makes assumptions about the analyticity of the amplitude and faces difficulties above production thresholds [32].

Let us finally comment on Ref. [26], where it was argued that in general *all* off-shell effects will necessarily be canceled. This total cancellation was inferred from a one-loop calculation involving pions and a sigma as dynamical degrees of freedom by investigating the s -wave $\pi\sigma$ intermediate state in the direct channel. However, angular momentum conservation forbids a net result for $J=0$ in real Compton scattering, no matter where it originates from. In other words, the absence of off-shell effects in the s wave can only serve as a consistency check of the calculation, but is no proof of a total cancellation.

B. The minimal substitution RCS amplitude

We now turn to the possibility of generating a RCS amplitude through minimal substitution. As we have shown above, the minimal-substitution vertex and the general vertex are the same for a real photon. Thus the exact class- A term and the class- A amplitude using the vertex generated by minimal substitution are the same. However, the class- A term by itself is not gauge invariant, only the sum of the class- A and class- B amplitudes. One can derive this class- B amplitude by a microscopic calculation, consistent with the calculations that yielded the class- A amplitude. However, we can also obtain a “minimal” class- B term $M_{\text{B,MS}}^{\mu\nu}$ by performing the minimal substitution in $-\Delta(p)$, as in Sec. II C, but now by determining the coefficients of *second* order in the electromagnetic field, i.e., by taking the second functional derivative. The amplitude corresponding to the tensor

$$M_{\text{MS}}^{\mu\nu} = M_{\text{A}}^{\mu\nu} + M_{\text{B, MS}}^{\mu\nu} \quad (50)$$

is then gauge invariant. The resulting tensor $M_{\text{B, MS}}^{\mu\nu}$ is

$$\begin{aligned} M_{\text{B, MS}}^{\mu\nu} = e^2 & \left\{ -2g^{\mu\nu} \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2} - (2P + q)^\nu \right. \\ & \times (2P + q')^\mu \frac{1}{s - p'^2} \left[\frac{\Delta^{-1}(p + q) - \Delta^{-1}(p)}{s - p^2} \right. \\ & \left. \left. - \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2} \right] - (2P - q')^\mu \right. \\ & \times (2P - q)^\nu \frac{1}{u - p^2} \left[\frac{\Delta^{-1}(p - q') - \Delta^{-1}(p')}{u - p'^2} \right. \\ & \left. \left. - \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2} \right] \right\}, \quad (51) \end{aligned}$$

which can be shown to be crossing symmetric. With the initial and final meson on mass shell

$$\begin{aligned} M_{\text{B, MS}}^{\mu\nu} = e^2 & \left\{ -2g^{\mu\nu} - (2P + q)^\nu (2P + q')^\mu \frac{1}{s - m^2} \right. \\ & \times [F(0, s, m^2) - F(0, m^2, m^2)] \\ & - (2P - q')^\mu (2P - q)^\nu \frac{1}{u - m^2} \\ & \left. \times [F(0, u, m^2) - F(0, m^2, m^2)] \right\}. \quad (52) \end{aligned}$$

By comparing to Eq. (36), we see that the last two terms cancel corresponding class-A terms and leave us with a class-A tensor for Compton scattering from an on-shell particle $M_{\text{pole}}^{\mu\nu}$ while the first term is the contact term one obtains in scalar QED. As a result, the minimal tensor is free of any “off-shell” properties and identical to the Born tensor

$$M_{\text{MS}}^{\mu\nu} = M_{\text{Born}}^{\mu\nu}. \quad (53)$$

This result for the minimal-substitution amplitude was already mentioned in Ref. [37] when deriving the virtual Compton scattering amplitude for a “Dirac proton” by minimal substitution. Again, as for the electromagnetic vertex for a real photon, this simple result for RCS from a spin-0 meson is an exception.

Minimal substitution in other cases *does* yield results with explicit reference to off-shell properties of the particles involved. An example is the minimal substitution into the πNN vertex in pion photoproduction, as was discussed by Ohta [15]. It was also examined for pion photoproduction and electroproduction by Bos *et al.* [14], who contrasted the minimal-substitution amplitude with the exact result of a model calculation.

IV. ELECTROMAGNETIC STRUCTURE IN EFFECTIVE FIELD THEORIES

Clearly a way to deal with particle structure and reaction mechanism in a consistent way and free of the shortcomings discussed above, is to start from a microscopic (effective) field theory. This will generate gauge-invariant amplitudes and take into account vertex structure, particle self-energies, etc. One may be tempted to assume that use of the minimal substitution on the Lagrangian level is a reliable procedure. Interestingly enough, this is not as simple and nonminimal terms involving field-strength tensors play an important role. To demonstrate this, we will discuss two different ways of obtaining the interaction of a charged pion with an external electromagnetic probe through minimal substitution in the context of an effective field theory—namely, chiral perturbation theory [38]. In the first procedure we start from an effective field theory with a global U(1) symmetry and calculate the renormalized propagator of the pion to one loop. We then generate a pion electromagnetic vertex via the “replacement” $p^\mu \rightarrow p^\mu - e_\pi A^\mu$ in the (negative of the) inverse of the renormalized propagator of a charged pion. This is analogous to the minimal-substitution procedure to obtain a vertex discussed in Sec. II.

In the second version, we make the minimal substitution on the Lagrangian level. We extend the same globally invariant effective field theory to include the electromagnetic interaction by replacing ordinary derivatives by appropriate covariant derivatives in the Lagrangian. With this Lagrangian we then proceed to calculate the electromagnetic vertex to one loop.

For the first method, we start from a *globally* invariant Lagrangian that contains no coupling to an electromagnetic field. The Lagrangian we use is [39]

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\text{global}} &= \mathcal{L}_2^{\text{global}} + \mathcal{L}_4^{\text{global}}, \\ \mathcal{L}_2^{\text{global}} &= \frac{F_0^2}{4} \text{Tr} [\partial_\mu U (\partial^\mu U)^\dagger + m_{\pi,2}^2 (U^\dagger + U)], \\ \mathcal{L}_4^{\text{global}} &= \frac{l_1}{4} \{\text{Tr} [\partial_\mu U (\partial^\mu U)^\dagger]\}^2 \\ &+ \frac{l_2}{4} \text{Tr} [\partial_\mu U (\partial_\nu U)^\dagger] \text{Tr} [\partial^\mu U (\partial^\nu U)^\dagger] \\ &+ \frac{l_3 m_{\pi,2}^4}{16} [\text{Tr}(U^\dagger + U)]^2 - \frac{l_7 m_{\pi,2}^4}{16} [\text{Tr}(U^\dagger - U)]^2. \end{aligned} \quad (54)$$

The Goldstone boson, i.e., pion fields are contained in an SU(2)-valued matrix U . The constant F_0 is the pion-decay constant in the chiral limit, the l_i are low-energy constants not determined by chiral symmetry. Equation (54) generates the most general strong interactions of low-energy pions at $O(p^4)$ in the quark-mass and momentum expansion including chiral symmetry breaking effects due to the quark

$$---\text{blob}--- = \frac{1}{Z} (--- + ---\text{blob} + ---\text{blob}^2 + ---\text{blob}^3 + \dots)$$

FIG. 3. Renormalized pion propagator. Z is the wave function renormalization constant.

masses. Note that in limit of a vanishing quark mass, Eq. (54) is equivalent to Eq. (2) of Weinberg's original paper on ChPT [40].

In the calculations that follow, we used the representation

$$U(x) = \frac{\sigma(x) + i\vec{\tau} \cdot \vec{\pi}(x)}{F_0}, \quad \sigma^2(x) + \vec{\pi}^2(x) = F_0^2. \quad (55)$$

The equivalence theorem guarantees that physical observables do not depend on the specific choice of parametrization of U [41]. However, separate building blocks, such as vertices and propagator, in general, exhibit different off-shell behavior depending on the choice of interpolating field [32–34].

To proceed in analogy with the discussion in Sec. II, we first use the Lagrangian in Eq. (54) to determine the renormalized propagator at $O(p^4)$ in the momentum expansion (see Figs. 3 and 4) and then generate a pion electromagnetic vertex through minimal substitution. To one loop, we obtain for the unrenormalized self-energy [29]

$$\Sigma(p^2) = A + Bp^2, \quad (56)$$

where

$$A = \frac{3}{2} \frac{m_{\pi,2}^2}{F_0^2} I(m_{\pi,2}^2, \mu^2) + 2l_3 \frac{m_{\pi,2}^4}{F_0^2}, \quad B = -\frac{I(m_{\pi,2}^2, \mu^2)}{F_0^2}, \quad (57)$$

and $I(M^2, \mu^2)$ refers to the dimensionally regularized one-loop integral

$$I(M^2, \mu^2) \equiv \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - M^2 + i0^+} \\ = \frac{M^2}{16\pi^2} \left[R + \ln\left(\frac{M^2}{\mu^2}\right) \right] + O(4-d), \quad (58)$$

$$R = \frac{2}{d-4} - [\ln(4\pi) + \Gamma'(1) + 1]. \quad (59)$$

The renormalized mass and the wave function renormalization constants, respectively to $O(p^4)$ and $O(p^2)$, are given by

$$\text{blob} = \text{loop}(2) + \text{blob}(4)$$

FIG. 4. One-particle-irreducible pion self-energy at $O(p^4)$. The vertices are derived from $\mathcal{L}_2$ and $\mathcal{L}_4$, denoted by 2 and 4 in the interaction blobs, respectively.

$$m_{\pi,4}^2 = m_{\pi,2}^2(1+B) + A, \quad (60)$$

$$Z = 1 + B. \quad (61)$$

At $O(p^4)$, the full renormalized propagator is given by

$$\Delta_R(p) = \frac{1}{Z} \frac{1}{p^2 - m_{\pi,2}^2 - \Sigma(p^2) + i0^+} = \frac{1}{p^2 - m_\pi^2 + i0^+}, \quad (62)$$

where we have replaced the $O(p^4)$ expression for the squared pion mass by the empirical value, the difference being of $O(p^6)$. Minimal substitution into the (negative) inverse propagator then leads to the following vertex of a positively charged pion:

$$\Gamma^\mu(p', p) = e(p' + p)^\mu. \quad (63)$$

Clearly, at $O(p^4)$ the minimal-substitution recipe only generates the interaction of a pointlike charged spin-0 field, without any form factors or q^2 dependence.

We now proceed according to the second method to obtain an electromagnetic vertex. We first extend the *Lagrangian* in Eq. (54) through minimal substitution²

$$\partial_\mu U \mapsto D_\mu U = \partial_\mu U + \frac{i}{2} e A_\mu [\tau_3, U]. \quad (64)$$

The calculation of the pion self-energy to order $O(p^4)$ based on this Lagrangian, of course, yields the same result as before, i.e., Eqs. (56)–(59). However, a different result for the electromagnetic vertex to order $O(p^4)$ is obtained using the “minimal Lagrangian.” The relevant diagrams are shown in Fig. 5. Note that the minimal-substitution procedure of Eq. (64) does *not* generate a tree-level contribution at $O(p^4)$, schematically shown in Fig. 6, since the candidate terms proportional to l_1 and l_2 involve at least four pion fields. The result for the *unrenormalized* one-particle-irreducible vertex reads

$$\Gamma^\mu(p', p) = e \left\{ (p' + p)^\mu \left[1 + \frac{I(m_\pi^2, \mu^2)}{F_\pi^2} \right] \right. \\ \left. + (p' + p)_\nu (q^\mu q^\nu - g^{\mu\nu} q^2) g(q^2, p'^2, p^2) \right\}, \quad (65)$$

where

²This corresponds to gauging the relevant U(1) subgroup of the global $SU(2)_L \times SU(2)_R$.

FIG. 5. One-particle-irreducible electromagnetic vertex at $O(p^4)$ obtained from the “minimal Lagrangian.” The vertices are derived from $\mathcal{L}_2$ denoted by 2 in the interaction blobs.

$$g(q^2, p'^2, p^2) = \frac{1}{6(4\pi F_\pi)^2} \left[\ln\left(\frac{m_\pi^2}{\mu^2}\right) + R + \frac{1}{3} + \left(1 - 4\frac{m_\pi^2}{q^2}\right) J^{(0)}\left(\frac{q^2}{m_\pi^2}\right) \right], \quad (66)$$

i.e., at order $O(p^4)$ does not depend on p'^2 and p^2 . Note that this result still depends on the renormalization scale μ . The (infinite) constant R was defined in Eq. (59) and $J^{(0)}(x)$ is the well-known integral

$$J^{(0)}(x) = \int_0^1 dy \ln[1 + x(y^2 - y) - i0^+].$$

All quantities at $O(p^4)$ have been replaced by their empirical values ($F_\pi = 92.4$ MeV).

Clearly with this procedure, based on the Lagrangian generated through minimal substitution, we now do obtain a vertex with internal structure. In particular, through the presence of the separately gauge-invariant term involving the function g we now do have a dependence on q^2 . The vertex obtained in this way is thus completely different from the first method, resulting in Eq. (63).

However, there remains a problem. After multiplication of the unrenormalized vertex, Eq. (65), with the wave function renormalization constant Z , Eq. (61), the result still contains infinite contributions proportional to R , even for on-shell pions. Only at the real-photon point does the result have the correct normalization, satisfying the Ward-Takahashi identity in combination with the propagator of Eq. (62). For $q^2 \neq 0$, these infinite terms as well as terms that depend on the renormalization scale remain.

In order to solve this puzzle, we observe that the most general *locally invariant* effective Lagrangian, which includes the coupling to an electromagnetic field, at $O(p^4)$ necessarily also contains “nonminimal terms” involving

FIG. 6. Generic tree-level contribution at $O(p^4)$.

field-strength tensors, such as the l_5 and l_6 structures of the effective Lagrangian of Gasser and Leutwyler [38]

$$\begin{aligned} \mathcal{L}_4^{\text{GL}} = & \dots + l_5 \left[\text{Tr}(F_{\mu\nu}^R U F_L^{\mu\nu} U^\dagger) - \frac{1}{2} \text{Tr}(F_{\mu\nu}^L F_L^{\mu\nu} \right. \\ & \left. + F_{\mu\nu}^R F_R^{\mu\nu}) \right] + i \frac{l_6}{2} \text{Tr}[F_{\mu\nu}^R D^\mu U (D^\nu U)^\dagger \\ & + F_{\mu\nu}^L (D^\mu U)^\dagger D^\nu U] + \dots \end{aligned} \quad (67)$$

Inserting the relevant expression for the field strength tensors

$$F_L^{\mu\nu} = F_R^{\mu\nu} = -\frac{e}{2} \tau_3 (\partial^\mu A^\nu - \partial^\nu A^\mu), \quad (68)$$

the l_6 term of Eq. (67) generates an additional separately gauge-invariant contact contribution (see Fig. 6)

$$\begin{aligned} \Delta \Gamma^\mu(p', p) = & e \left[-(p' + p)^\mu \frac{q^2}{F_\pi^2} l_6 + (p' - p)^\mu \frac{(p'^2 - p^2)}{F_\pi^2} l_6 \right] \\ = & e (p' + p)_\nu l_6 \frac{q^\mu q^\nu - g^{\mu\nu} q^2}{F_\pi^2}. \end{aligned} \quad (69)$$

Adding the contributions from Eqs. (65) and (69) and multiplying the result by the wave function renormalization constant yields [42]

$$\begin{aligned} \Gamma_R^\mu(p', p) = & e \left\{ (p' + p)^\mu F(q^2) \right. \\ & \left. + (p' - p)^\mu \frac{p'^2 - p^2}{q^2} [1 - F(q^2)] \right\}, \end{aligned} \quad (70)$$

with the form factor $F(q^2)$ at $O(p^4)$ [see Eq. (15.3) of Ref. [38]]

$$\begin{aligned} F(q^2) = & 1 - l_6^r \frac{q^2}{F_\pi^2} - \frac{1}{6} \frac{q^2}{(4\pi F_\pi)^2} \left[\ln\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{1}{3} \right. \\ & \left. + \left(1 - 4\frac{m_\pi^2}{q^2}\right) J^{(0)}\left(\frac{q^2}{m_\pi^2}\right) \right], \end{aligned} \quad (71)$$

where we introduced $l_6^r = l_6 + R/96\pi^2$. To this order, the electromagnetic form factor shows no off-shell dependence. The explicit dependence on the renormalization scale μ cancels with a corresponding scale dependence of the parameter l_6^r . Clearly, the vertex in Eq. (70) and the propagator in Eq. (62) now satisfy the Ward-Takahashi identity for arbitrary q^2 .

In the discussion in this section, the first method started out from a globally invariant effective Lagrangian $\mathcal{L}_{\text{eff}}^{\text{global}}$ that was used to obtain the renormalized propagator. Minimal substitution into the inverse of this propagator then yields to order $O(p^4)$ an electromagnetic vertex that does not reflect the structure of the pion through a q^2 dependence. This method only served to illustrate the general discussion in Sec. II in the context of chiral perturbation theory.

The second method, based on a Lagrangian obtained through minimal substitution into $\mathcal{L}_{\text{eff}}^{\text{global}}$, when used to the same order $O(p^4)$, does yield a vertex with q^2 -dependent form factors. However, we saw that it does not lead to a consistent order-by-order renormalizable theory. Already at the one-loop level nonminimal terms, i.e., terms not generated through minimal substitution in the globally invariant Lagrangian, are mandatory. This is a specific example of a general result obtained by Leutwyler [43]: investigating Green functions and their Ward identities that express the symmetries of the underlying theory requires, in the framework of effective field theory, the most general *locally* invariant Lagrangian (even if the symmetry results from a global underlying symmetry) [43]. In the present context, the ChPT Lagrangian leads to separately gauge-invariant contributions to the vertex that absorb divergences appearing in one-loop calculations of electromagnetic processes.

The above example has shown that a meaningful calculation of the electromagnetic structure of a hadron in an *effective* field theory needs nonminimal terms. These are not generated by minimal substitution on the Lagrangian level. The situation would be different in a truly fundamental theory based on pointlike particles, such as the standard model, where the underlying electromagnetic coupling of bosons and fermions is minimal.

V. SUMMARY AND CONCLUSIONS

Experiments are presently being carried out at the modern electron accelerators to investigate details of the structure of hadrons and to examine microscopic aspects of the reaction mechanism. The majority of the theoretical descriptions of these reactions, such as pion electroproduction or (virtual) Compton scattering on a nucleon, are phenomenological. They are based on Born-term or pole diagrams that use free hadron properties. Various improvements are added, such as final-state interactions, resonance terms and form factors.

There are several shortcomings of this approach. The two-step reactions under consideration necessarily involve intermediate particles. Microscopic models will yield for their strong and electromagnetic vertices a much more general structure than that of a free hadron. Concomitantly, also the propagators must have a self-energy. And, last but not least, there will be a new class of irreducible diagrams for each reaction, which do not occur in the pole terms.

Not surprisingly the commonly used approaches therefore encounter problems when form factors are introduced into the reaction amplitudes: the resulting expressions are not gauge invariant. It has been customary to deal with this particular problem by invoking *ad hoc* recipes. Examples are restrictive assumptions about the individual form factors—even though these are the objects one wants to study—or the addition of terms that make the amplitude gauge invariant.

While the approaches based on (extended) Born-term amplitudes are often simple and intuitively appealing, they should be improved in view of the detailed questions one wants to answer through the interpretation of the measurements with the new generation of electron accelerators. We, therefore, studied here general features of the theoretical de-

scription of electromagnetic reactions, focussing, in particular, on the requirement of gauge invariance. When this condition is applied in its stronger form, the WT identity, it leads to several general statements about the vertices and their form factors. We looked at these consequences in some detail, since they put the different recipes commonly used into perspective and show that a consistent microscopic treatment of all aspects of the reaction is required.

We started by discussing the general structure of the electromagnetic vertex of hadrons with spin 0 and 1/2, and the restrictions imposed by covariance and gauge invariance. By using the WT identity, we showed that the most general form of these vertices, which is manifestly covariant, has two features in common. First, there is a q^2 -independent term related through gauge invariance to the hadron propagator. Its off-shell dependence is determined by the dressed propagator. A second and very important feature we pointed out is that the q^2 dependence of the pion and nucleon vertices resides entirely in separately gauge-invariant terms, and thus is *not* related to the propagator or otherwise constrained by the WT identity. This new insight clearly identifies a serious shortcoming of the commonly used *ad hoc* recipes that make assumptions about the q^2 dependence of form factors for the sake of making a reaction amplitude gauge invariant.

In phenomenological models a frequently used prescription to obtain a gauge-invariant amplitude is the “minimal substitution.” For example, given the self-energy of a particle, it allows one to obtain a vertex that will satisfy the Ward-Takahashi identity. It is *a priori* clear that this method can only yield a very limited result: it is blind to neutral particles and, for example, cannot provide an electromagnetic coupling to a neutron. But even where it does yield a result, we showed that it is restricted in its operator structure and the dependence on scalar variables. We found that the pion vertex for a *real* photon is an exception. Minimal substitution into the pion propagator yields the exact result, including the dependence on the invariant mass of the off-shell pions. As we saw, this is because the “nonminimal” term in the vertex does not contribute for a real photon. For virtual photons, a nonminimal term can contribute, but the minimal substitution, of course, fails to produce it. As a result this method predicts no dependence on q^2 ; this latter statement applies also to the spin-1/2 electromagnetic vertex. Nevertheless, minimal substitution has been used in some instances for virtual photons, see, e.g., [37].

Another often used method to enforce gauge invariance in the case of pion electroproduction is to assume that the pion electromagnetic form factor and the nucleon isovector form factor are identical, $F_1^V(q^2) = F_\pi(q^2)$, or to assume the pion form factor that applies to the t channel also can be used for the s or u channel. This is clearly an assumption one wants to avoid when trying to extract the pion electromagnetic form factor from pion electroproduction. This kind of problem can be expected in all reactions involving different hadrons. We showed how a proper treatment of vertex and propagator makes these assumptions unnecessary.

After the discussion of the electromagnetic vertex of a hadron, we turned to the description of two-step processes, using real Compton scattering off a pion as an example and

examined how the general structure of vertices and propagators carries through to the total amplitude. We showed how for real photons one arrives through the requirement of gauge invariance and of discrete symmetries, at an amplitude consisting of the Born amplitude of a point particle and a separately gauge invariant term. The Born term is free of any reference to off-shell properties and was shown to be produced by the minimal substitution into the pion self-energy. Off-shell effects were seen to enter through the separately gauge-invariant terms. Due to photon-crossing symmetry, these off-shell contributions start appearing in the Compton amplitude only in second order, the polarizabilities. (For a discussion of virtual Compton scattering and systematics of the q^2 dependence we refer to Refs. [27,28].) We stressed that these higher-order terms contain “off-shell behavior” of the pion *as well as* contributions from intermediate states other than a single pion. The total cancellation of off-shell effects observed in the $J=0$ channel in Ref. [26] does not prove in general a complete cancellation of off-shell effects. It only occurs since due to angular momentum conservation there is no net $J=0$ contribution at all. Finally, we stressed that “off-shell effects” cannot uniquely be associated with features of a vertex or a propagator; they can, for example, be shifted from a vertex to reaction-specific contact terms of an effective theory.

A logical way to deal consistently with hadron structure in vertices, propagators, and irreducible contributions or contact terms, is by starting out from a (effective) field theory. We demonstrated this by considering the pion electromagnetic vertex to one loop in chiral perturbation theory. The Ward-Takahashi identity is satisfied in a simple fashion. An important point is that even on this level nonminimal terms are crucial and that the minimal substitution leads to an *inconsistent* theory. This observation again signals the dangers of working with the minimal substitution on the phenomenological level.

We have here focused mainly on hadron structure in connection with the electromagnetic interaction. Clearly, analogous considerations about the general vertex structure also apply to the strong vertices. This has received less attention in the literature and most of the prescriptions invented for reactions involving extended, off-shell hadrons have been designed to ensure a gauge-invariant amplitude. The microscopic field theoretical approaches now being used for strong interaction processes, such as πN [44–47] or NN [48–51] scattering can be extended to a meaningful interpretation also of the electromagnetic reactions considered here, allowing one to consistently deal with all aspects of internal structure and making the introduction of *ad hoc* assumptions superfluous.

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APPENDIX: THE TENSOR FOR COMPTON SCATTERING OFF A SPIN-0 PARTICLE

In general, the Compton tensor of a spin-0 particle can be expressed as

$$\begin{aligned} M^{\mu\nu} &= \sum_{i=1}^{10} M_i T_i^{\mu\nu} \\ &= M_1 g^{\mu\nu} + M_2 P^\mu P^\nu + M_3 (P^\mu q^\nu - q'^\mu P^\nu) \\ &\quad + M_4 (P^\mu q'^\nu + q'^\mu P^\nu) + M_5 (P^\mu q'^\nu - q^\mu P^\nu) \\ &\quad + M_6 (P^\mu q'^\nu + q^\mu P^\nu) + M_7 (q^\mu q'^\nu + q'^\mu q^\nu) \\ &\quad + M_8 (q^\mu q^\nu - q'^\mu q'^\nu) + M_9 q^\mu q'^\nu + M_{10} q'^\mu q^\nu, \end{aligned} \quad (A1)$$

where for real Compton scattering, the functions M_i depend on two kinematical invariants, which we choose as

$$x = \frac{1}{2}(q + q') \cdot P, \quad (A2)$$

$$y = q \cdot q'. \quad (A3)$$

Note that for $p^2 = p'^2 = m^2$ one has $x = q \cdot P = q' \cdot P$. The tensor we are considering must be symmetric under photon crossing

$$q \leftrightarrow -q', \mu \leftrightarrow \nu, \quad (A4)$$

which also implies $s \leftrightarrow u$ and $t \leftrightarrow t$. As a result, we have for the functions M_i that

$$\begin{aligned} M_i(x, y) &= \pm M_i(-x, y), \quad + : i = 1, 2, 3, 5, 7, 9, 10, \\ &\quad - : i = 4, 6, 8. \end{aligned} \quad (A5)$$

Invariance under pion crossing in combination with charge-conjugation invariance [52]

$$M^{\mu\nu}(P, q, q') = M^{\mu\nu}(-P, q, q') \quad (A6)$$

leads to

$$\begin{aligned} M_i(x, y) &= \pm M_i(-x, y), \quad + : i = 1, 2, 7, 8, 9, 10, \\ &\quad - : i = 3, 4, 5, 6. \end{aligned} \quad (A7)$$

The combination of Eqs. (A5) and (A7) then yields

$$M_3 = M_5 = M_8 = 0. \quad (A8)$$

Furthermore, we can extract appropriate powers of x such that the invariant amplitudes are functions of x^2 only. For real photons, terms proportional to either q^μ or q'^ν can be omitted and finally the tensor can thus be written as

$$M^{\mu\nu} = c_1 g^{\mu\nu} + c_2 P^\mu P^\nu + c_3 x (P^\mu q^\nu + q'^\mu P^\nu) + c_4 q'^\mu q^\nu, \quad (A9)$$

where $c_i = c_i(x^2, y)$. We can rearrange Eq. (A9) in terms of two structures that are not gauge invariant and two that are separately gauge invariant

$$\begin{aligned}
M^{\mu\nu} &= \sum_{i=1}^4 a_i T_i^{\mu\nu} \\
&= a_1 g^{\mu\nu} + a_2 P^\mu P^\nu + a_3 [4x^2 g^{\mu\nu} - 4x(P^\mu q^\nu + q'^\mu P^\nu) \\
&\quad + 4y P^\mu P^\nu] + a_4 \frac{1}{2} (y g^{\mu\nu} - q'^\mu q^\nu). \quad (A10)
\end{aligned}$$

Since the total Compton scattering amplitude is symmetric under photon crossing and $M_A^{\mu\nu}$ is explicitly crossing symmetric, so must $M_B^{\mu\nu}$ also be. Similarly, since the pole terms $M_{pole}^{\mu\nu}$ are crossing symmetric, also $\Delta M^{\mu\nu}$ must have this property. By the analogous reasoning as for $M^{\mu\nu}$ above, it can be seen that the same tensors provide also an appropriate basis for the expansion of $\Delta M^{\mu\nu}$ and $M_B^{\mu\nu}$.

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