Chapter 4

The principles

In Chapter 2 we introduced and discussed the meaning of the principles of the preservativity logic of HA known so far. In this and the next chapter we consider these principles from a modal point of view. In this chapter we study them separately and in the next chapter together. Here we describe to which frame properties the principles correspond and prove that all principles but Löb’s Preservativity Principle are canonical. Since every canonical logic is complete (Section 3.4.3), this implies that besides Löb’s Preservativity Principle, all these principles are complete with respect to a certain class of frames. Except for Löb’s Preservativity Principle and Visser’s Scheme, we show that all these principles have the finite model property as well, i.e. they are complete with respect to a certain class of finite frames. For the study of classical modal logics via frame characterizations and the like, we refer the reader to (van Bentham 1983)(van Bentham 1984)(Chagrov and Zakharyaschev 1997)(Blackburn, de Rijke and Venema 2001).

In Section 4.4 we show that iLe is conservative over iP4 with respect to formulas in L□. Thus in the absence of other principles, the Disjunctive Principle does not capture more of the Disjunction Property than Leivant’s Principle (compare the discussion on the Disjunctive Principle in Section 2.3). However, in the next chapter we will see that this no longer holds in the presence of principles like the Formalized Markov Scheme. Namely, in Section 5.4, we show that iPH derives □((□A → A) ∨ ¬¬□B) → □(A ∨ □B), while the logic iH does not derives this principle, although it contains Leivant’s Principle and the Formalized Markov Scheme.

We will see in Section 4.7 that besides the principles Vp_n, none of the preservativity principles derive one another, and that the same holds for all provability principles. In Section 4.6.1 we show that Vp_m does not derive Vp_n for n > m. However, sometimes two principles interfere in a different way. For example, Montagna’s Principle and Visser’s Scheme are both canonical, i.e. their canonical models have respectively the Ma- and the Vp∞-property to which these principles correspond. But the canonical model for the logic iP MV given by both these principles has
a stronger frame property than just these two properties, as will be shown in Corollary 4.6.3.

The results that will be used in Chapter 5 in the completeness proof for the logic iP\(\Phi\) given by all principles together, are the correspondence for the principle \(Lp\) (Lemma 4.3.1), the canonicity of the logics iP4 (Proposition 4.2.1) and iP\(\Phi\)M (Proposition 4.2.1), and the mentioned completeness proof for the logic iP\(\Psi\) (Corollary 4.6.3). In Chapter 5 we also give a completeness proof for the logic iP\(\Omega\) given by the first known principles of the provability logic of HA. There we use the following results from this chapter: the completeness proof for the logic iP\(\Lambda\) (Proposition 4.6.7), and the completeness proof with respect to finite frames for the logic iP\(\Xi\le\) (Proposition 4.4.1).

In Section 4.1 we show that the base logics iP and iK are complete with respect to their given frame semantics. The completeness proofs for iP, iP4 and iP\(\Lambda\) are similar to the ones in classical modal logic. The proofs for iK, iP4 and iP\(\Lambda\) occur already in the literature (Božić and Dösen 1984)(Kirov 1984)(Ursini 1979b). We treat a preservativity principle and its corresponding counterpart, like \(Lp\) and \(L\), in one Section. The only exception is Leivant’s Principle. Although it is derivable from 4\(p\) we treat it in a separate section because in this way all ‘standard’ proofs, for iP, iK, 4\(p\), \(Lp\), 4, \(L\), precede the more interesting and non-standard proofs for \(Le\), \(Mp\), \(Vp\) and \(Ma\).

We recall the known principles of the preservativity logic of HA that were discussed in Section 2.2.

\[
\begin{align*}
\Box A \equiv_{ad} & \top \vdash A \\
P1 & A \vdash B \land B \vdash C \rightarrow A \vdash C \\
P2 & C \vdash A \land C \vdash B \rightarrow C \vdash (A \land B) \\
Dp & A \vdash C \land B \vdash C \rightarrow (A \lor B) \vdash C \quad \text{(Disjunctive Principle)} \\
4p & A \vdash \Box A \\
Lp & (\Box A \rightarrow A) \vdash A \quad \text{(Löb’s Preservativity Principle)} \\
Mp & A \vdash B \rightarrow (\Box C \rightarrow A) \vdash (\Box C \rightarrow B) \quad \text{(Montagna’s Principle)} \\
Vp_n & (\land_{i=1}^n (A_i \rightarrow B_i) \rightarrow A_{n+1} \lor A_{n+2}) \vdash (\land_{i=1}^n A_i \rightarrow B_i)(A_1, \ldots, A_{n+2}) \\
Vp & Vp_1, Vp_2, Vp_3, \ldots \quad \text{(Visser’s Scheme)}
\end{align*}
\]

The fragment of the provability logic of HA treated in Section 2.5 consists of the
4.1. The base of preservativity logic

following principles.

\[ K \quad \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \]

\[ 4 \quad \Box A \rightarrow \Box \Box A \]

\[ L \quad \Box (\Box A \rightarrow A) \rightarrow \Box A \quad \text{(Löb's Principle)} \]

\[ Le \quad \Box (A \lor B) \rightarrow \Box (A \lor \Box B) \quad \text{(Leivant's Principle)} \]

\[ Ma \quad \Box \neg \neg (\Box A \rightarrow \lor \Box B_i) \rightarrow \Box (\Box A \rightarrow \lor \Box B_i) \]

(Formalized Markov Scheme)

Preservativity logic has the rules Modus Ponens and the

Preservation Rule \[ \vdash (A \rightarrow B), \text{ then } \vdash A \rightarrow B. \]

In the case of provability logic the Preservation Rule is replaced by

Necessitation \[ A / \Box A. \]

The logic \( IP \) is given by the axioms \( Taut, Dp, P1 \) and \( P2 \). The logic \( iK \) is given by the axioms \( Taut \) and \( K \).

4.1 The base of preservativity logic

In this section we show that the frames defined in Section 3.4.2 are exactly the frames we need for the semantical base preservativity logic \( IP \) and for the base logic \( iK \) of provability logic.

4.1.1. Proposition. \( \vdash_{IP} A \) iff \( A \) is valid on all finite frames.

Proof We treat the direction from right to left. Suppose \( IP \not\vdash A \). We have to show that there is a model for \( IP \) which does not force \( A \). Let \( X \) be a finite adequate set containing \( A \). We prove that the canonical \( X \)-model is such a model. Observe that the canonical \( X \)-model is indeed a model, i.e. \( (\leq;R) \subseteq R \), and that every model satisfies the axioms of \( IP \). It is easy to see that there is an \( X \)-saturated set (hence a node in this model) which does not contain \( A \). Therefore, to see that \( A \) is not valid on this model it suffices to show that

\[ \forall B \in \forall w : B \in w \text{ iff } w \vdash B. \]

This can be easily shown by formula induction. We only treat implication and preservation for the direction from right to left. Suppose \( B = (C \rightarrow D) \) and \( B \not\in w \). If \( w \cup \{C\} \) would derive \( D \), then also \( w \vdash (C \rightarrow D) \). Thus \( w \cup \{C\} \not\vdash D \).

This implies that \( w \cup \{C\} \) is consistent. Let \( v \) be an \( X \)-saturated extension of
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4.1.1 The base of provability logic

The defined semantics is correct for the base logic iK of provability logic:

4.1.2 Proposition. In $\mathcal{L}_\Box$: $\vdash_{iK} A$ iff $A$ is valid on all finite brilliant frames.

Proof This proof is similar to the completeness proof for iP above. The only difference is that one has to observe that the canonical X-model is brilliant in this case, see Section 3.4.6.

4.2 The principle 4p

The logic iP4 is axiomatized over iP by

$$4p \quad A \rightarrow \Box A.$$  

We show that iP4 is complete with respect to the class of gathering frames. We call a model or a frame gathering if it satisfies

$$(gathering) \quad wRvRu \rightarrow v \leq u.$$  

We show that iK4 is complete with respect to a different class of frames (Section 4.2.1). Although the Leivant Principle is derivable in iP4 we treat it in a separate section. The reason for this given at the beginning of this chapter.

4.2.1 Proposition.

(i) The principle 4p corresponds to gatheringness.

(ii) The logic iP4 is canonical.

(iii) $\vdash_{iP4} A$ iff $A$ valid on all finite gathering frames.

Proof The three statements are easy to prove. We leave (i), (ii) and the direction from left to right of (iii) to the reader. For the the direction from right to left of the last statement it suffices to observe that for any finite adequate set which contains $\Box B$ for any nonboxed $B \in X$, the iP4-canonical X-model is gathering. $\Box$
4.2. The principle 4

4.2.1 The principle 4

The logic $iK4$ is axiomatized over $iK$ by

$$
\Box A \rightarrow \Box \Box A.
$$

We show that $iK4$ is complete with respect to finite transitive frames; this is similar to the situation in the classical case. We will see however that it corresponds to a weaker property than transitivity $(R; R \subseteq R)$, namely

(semi-transitivity) \( (R; R) \subseteq \bar{R}. \)

As is explained in Section 3.4.6 one cannot have both gatheringness and brilliancy. Here we see that for 4 we have brilliancy and for 4p we have gatheringness. Note that in the same way as in the classical case it can be shown that $\vdash_{iL} 4$ (see Section 4.3).

4.2.2 Proposition. In $\mathcal{L}\Box$:

(i) 4 corresponds to semi-transitivity.

(ii) $iK4$ is canonical.

(iii) $\vdash_{iK4} A$ iff $A$ is valid on all finite transitive brilliant frames.

Proof The first two statements and the direction from left to right of the third statement are left to the reader. We treat the direction from right to left of (iii). Assume $\not\vdash_{iK4} A$. Let $X$ be a finite adequate set which contains $A$ and let $\Gamma$ be an $X$-saturated set such that $\Gamma \not\vdash_{iK4} A$. Consider the model $(W, \preceq, R, V)$ such that $W, \preceq$ and $V$ are like in the $iK$-canonical $X$-model and $R$ is defined via

$$wRv \equiv_{ad} \forall B \in w (B, \Box B \in v).$$

It is clear that this frame is transitive. Therefore, to see that this model refutes $A$, we only have to show that $(w \models B$ iff $B \in w)$ holds for all $B \in X$. The only step which is different from the completeness proof for $iK$, is the direction from left to right for the case $B = \Box C$. Suppose $\Box C \notin w$. It is not difficult to see that the following property is an $iK4$-extendible property w.r.t. $X$ (compare Lemma 3.4.3),

\[ *(x) \quad x \not\models C. \]

Observe that $*\{(D, \Box D \mid \Box D \in w}\}$ holds. It is easy to see that any $*$-extension of the set $\{D, \Box D \mid \Box D \in w\}$ is an $X$-saturated set $v$ such that $C \notin v$, and $wRv$. This proves $w \not\models \Box C$. \[ \square \]
Chapter 4. The principles

4.3 Löb’s Preservativity Principle

The logic iPL is axiomatized over iP by Löb’s Preservativity Principle

\[ Lp \quad (\Box A \rightarrow A) \vdash A. \]

We show that \( Lp \) corresponds to the gathering conversely well-founded frames. We call a frame conversely well-founded if the modal relation on the frame is conversely well-founded. We do not know if iPL is also complete with respect to these frames. If we restrict ourselves to the language \( L_\Box \), then Löb’s Principle is complete with respect to the gathering conversely well-founded frames, which is shown in Section 4.3.1. However, the ‘trick’ used in this completeness proof for iL breaks down for iPL in the absence of the principle Mp. The completeness proof for iL is similar to the one in classical logic. We have included it for completeness’ sake.

Classically we have, in \( L_\Box \), that iL \( \vdash 4 \). Here we also have

\[ \vdash_{iPL} L \quad \text{and} \quad \vdash_{iPL} 4p \quad \text{and} \quad \vdash_{iP(4p\in L)} Lp. \]

The first deduction is trivial. The second one has a similar proof as the above mentioned analogue in \( L_\Box \):

\[ \vdash_{iPL} A \rightarrow (\Box(\Box A \land A) \rightarrow \Box A \land A)) \]
\[ \vdash A(\Box(\Box A \land A) \rightarrow \Box A \land A)) \]
\[ \vdash A(\Box A \land A) \]
\[ \vdash A \]

The third derivation runs as follows.

\[ \vdash_{iL} \quad \Box(\Box A \rightarrow A) \rightarrow \Box A \]
\[ \vdash_{iL} \quad \Box(\Box A \rightarrow A) \rightarrow \Box A \]
\[ \vdash_{iP4} \quad (\Box A \rightarrow A) \rightarrow \Box(\Box A \rightarrow A) \land (\Box A \rightarrow A) \]
\[ \vdash_{iP(4p\in L)} \quad (\Box A \rightarrow A) \rightarrow (\Box A \land (\Box A \rightarrow A)) \]
\[ \vdash_{iP(4p\in L)} \quad (\Box A \rightarrow A) \rightarrow A \]

4.3.1. Lemma. The principle \( Lp \) corresponds to gatheringness plus converse well-foundedness of the modal relation.

Proof Left to the reader. \( \Box \)
4.3.1 Löb's Principle

The logic il is axiomatized over iK by Löb's Principle

$L: \square(\square A \rightarrow A) \rightarrow \square A.$

In a manner similar to the classical case, we show that il is complete with respect to the finite transitive conversely well-founded brilliant frames. We call these frames \(L\)-frames. We saw that although the principle 4 is complete with respect to transitive frames it corresponds to the weaker property of being semi-transitive. A similar difference occurs in the case of \(L\). This is not surprising, since il derives the principle 4 (the proof of this fact is similar to the one that \(Lp\) derives \(4p\) above).

4.3.2. Proposition. In \(L\):

(i) \(L\) corresponds to semi-transitivity plus converse well-foundedness.

(ii) \(\vdash_{il} A\) iff \(A\) is valid on all finite transitive conversely well-founded brilliant frames.

Proof

(i) We only treat the direction from left to right. First assume that \(F\) is not semi-transitive; choose \(w, v, u\) such that

\[wRvRu \land \forall v'(wRv' \rightarrow v' \not\in u).\]  

(4.1)

Consider the model on \(F\) given by the valuation \(x \vdash p \equiv_{af} (x \not\in v \land x \not\in u)\).

We have to see that \(w \vdash \square(\square p \rightarrow p)\). Therefore, take \(x, y\) with \(wRx, x \not\in y\) and \(y \vdash \square p\). Because \(u \not\vdash p\), \(v \not\vdash \square p\) and thus \(y \not\in v\). Also \(y \not\in u\) by (4.1), hence \(y \vdash p\). But clearly \(w \not\vdash \square p\) since \(v \not\vdash p\).

For the second part, assume \(F\) is a semi-transitive but non conversely well-founded frame. Let \(w_0Rw_1Rw_2\ldots\) be a chain in \(F\). Define a valuation on \(F\) via

\[x \vdash p \equiv_{af} \forall i(x \not\in w_i).\]

In this model on \(F\), if \(w_0Rx\) and \(x \vdash \square p\) then \(x \not\in w_i\), for all \(i\), as \(x \not\in w_i\) implies \(xRw_{i+1}\). Hence \(w \vdash \square(\square p \rightarrow p)\). But not \(w \vdash \square p\).

(ii) The direction from right to left. Let \(A\) be such that \(\not\vdash_{il} A\). Let \(X\) be a finite adequate set containing \(A\), such that there is an \(X\)-saturated \(\Gamma\) for which \(\Gamma \vdash_{il} A\). We build a model \((W, \prec, R, V)\), which does not make \(A\) valid. \(W\) is the set of \(X\)-saturated sets. \(\prec, V\) are defined in the same way as for the il-canonical model.

But define

\[wRv \equiv_{af} \forall B \in w(\square B, B \in v) \land \exists D \in v(\square D \not\in w).\]

This makes \(W\) into a finite, transitive, conversely well-founded brilliant frame. We show that \(w \vdash B\) iff \(B \in w\), for \(B \in X\). We only treat the left to right direction for the case \(B = \square C\). Assume \(\not\Box C \not\in W\). Let \(\Delta = \{D, \Box D \mid \Box D \in w\} \cup \{\Box C\}\). From \(\Delta \vdash C\) it would follow that \(w \vdash \Box(\Box C \rightarrow C)\), and hence \(w \vdash \Box C\), which is false. Therefore, \(\Delta \not\vdash C\). Let \(v\) be an \(X\)-saturated extension of \(\Delta\) which does not derive \(C\), then \(wRv\), and therefore \(w \not\vdash \Box C\). 

\(\square\)
4.4 Leivant’s Principle

The logic $\text{iLe}$ is axiomatized over $iK$ by Leivant’s Principle

$$\text{Le} \quad \Box(A \lor B) \rightarrow \Box(A \lor \Box B).$$

Although the Leivant Principle is derivable in $iP_4$ (Section 3.3) we have not treated it in the section on the principle 4p. The reason for this is given at the beginning of this chapter. As was explained in Sections 2.2 and 2.5, Leivant’s Principle and the Disjunctive Principle are related to the Disjunction Property. In Section 4.4.1 we show that from the viewpoint of provability logic the Disjunctive Principle, in combination with the principle 4p, does not capture more than Leivant’s Principle, i.e. we show that the former is conservative over the latter.

In this section we show that $\text{iLe}$ is complete with respect to finite transitive frames which have the $\text{Le}$-property

$$(\text{Le}-\text{property}) \quad wRv \rightarrow \exists x(wRx \leq v \land \forall u(vRu \rightarrow x \leq u)).$$

This completeness proof will be the first non-standard proof so far. One cannot use classical analogies, since in the context of natural classical modal logics Leivant’s principle does not occur (Section 2.2). We will see that $\text{Le}$ corresponds to the property

$$(\text{Le}^\infty-\text{property}) \quad wRvRu \rightarrow \exists x(wRx \land x \leq v \land x \leq u).$$

However, on finite frames $\text{Le}$ corresponds to the Le-property. The proof of this fact will explain how this difference occurs when no infinite frames are allowed.

Finally we show that $\text{iLe}$ is also complete with respect to finite gathering frames. This implies that

for all $A \in \mathcal{L}$: $\vdash_{iP_4} A$ iff $\vdash_{\text{iLe}} A$.

Note that one loses the brilliancy in this case. One cannot have both gatheringness and brilliancy: in these frames $\Box \neg \Box \bot$ is valid, which clearly is not arithmetically valid. A node $x \leq v$ for which $\forall u(vRu \rightarrow wRx \leq u)$ holds is called a leivant-node for the pair $(w, v)$.

We remind the reader of the following consequence of $\text{iLe}$ that we will often use (it is proved in Section 3.1):

$$\vdash_{\text{iLe}} \Box(A \lor B) \rightarrow \Box(A \lor \Box B). \tag{4.2}$$

Clearly $\text{iLe} \vdash 4$. Observe that this is reflected in the corresponding frame properties; an $\text{Le}^\infty$-frame is semi-transitive.

In Chapter 5 on the completeness of $iH$ we will need the fact that $\text{iLe}$ is complete with respect to the finite transitive conversely well-founded brilliant $\text{Le}$-frames. (Recall that $\text{iLe}$ is $iKLLLe$. The principle $L$ is treated in the previous section.) Since this proof is similar to the completeness proof for $\text{iLe}$ we treat it at this place.
4.4. Leivant's Principle

4.4.1. Proposition. In \(\mathcal{L}_\Box\):

(i) \(Le\) corresponds to the \(Le^\infty\)-property.

(ii) On finite frames \(Le\) corresponds to the \(Le\)-property.

(iii) \(iLe\) is canonical.

(iv) \(\vdash iLe A\) iff \(A\) is valid on all finite brilliant \(Le\)-frames.

(v) \(\vdash iLL A\) iff \(A\) is valid on all finite transitive conversely well-founded brilliant \(Le\)-frames.

Proof We only treat the direction from left to right of (i) and (ii) and the direction from right to left of (v). The proof of (iv) is similar to the one of (v). The rest of the statements are easy.

(i) Assume that a frame \(\mathcal{F}\) does not have the \(Le^\infty\)-property. Take \(wRvRu\) such that

\[\forall x\neg (wRx \land x \leq u \land x \leq u).\]  

(4.3)

Now define a model on \(\mathcal{F}\) via

\[y \vdash p \equiv_{ad} \exists x \leq y(wRx \land x \leq u)\]
\[y \vdash q \equiv_{ad} \exists x \leq y(wRx \land x \not\leq u).\]

With this valuation clearly \(w \vdash \Box (p \lor q)\). It is also easy to see that \(u \not\vdash q\). Moreover, \(v \not\vdash p\). For, if \(v \vdash p\), then there is a node \(x \leq v\) such that \(wRx \land x \not\leq u\). This contradicts (4.3). Hence \(v \not\vdash p\), thus \(v \not\vdash p \lor \Box q\). Therefore, \(w \not\vdash \Box (p \lor \Box q)\).

(ii) The direction from left to right. It suffices to show that a finite \(Le^\infty\) frame is an \(Le\)-frame. Let \(\mathcal{F}\) be a finite \(Le^\infty\)-frame. Consider \(wRv\). We show that there is a node \(x\) such that \((wRx \leq v \land \forall u(vRu \rightarrow x \leq u))\). Let \(n\) be the number of successors of \(v\). If \(n = 0\), then we can take \(v = x\), and we are done. Therefore, assume \(n \geq 1\). We show that there is an enumeration \(u_1, \ldots, u_n\) of the successors of \(v\) such that there is a sequence \(x_1 \triangleright x_2 \triangleright \ldots \triangleright x_n\) of nodes, for which

\[wRx_i \leq v \land x_i \leq u_i.\]

Clearly, \(x_n\) has the desired properties, i.e. we can take \(x = x_n\), since \(\forall i(x_n \leq u_i)\), and \(wRx_n \leq v\) by construction.

We construct these sequences as follows. Let \(u_1\) be a successor of \(v\). Since \(wRuRu_1\), by the \(Le^\infty\)-property, there is a node \(x_1\) such that \(wRx_1 \leq v \land x_1 \leq u_1\). Assume that for \(j \leq i\), \(x_j\) and \(u_j\) are defined. Let \(u_{i+1}\) be a successor of \(v\) distinct from \(u_1, \ldots, u_i\). Since \(wRx_i \leq vRu_{i+1}\) holds, we also have \(wRx_iRu_{i+1}\). Hence by the \(Le^\infty\)-property there is a node \(x_{i+1}\) such that

\[wRx_{i+1} \leq x_i \land x_{i+1} \leq u_{i+1}.\]
Since $x_{i+1} \preceq x_i \preceq v$, we also have $x_{i+1} \preceq v$. Thus $x_{i+1}$ has the desired properties.

(v) The direction from right to left. An adequate set $X$ is called Le-adequate if it is the single closure under $\Box$ of an adequate set which is of the form \{\$Z \mid Z \subseteq X_0\}$ for some set $X_0$ which does not contain formulas of the form $A \lor B$. $X_0$ is called the base of $X$. If $\forall i \in \text{Le} A$, there is a finite Le-adequate set $X$ which contains $A$, and such that there is an $X$-saturated set $\Gamma$ which does not derive $A$. For the base of $X$ just take the set $X_0$ of all subformulas of $A$ (and their negations) minus the disjunctive ones. Consider the model $(W, \preceq, R, V)$, where $W$, $\preceq$ and $V$ are defined as on the ilLe-canonical $X$-model, and $R$ is defined via

$$wRv \equiv_{def} \forall Z \subseteq X_0 \left( \text{if } \Box(\forall Z) \in w \text{ then } \exists Z_i \in Z(Z_i, \Box Z_i \in v) \right) \wedge \exists \Box B \in v \left( \Box B \not\in w \right).$$

First, we show that

for all $B \in X$: $w \Vdash B$ iff $B \in w$. (4.4)

And second we show that the given frame is an Le-frame. This will complete the proof, because is easy to see that it is a transitive conversely well-founded brilliant frame. Just consider singleton sets $Z$ in the definition of $R$.

The proof of (4.4). We need some notation. Let $\sigma$ range over all functions on \{\$Z \mid Z \subseteq X_0 \wedge Z \neq \emptyset\}$ for which $\sigma Z \in Z$. For any set $x$, $x_\sigma$ denotes the set \{\$Z, \Box \sigma Z \mid \Box(\forall Z) \in x\}. Note that if $w$ and $v$ are $X$-saturated, then

$$wRv \text{ iff } \exists \sigma \exists \Box B \not\in w (w_\sigma \cup \{\Box B\} \subseteq v).$$

For the proof of (4.4) the only nontrivial step is the direction from left to right in the case that $B = \Box C$. Assume $\Box C \not\in w$. It is clear that the property

$$*(x) \quad x \not\Vdash C$$

is an iLLe-extendible property w.r.t. $X$. We show that if for all $\sigma$ we have $w_\sigma \cup \Box C \vdash C$, then $w \vdash \Box C$. Then we can conclude that there is a $\sigma$ such that $*(w_\sigma, \Box C)$. Clearly, any $*$-extension of $w_\sigma, \Box C$ is an $X$-saturated set $v$ such that $C \not\in v$ and $wRv$. This would show that $w \not\Vdash \Box C$.

Arguing by contradiction suppose that for all $\sigma$ we have $w_\sigma, \Box C \vdash C$. Let $Z_1, \ldots, Z_n$ be all the subsets $Z$ of $X_0$ for which $\Box(\forall Z) \in w$.

$$\forall \sigma (w_\sigma \vdash \Box C \rightarrow C)$$

$$\forall \sigma (\Box \sigma Z_1, \ldots, \Box \sigma Z_n \vdash \Box C \rightarrow C)$$

$$\forall \sigma \forall B \in Z_1 (\Box B, \Box \sigma Z_2, \ldots, \Box \sigma Z_n \vdash \Box C \rightarrow C)$$

$$\forall \sigma (\lor B \in Z_1, \Box B, \lor \sigma Z_2, \ldots, \lor \sigma Z_n \vdash \Box C \rightarrow C)$$

$$\vdots$$

$$\land (\lor B \in Z_1, \Box B), \ldots, \land (\lor B \in Z_n, \Box B) \vdash \Box (\lor C \rightarrow C).$$
4.4. Leivant’s Principle

As $\vdash_{iL} \Box(V Z) \rightarrow \Box(\bigvee_{B \in Z} \Box B)$, this implies that $w \vdash \Box(\Box C \rightarrow C)$. Hence by $L$ also $w \vdash \Box C$, a contradiction. This concludes the proof of (4.4).

To verify the $Le$-property, let $wRv$, $vR = \{u_1, \ldots, u_n\}$ and $u = v \cap u_1 \cap \ldots \cap u_n$. We have to find a node $x$ such that $wRx$ and $x \subseteq u$. Let

$$E = \bigvee\{D \in X_0 \mid u \not\subseteq \Box D\}.$$ 

In order to find $x$, we will construct $w_\sigma$ and $B$ such that

$$w_\sigma, \Box B \not\subseteq E, \; w_\sigma \subseteq v, \; \Box B \in v \cap w \not\subseteq.$$ (4.6)

Assuming (4.6) to be satisfied, we will show that there is a $y$ such that

$$w_\sigma, \Box B \subseteq y \subseteq u, \; \forall Z \subseteq X_0 (\text{if } y \vdash \bigvee Z \text{ then } \exists Z_i \in Z(Z_i \in y)).$$ (4.7)

Then $x = \{D \in X \mid y \vdash D\}$ is $X$-saturated, $x \subseteq u$ and by (4.5) also $wRx$. So it meets our conditions. Therefore, it remains to prove (4.6) and (4.7).

Assuming (4.6) we show (4.7) as follows. Let $Y_1, Y_2, \ldots$ be an enumeration of all the subsets of $X_0$ with infinite repetition. We construct sets $x_n$, such that the required $y = \bigcup_n x_n$.

$$x_1 = w_\sigma \cup \{\Box B\} \quad \text{if } x_i \not\vdash \bigvee Y_i$$

$$x_{i+1} = \begin{cases} x_i \cup \{\bigvee Y_i, D\} & \text{if } x_i \vdash \bigvee Y_i, \text{ and } D \in Y_i \text{ is such that } x_i \cup \{\bigvee Y_i, D\} \not\subseteq E. 
\end{cases}$$

Now all $x_i$ are subsets of $u$. The set $x_1$ is a subset of $u$ since $\Box B \subseteq v$. Also $w_\sigma \subseteq u$; formulae in $w_\sigma$ come in pairs $D, \Box D$ with $D \in X_0$. Suppose $D, \Box D \in w_\sigma$ and either of $D, \Box D \not\subseteq u$. Then $u \not\subseteq \Box D$, so $D \vdash E$, and thus $w_\sigma \vdash E$, which is not the case. So $D, \Box D \subseteq u$. Assume we have already shown $x_i \subseteq u$. If $x_i = x_{i+1}$ it is trivial; so let $x_{i+1} = x_i \cup \{\bigvee Y_i, D\}$, and assume, arguing by contradiction, that $x_{i+1} \not\subseteq u$. Since $x_i \vdash \bigvee Y_i$, so $u \vdash \bigvee Y_i$, thus $\bigvee Y_i \subseteq u$, this implies that $D \not\subseteq u$. Hence $u \not\vdash D$; but then $D \vdash E$, and $x_{i+1} \vdash E$, a contradiction.

Now we turn to the proof of (4.6). We have two cases.

**Case $\square E \not\subseteq v$.** If $\square E \not\subseteq v$, then for all $\square B \in v$, all $w_\sigma \subseteq u$ we have $w_\sigma, \Box B \not\subseteq E$; for otherwise we have $\square E \in v$, since $v \vdash (\bigvee w_\sigma \land \Box B)$. Since $wRx$ by (4.5), there is a $w_\sigma \subseteq v$ and a $\Box B \in v \cap w \not\subseteq$, hence we are done.

**Case $\square E \subseteq v$.** We show that there is a $w_\sigma \subseteq v$ such that $w_\sigma, \Box E \not\subseteq E$. This would prove (4.6) with $\Box E$ for the $\Box B$, since it is easy to see that $\Box E \not\subseteq w$. Arguing by contradiction, let us assume that for all $w_\sigma \subseteq v$ we have $w_\sigma, \Box E \subseteq E$; then we can derive the incorrect statement $w \vdash \square E$, as follows. Again, let $Z_1, \ldots, Z_n$ be all subsets $Z \subseteq X_0$ for which $\Box (\bigvee Z) \in w$. First note that

$$Z_i = \bigvee_{D \in Z_i \cap u} \bigvee_{D \in Z_i, D \not\subseteq u} D.$$
Hence also

$$Z_i \vdash E \lor \bigvee_{D \in Z_i \cap u} D.$$ 

And thus

$$w \vdash (E \lor \bigvee_{D \in Z_i \cap u} \Box D).$$

Now we reason as follows.

$$\forall \sigma \ (\text{if } w_\sigma \subseteq v, \text{ then } w_\sigma, \Box E \vdash E)$$

$$\forall \sigma \ (\text{if } w_\sigma \subseteq v, \text{ then } \Box \sigma Z_1, \ldots, \Box \sigma Z_n, \Box E \vdash E)$$

$$\forall \sigma \ (\text{if } w_\sigma \subseteq v, \text{ then } \bigvee_{B \in Z_i \cap u} \Box B, \Box \sigma Z_1, \ldots, \Box \sigma Z_n, \Box E \vdash E)$$

$$E \lor \bigvee_{D \in Z_i \cap u} \Box D, \ldots, E \lor \bigvee_{D \in Z_i \cap u} \Box D, \Box E \vdash E$$

$$\Box(E \lor \bigvee_{D \in Z_i \cap u} \Box D), \ldots, \Box(E \lor \bigvee_{D \in Z_i \cap u} \Box D) \vdash \Box(\Box E \rightarrow E)$$

$$w \vdash \Box(\Box E \rightarrow E)$$

$$w \vdash \Box E.$$ 

This completes the proof of $(v)$. 

4.4.1 Conservativity

As promised, we show that iLe is also complete with respect to finite gathering frames. This implies that iP4 is conservative over iLe with respect to formulas in $\mathcal{L}_{\Box}$. (A theory $T$ is called conservative over $T'$ with respect to formulas in $\mathcal{L}$ if $T'$ derives every formula in $\mathcal{L}$ that $T$ derives.) As was explained in Section 4.4, we cannot have both gatheringness and brilliancy.

As explained before, the fact that iLe is conservative over iP4 with respect to formulas in $\mathcal{L}_{\Box}$, shows that in the absence of other principles, the Disjunctive Principle does not capture more of the Disjunction Property than Leivant's Principle. In the next chapter we will see that this no longer holds in the presence of principles like the Formalyzed Markov Scheme: in Section 5.4, we show that iPH derives $\Box((\Box A \rightarrow A) \lor \neg \Box B) \rightarrow \Box(A \lor \Box B)$, while the logic iH does not derives this principle, although it contains Leivant's Principle and the Formalized Markov Scheme.

4.4.2 Proposition. $\vdash_{iLe} A$ iff $A$ is valid on all finite gathering frames.

Proof Suppose $\vdash_{iLe} A$. Let $M = (W, \ll, R, V)$ be a finite brilliant Le-model which does not validate $A$ in some node $b$. We define a new relation $R' \subseteq R$ on $W$ such that the model $M' = (W, \ll, R', V)$ has a gathering frame and validates the same formulas as $M$. 
4.5. Montagna's Principle

Intuitively we 'erase' those modal relationships $R$ between elements which violate the gatheringness of the frame, i.e. between nodes $w, v$ such that there is a $vRu$ with $v \not\approx u$. That is, we define

$$wR'v \equiv_{df} wRv \text{ and } \forall u(vRu \to v \lessdot u).$$

To prove that $M, w \vdash B$ iff $M', w \vdash B$, is straightforward once one knows

$$wRv \rightarrow w(R'v \lessdot v).$$

We will show this last fact. Let $S(x)$ be short for $\forall u \in W(xRu \rightarrow x \lessdot u)$. Now, assume $wRv$. We show that there is a node $v'$ with $wR'v' \lessdot v$, that is, with $wRv' \lessdot v$ and $S(v')$. The idea is as follows. By the Leivant property there is a node $x_1$ below $v$ and all its successors (in $M$), and such that $wRx_1$. If $x_1 = v$, we have $wRv$ and are done. If $x_1 \neq v$ we consider a node $x_2$ below $x_1$ and all its successors, and such that $wRx_1$, which again exists by the Leivant property. If $x_2 = x_1$, we can take $v' = x_1$. If $x_2 \neq x_1$ we consider a node $x_3$ which is below $x_2$ and all its successors, and such that $wRx_3$, etc.

More formally, we construct a sequence of elements $v = x_1 \succ x_2 \succ \ldots$ in $W$ such that for all $i$, it holds that $wRx_i$. And such that if $S(x_i)$ does not hold, then $(x_i \neq x_{i-1})$. As $M$ is finite this implies we can find an element $x_i$ with the desired properties. We show how to construct $x_{n+1}$ from $x_n$. If $S(x_n)$ holds, put $x_{n+1} = x_n$. If $S(x_n)$ does not hold, $x_{n+1}$ is a node which is below $x_n$ and its successors, and moreover such that $wRx_{n+1}$. 

4.4.3. Corollary. The logic $iP4$ is conservative over $iL_e$ with respect to formulas in $L_\Box$.

4.5 Montagna's Principle

The logics $iPM$ is axiomatized over $iP$ by Montagna's Principle

$$M_{p} \quad A \rightarrow (\Box C \rightarrow A) \rightarrow (\Box C \rightarrow B).$$

We show that $M_p$ corresponds to the $M_p$-property defined as

$$(M_p\text{-property}) \quad wRv \lessdot u \rightarrow \exists x(wRx \land v \lessdot x \lessdot u \land xR \subseteq uR).$$

Then we prove that $iPM$ is canonical.

If a principle corresponds to a frame property in which expressions like $xR \subseteq yR$ occur, like Montagna's Principle, then for a proof of its canonicity we need to know what $xR \subseteq yR$ means on the canonical model, i.e. in terms of saturated sets. This is the content of the following lemma. Note the difference with the language $L_\Box$, in which case $w_\Box \subseteq v_\Box$ iff $vR \subseteq wR$. The proof is similar to the proof of the following lemma.
4.5.1. Lemma. In any canonical model: \( vR \subseteq wR \) iff \( \square A \subseteq \square v \).

Proof First the direction from left to right. Suppose \( \square A \in w \) while \( \square A \not\in v \). By Lemma 3.4.3, the property

\[ *(x) \quad v \not\models xR A, \]

is an extendible \( v \)-successor property. Note that \( *(\{T\}) \) holds, and let \( u \) be any \( * \)-extension of \( \{T\} \). Clearly, \( vRu \), and \( A \not\in u \) hence \( w(R; \leq) u \) cannot hold.

For the other direction, assume \( w\Box \subseteq v\Box \) and \( vRu \). We have to construct a node \( u' \) such that \( wRu' \subseteq u \). By Lemma 3.4.3, the property

\[ *(x) \quad \text{for all } A: w \models xR A \text{ implies } A \in u, \]

is an extendible \( w \)-successor property. Clearly, \( *(\{T\}) \) holds. Therefore, any \( * \)-extension of \( \{T\} \) will do for \( u' \). \( \square \)

4.5.2. Proposition.

(i) The principle \( Mp \) corresponds to the the \( Mp \)-property.

(ii) The logic \( iP M \) is canonical.

Proof We prove part (ii) of the proposition and leave (i) to the reader. Consider \( wRv \leq u \) in the \( iP M \)-canonical model. Define the property

\[ *(x) \quad \text{for all } A: w \models xR A \text{ implies } A \in u. \]

It is easy to see that \( *(\cdot) \) is an \( iP M \)-extendible \( w \)-successor property. Thus if \( *(v \cup v\Box) \) holds, then any \( * \)-extension of \( v \cup v\Box \) is a node \( x \) such that \( wRx \) (Section 3.4.4) and \( v \leq x \leq u \) and \( xR \subseteq uR \) hold (Lemma 4.5.1). Thus it remains to show that \( *(v \cup u\Box) \) holds. This follows from the fact that for all finite subsets \( \Gamma \subseteq v \) and \( \Delta \subseteq u\Box \), and for all \( B \) we have that \( w \models \Gamma, \Delta \models B \) implies \( B \in u \). Therefore, suppose that for some such \( \Gamma, \Delta, B \) it does hold that \( w \models \Gamma, \Delta \models B \). Replace \( \Delta \) by the equivalent \( \Box A \) where \( A = (\bigwedge C \Box C \in \Delta) \). Then

\[ w \models \Gamma, \Box A \models B \]

\[ (\Box A \rightarrow \bigwedge \Gamma \land \Box A) \models (\Box A \rightarrow B) \]

\[ \Gamma \models (\Box A \rightarrow B). \]

This implies that \((\Box A \rightarrow B) \in v\), whence that \( B \in u \). \( \square \)
4.6 Visser's Scheme

The logic $iPV$ is axiomatized over $iP$ by Visser's Scheme which consists of the principles $V_{p_1}, V_{p_2}, \ldots$, where

$$V_{p_n} \left( \bigwedge_{i=1}^{n} (A_i \rightarrow B_i) \rightarrow A_{n+1} \lor A_{n+2} \right) \vdash \left( \bigwedge_{i=1}^{n} A_i \rightarrow B_i \right)(A_1, \ldots, A_{n+2}).$$

Recall that $(A)(B_1, \ldots, B_n)$ is defined as

$$(A)(B, C_1, \ldots, C_n) \equiv_{def} (A)(B) \lor (A)(C_1, \ldots, C_n)$$
$$(A)(\bot) \equiv_{def} \bot$$
$$(A)(B \land B') \equiv_{def} (A)(B) \land (A)(B')$$
$$(A)(\Box B) \equiv_{def} \Box A$$
$$(A)(B) \equiv_{def} (A \rightarrow B)$$

where $B$ not of the form $\bot$, $(C \land C')$ or $\Box C$.

In Section 2.3 we discussed the meaning of Visser's Scheme and its relation with the admissible rules of $HA$. In Section 3.2 we discussed the workings of the scheme. In this section we show that Visser's Scheme $Vp$ corresponds to a certain frame property $Vp^\infty$, and that the $iPV$-canonical frame has a stronger frame property. This proves that the logic $iPV$ is complete. We also prove that in combination with $Mp$ the logic is complete with respect to a class of frames which have a more elegant property, which will be called the $Vp$-property. In Section 4.6.1 we show that $iPV_n$ does not derive $Vp_{(n+1)}$, a result which does not play a role in the completeness proof of $iPH$. Of course, this result shows that Visser's Scheme is an essentially infinite collection of principles. In Section 4.6.2 we treat the Formalized Markov Scheme, which is derivable in $iPMV$.

For the frame characterization of Visser's Scheme we need the notion of a tight predecessor. We will first give the intuition behind it. Let $\bar{v}, \bar{u}$ range over finite sets of nodes, and write e.g. $x \preceq \bar{v}$ for 'for all $v \in \bar{v}(x \preceq v)'.

Consider two main instances of Visser's Scheme (see also Section 2.2):

$$\left( \bigwedge_{i=1}^{n} (A_i \rightarrow B_i) \rightarrow A_{n+1} \lor A_{n+2} \right) \vdash \left( \bigvee_{j=1}^{n} \left( \bigwedge_{i=1}^{n} (A_i \rightarrow B_i) \rightarrow A_j \right) \right)$$

(4.8)

$$\left( \bigvee_{i=1}^{n} \neg \Box A_i \right) \vdash \left( \bigvee_{i=1}^{n} \Box A \right)$$

(4.9)

The second one is treated in Section 3.3. The first one arises if we restrict Visser's Scheme to pure propositional variables, the second one if we restrict it to boxed...
formulas and \( \perp \). These two principles are related to two parts of the frame characterization of Visser’s Scheme. It is easy to see that (4.9) is valid on frames which satisfy

\[
\forall R \forall \forall x (v \leq x \land x \forall \forall y \geq x \exists z \in \forall (z \leq y)).
\] (4.10)

Formula (4.8) holds on frames which satisfy

\[
\forall R \forall v \forall x (v \leq x \leq \forall \land \forall y \geq x \exists z \in \forall (z \leq y)).
\] (4.11)

We show this for \( n = 3 \). If for nodes \( \forall R \forall v \) in such a frame we have \( \forall \models ((p_1 \rightarrow q) \rightarrow p_2 \lor p_3) \), and not \( \forall \models (p_1 \rightarrow q) \rightarrow p_2 \) then there are nodes \( u_1, u_2, u_3 \gg v \) that force \( (p_1 \rightarrow q) \) and such that \( u_i \) does not force \( p_i \). Let \( \forall = \{u_1, u_2, u_3\} \) and let \( x \) be the node such that \( v \ll x \ll \forall \) and such that for all \( y \geq x \), it holds that \( u_i \ll y \) for some \( i \). Observe that \( x \) forces \( (p_1 \rightarrow q) \) but that it does not force \( (p_2 \lor p_3) \), contradicting the assumption that \( v \) forces \( ((p_1 \rightarrow q) \rightarrow p_2 \lor p_3) \). For arbitrary \( n \) the reasoning is the same.

The combination of the two frame properties above leads to the frame property with respect to which \( \forall \) is complete. However, \( \forall \) does correspond to a weaker property, which will be called the \( \forall \)-property. This is best illustrated by the discussion on formula (4.8) above. Namely, one can weaken (4.11) by requiring that all nodes \( y \) above \( x \) are either below all nodes in \( \forall \) or above at least one node in \( \forall \):

\[
\forall R \forall v \forall x (v \leq x \leq \forall \land \forall y \geq x (y \leq \forall \lor \exists z \in \forall (z \leq y))).
\]

The same reasoning as above shows that \( \forall \) is still valid on frames with this property.

**Tight predecessors (in modal logic)**

We say that a node \( x \) in \( K \) is a **semi-tight predecessor of \( \forall \) holding \( u \)**, if

\[
x \ll \forall x \forall \forall y \geq x (\exists z \in \forall (z \ll y) \lor (y \ll \forall \land \forall y u)).
\]

It is called a **tight predecessor of \( \forall \) holding \( u \)** if in addition there holds the stronger

\[
\forall y \geq x \exists z \in \forall (z \ll y).
\]

If in addition we have

\[
v \forall \subseteq x \forall \land \forall y \geq x \exists z \in \forall (z \ll y),
\]

then \( x \) is called a **tight predecessor of \( \forall \) for \( v \)**.

We call a frame (model) a **\( \forall \)-frame (model)** if it has the **\( \forall \)-property**:

\(( \forall \)-property\) for all finite sets of nodes \( \forall, \forall \) : \( \forall R \forall v \ll \forall \land \forall R \forall \rightarrow
\]

\[
\exists x \geq v (x \text{ is a semi-tight predecessor of } v \\text{ holding } \forall).
\]
An inspection of the $Vp^\infty$-property will convince the reader that there are hardly any finite models that have this property.

Observe that if one reads tight for semi-tight in the $Vp^\infty$-property, it expresses (4.10) if $\bar{v}$ is empty, and (4.11) if $\bar{u}$ is empty. To show that $iPV$ is complete with respect to $Vp^\infty$-frames we need the following lemma (compare Lemma 4.5.1 and the discussion just before it).

4.6.1. Lemma. In any canonical model: $w\bar{R}v$ iff for all $\square A \in w$, $A \in v$.

Proof Only the direction from right to left. Let $*(\cdot)$ be the property 

$*(y)$ for all $A$: $w \vdash y \supset A$ implies $A \in v$.

By Lemma 3.4.3, $*$ is an extendible $w$-successor property. It is easy to see that $*(u_0)$ holds, where $u_0 = \{A \mid \square A \in w\}$. Let $u$ be the $*$-extension of $u_0$. Clearly, $wRu \not\leq v$ holds.

4.6.2. Proposition.

(i) Visser's Scheme corresponds to the $Vp^\infty$-property.

(ii) The logic $iPV$ is canonical.

(iii) The canonical model $iPV$ satisfies the following property which is stronger than the $Vp^\infty$-property:

for all finite sets of nodes $\bar{v}, \bar{u}$: $wRv \land v \not\leq \bar{v} \land vR\bar{u} \rightarrow 

\exists x \geq v (x$ is a tight predecessor of $\bar{v}$ holding $\bar{u})$.

Proof We often use lemma 3.2.1 (i) without mentioning. (i) First we show that $Vp$ holds on a $Vp^\infty$-frame. Suppose $wRv$ and $v \not\models (A)(D_1, \ldots, D_{n+2})$ hold, for some $A = \bigwedge_{i=1}^n(D_i \rightarrow E_i)$, on some $Vp^\infty$-frame. We show that $v \not\models (A \rightarrow D_{n+1} \lor D_{n+2})$. Assume $D_i = B_i \land \Box C_i$, where $B_i$ is not of the form $\Box C$. From the assumption it follows that $v \not\models (A \rightarrow B_i) \land \Box C_i$, whence either $v \not\models \Box C_i$ or $v \not\models (A \rightarrow B_i)$. Therefore, there are finite sets of nodes $\bar{v}$ and $\bar{u}$ such that for all $i$ we have that either there is a node $x \in \bar{u}$ with $vRx$ and $x \not\models C_i$ or there is a node $x \in \bar{v}$ with $x \not\models x, x \models A$ and $x \not\models B_i$. Let $\bar{v}$ and $\bar{u}$ be a smallest pair of sets with these properties. Let $u \models v$ be a semi-tight predecessor of $\bar{v}$ holding $\bar{u}$. We show that $u \models A$ and $u \not\models (D_{n+1} \lor D_{n+2})$. This will prove that $v \not\models (A \rightarrow D_{n+1} \lor D_{n+2})$.

To see that $u \not\models (D_{n+1} \lor D_{n+2})$, note that for $i = n+1, n+2$ we have that either there is node $x \in \bar{u}$ with $x \not\models C_i$ or there is a node $x \in \bar{v}$ with $x \models A$ and $x \not\models B_i$. In the first case we have that $u\bar{R}x$, and hence $u \not\models \Box C_i$. In the second case we have that $u \not\models x$ and thus $u \not\models (A \rightarrow B_i)$. Hence in both cases we can conclude $u \not\models D_i$. To see that $u \models A$, consider $y \models u$. Then either $y \leq \bar{v}$ and $y\bar{R}\bar{u}$,
or $z \preceq y$ for some $z \in \hat{v}$. In the last case $y$ forces $A$ because all nodes in $\hat{v}$ force $A$.

In the first case, it suffices to show that for all $i \leq n$, we have that $y \not\models B_i \land \Box C_i$. Note that for all $i \leq n$ either there is node $x \in \hat{u}$ with $x \not\models C_i$ or there is a node $x \in \hat{u}$ with $x \models A$ and $x \not\models B_i$. In the first case we have that $y \models \Box x$ holds, and whence $y \not\models \Box C_i$. In the second case we have $y \preceq x$, and therefore $y \not\models B_i$. Hence in both cases we can conclude $y \not\models \Box D_i$.

The other part of (i) follows from part (i) of Lemma 4.6.4; a frame which does not have the $Vp^\infty$-property does not have the $Vp_n$-property, for some $n$.

(ii) This follows from (iii).

(iii) Consider nodes $wRv$, $v \preceq v_1, \ldots, v_m$ and $vRu_1, \ldots, u_n$, in the iPV-canonical model. Let $\hat{v}, \hat{u}$ denote $v_1 \cap \ldots \cap v_m$ and $u_1 \cap \ldots \cap u_n$ respectively. First note that in general $\hat{v}$ and $\hat{u}$ are not saturated. Therefore, they are not necessarily nodes in the canonical model. Let

$$\Delta = \{(E \land \Box E' \rightarrow F) \mid F \in \hat{v} \land (E \not\in \hat{v} \lor E' \not\in \hat{u})\}.$$

(Thus in particular the implications $(E \rightarrow F)$ and $(\Box E \rightarrow F)$, for which $F \in \hat{v}$ and respectively $E \not\in \hat{v}$ and $E \not\in \hat{u}$, are in $\Delta$.) Note that $\Delta \subseteq \hat{v}$. Let $\ast(\cdot)$ be the property

$$\ast(x) \mid x \models A_1 \lor \ldots \lor A_m \lor \Box B_1 \lor \ldots \lor \Box B_n$$

implies $\exists i (A_i \in \hat{v} \lor B_i \in \hat{u})$.

Clearly, $\ast(\cdot)$ is an extendible property (Section 3.4.4). We show that $\ast(v \cup \Delta)$ holds. Let $C = A_1 \lor \ldots \lor A_m \lor \Box B_1 \lor \ldots \lor \Box B_n$ and suppose $v \cup \Delta \models C$. This implies that there is a conjunct $D = \bigwedge_{i=1}^k (E_i \rightarrow F_i)$ of implications in $\Delta$, such that $v \models (D \rightarrow C)$. Thus $(D \rightarrow C) \in v$, because $v$ is saturated. Since

$$(D \rightarrow C) \lor (D)(E_1, \ldots, E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n),$$

also $(D)(E_1, \ldots, E_k, A_1, \ldots, A_m, \Box B_1, \ldots, \Box B_n) \in v$. From the construction of $\Delta$ it follows that $v$ does not contain any of $(D \rightarrow E) \land \Box E'$, for $E_i = E \land \Box E'$. Therefore $v$ contains either $(D \rightarrow A_i)$ or $\Box B_i$ for some $i$. This proofs that $\ast(v \cup \Delta)$ holds. Let $u$ be the $\ast$-extension of $v \cup \Delta$. As described in Section 3.4.4, $u$ is saturated. We show that $u$ is a semi-tight predecessor of $\hat{v}$ holding $\hat{u}$. Clearly, $v \preceq u \preceq v_1, \ldots, v_n$ holds, and by Lemma 4.6.1 we have $uR\hat{u}_i$ for all $i$.

It remains to show that

$$\forall y \models u \exists i (z'v_i \preceq y).$$

Arguing by contradiction, suppose $u \prec u'$ for some saturated set $u'$. For all $i \leq m$, we choose a formula $A_i \in v_i$ outside $u'$. Then the formula $(A_1 \lor \ldots \lor A_m)$ is in $\hat{v}$ but not in $u'$. From the construction of $u$, and the fact that $u'$ is a superset of $u$, it follows that there is a formula $(E \land \Box E') \in u'$ such that either $E \not\in \hat{v}$ or $E' \not\in \hat{u}$. Now $(E \land \Box E' \rightarrow A_1 \lor \ldots \lor A_m)$ is an element of $\Delta$, thus also of $u$. Hence $(A_1 \lor \ldots \lor A_n)$ should be in $u'$, a contradiction. This proves that iPV is canonical. \[\Box\]
4.6. Visser’s Scheme

We saw that $Vp$ is complete with respect to a stronger frame property than the property to which it corresponds. On frames for which for every two nodes $x \leq y$ there is a node $x \preceq z \preceq y$ such that there is no node $z < z' \prec y$, the two frame properties coincide. Since the canonical model has such a frame, $(iii)$ follows in fact from $(ii)$.

In the presence of Montagna’s principle

Happily, in the presence of Montagna’s Principle, Visser’s Scheme has a more compact characterization. It is given by the following property

$$\text{(Vp-property)} \quad wRv \preceq v_1, \ldots, v_m \Rightarrow$$

$$\exists x(v \preceq x \preceq v_1, \ldots, v_m \land vR \subseteq xR \land \forall y \exists x(v_i \preceq y)).$$

Recall that in this case $x$ is called a tight predecessor of $v_1, \ldots, v_n$ for $v$. The following corollary of the previous lemma plus the canonicity of Montagna’s Principle (Proposition 4.5.2) shows that the logic $iPMV$ is complete with respect to $M_pVp$-frames.

4.6.3. Corollary. Any canonical model of a logic containing the principles $M_p$ and $Vp$ has the $Vp$-property.

Proof The proof is analogous to the proof of the canonicity of $iPV$ above. Consider $wRv \preceq v_1, \ldots, v_m$. The set $\Delta$ will now be

$$\Delta = \{(E \land \Box E' \rightarrow F) \mid F \in \hat{v} \land (E \not\in \hat{v} \lor \Box E' \not\in v)\}.$$

In the same way as in the proof of Proposition 4.6.2, define a property

$$\ast(x) \quad x \vdash A_1 \lor \ldots \lor A_n \lor \Box B_1 \lor \ldots \lor \Box B_n \text{ implies } \exists \hat{v}(A_i \in \hat{v} \text{ or } \Box B_i \in v).$$

and construct a node $u$ via this property. This leads to a node $u$ such that $v \preceq u \preceq v_1, \ldots, v_m$ and $u_\Box \subseteq v_\Box$. Applying Lemma 4.5.1 gives $vR \subseteq uR$. Following the proof of Proposition 4.6.2 it is easy to see that $u$ has the desired properties.

4.6.1 The independence of Visser’s Principles

In this section we show that $iPV_n$ does not derive $Vp_{n+1})$. The proof is rather unpleasant but we think that the result needs to be established. It implies that Visser’s Scheme is infinite in an essential way. The result does not play a role in the next chapter on the completeness of $iPH$.

This proof is based on the fact that $iPV_n$ is complete with respect to frames which satisfy the $Vp_n$-property:

$$\text{(Vp_n-property)} \quad \text{for all finite sets of nodes } \bar{v}_+ = \bar{v} \cup \bar{v}_-, \bar{v}_+ = \bar{v} \cup \bar{v}_-$$

such that $|\bar{v}| + |\bar{u}| \leq n, |\bar{v}_-| + |\bar{u}_-| \leq 2$: $wRv \land v \preceq \bar{v}_+ \land vR\bar{u}_+ \Rightarrow$

$$\exists x \exists z(v \preceq \bar{v}_+ \land xR\bar{u}_+ \land \forall y \exists x(\exists z \in \bar{v}_+(z \preceq y) \lor (y \preceq \bar{v} \land yR\bar{u}))).$$
Note that in the formula above, if $|\bar{v}_-|+|\bar{u}_-|=0$, then $x$ is just a tight predecessor of $\bar{v}$ holding $\bar{u}$. Thus a frame which satisfies all $Vp_n$-properties has the $Vp^\infty$-property (Section 4.6). This is what we expect, since Visser’s Scheme, which corresponds to the $Vp^\infty$-property, consists of all principle $Vp_n$.

Before we treat the completeness proof for Visser’s Principles we clarify the difference between the $Vp$-property and the not very elegant $Vp^-$-property. To make the discussion more transparent, we forget about the special treatment of boxed formulas in these schemes. Therefore, consider the following principle which is a special instance of $Vp_1$:

$$((A_1 \rightarrow B) \rightarrow A_2 \lor A_3) \lor \bigvee_{i=1}^{3}((A_1 \rightarrow B) \rightarrow A_i).$$

If we look for the minimal requirement on frames for which they validate this principle, we arrive at the following property

$$wRv \preceq v_1, v_2, v_3 \rightarrow \exists x (v \preceq x \preceq v_1, v_2, v_3 \land \forall y \succ x ((v_2 \preceq y \lor v_3 \preceq y) \lor y \preceq v_1)).$$

($v_1$ is $\bar{v}$ and $v_2, v_3$ is $\bar{u}$ in the definition of the $Vp_n$-property above.) We do not prove that the principle corresponds to this property, but it is instructive to see why the principle is valid on these frames. Suppose $v$ forces $((A_1 \rightarrow B) \rightarrow A_2 \lor A_3)$ but not $(A_1 \rightarrow B) \rightarrow A_i$. Select nodes $v_i \succ v$ such that $v_i \models (A_1 \rightarrow B)$ but $v_i \not\models A_i$. To derive a contradiction, we use the existence of a node $x$ such that $v \preceq x \preceq v_1, v_2, v_3$ and for all $y > x$, we have $((v_2 \preceq y \lor v_3 \preceq y) \lor y \preceq v_1)$. Namely, since $x \preceq v_2, v_3$, it follows that $x \not\models (A_2 \lor A_3)$. We show that $x \models (A_1 \rightarrow B)$, and then we have a contradiction with $v \preceq x$ because $v \models (A_1 \rightarrow B) \rightarrow A_2 \lor A_3)$. Therefore, consider $y \models x$ and assume $y \models A_1$. From $y \models x$ it follows that $y = x$ or $(y \preceq v_1 \lor v_2 \preceq y \lor v_3 \preceq y)$. Thus $(v_2 \preceq y \lor v_3 \preceq y)$. But then $y \models (A_1 \rightarrow B)$, and whence $y \models B$.

The example above showed that the sets $\bar{v}_-$ and $\bar{u}_-$ correspond to the formulas $A_{n+1}$ and $A_{n+2}$ in the principle $Vp_n$. In the example it could be that $v_1 = v_2$, in which case $\bar{v}_- = v_3$. This explains the requirement $|\bar{v}_-|+|\bar{u}_-| \leq 2$ in the $Vp_n$-property.

4.6.4. Lemma.

(i) The principle $Vp_n$ corresponds to the $Vp_n$-property.

(ii) The logic $iPV_n$ is canonical.

Proof (i) We only show that any frame which does not have the $Vp_n$-property has a valuation which refutes $Vp_n$. Consider such a frame $F$. We leave it to the reader to check that if, in the $Vp_n$-property above, we change the words ‘for all
4.6. Visser’s Scheme

finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-$, $\bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2$ to ‘for all finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-$, $\bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that

$$\forall x \in \bar{v}_+, \forall y \in \bar{v}_- \neg (x \bar{R} y) \quad \text{and} \quad \forall x \in \bar{v}_+ \forall y \in \bar{v}_- \neg (x \bar{R} y) \quad \text{and} \quad (4.12)$$

and $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2'$, we still have an equivalent property. Therefore, we can conclude that in $\mathcal{F}$ there are finite sets of nodes $\bar{v}_+ = \bar{v} \cup \bar{v}_-$, $\bar{u}_+ = \bar{u} \cup \bar{u}_-$ such that (4.12) holds and $|\bar{v}| + |\bar{u}| \leq n$, $|\bar{v}_-| + |\bar{u}_-| \leq 2$, for which

$$w \bar{R} v \leq \bar{v}_+ \land \bar{v} \bar{R} u_+ \land \forall u \not\equiv v(u \not\equiv \bar{v}_+ \lor \neg (u \bar{R} \bar{u}_+) \lor$$

$$(\exists u') \not\equiv u(\forall x \in \bar{v}_+(x \not\equiv u') \land (u' \not\equiv \bar{v}_+ \lor \neg (u' \bar{R} \bar{u}_+)))$$

(4.13)

Observe that if both $\bar{v}$ and $\bar{u}$ are empty, (4.13) cannot hold. For if so, then there exists a node $u' \vartriangleright v$ such that $u' \not\equiv \bar{v}$ or $\neg (u' \bar{R} \bar{u})$ holds, quod non. We have to consider three cases: (a) $\bar{v}_+ = \{v'\}$, $\bar{u}_+$ is empty, (b) $\bar{u}_+ = \{u'\}$ and $\bar{v}_+$ is empty, (c) both $\bar{v}_+$ and $\bar{u}_+$ contain at least one node or one of them contains at least two nodes.

(a) In this case (4.13) cannot hold (take $u = v'$).

(b) If $\bar{u}$ is empty, (4.13) cannot hold (take $u = v$). If $\bar{u} = \bar{u}_+$, by (4.13), for all $x \vartriangleright v$, if $x \bar{R} u'$ there exists $x' \vartriangleright x$ such that $x' \bar{R} u'$ does not hold. Define the valuation

$$x \models p \equiv_{\text{def}} x \not\equiv u'.$$

Clearly, in this model $v \not\models \Box p$ holds. We show that $v \models \neg \Box p$ holds, and this proves that $Vp_1$ does not hold on the frame. To see that $v \models \neg \Box p$, consider $x \vartriangleright y$. We have to show that $x \not\models \neg \Box p$. By assumption there exists a node $x' \vartriangleright x$ such that $x' \bar{R} u'$ does not hold. Hence $x' \models \Box p$, and thus $x \not\models \neg \Box p$.

(c) To define a valuation which is going to refute $Vp_n$ on $\mathcal{F}$, we want that either $\bar{u}_-$ contains at least one element or that $\bar{u}_-$ contains two elements. First we show how we can amend $\bar{v}_-$ and $\bar{u}_-$ in such a way that this holds, while keeping (4.13) and (4.12) valid. If $\bar{u}_-$ and $\bar{v}_-$ are empty, we take $x_1, x_2 \in \bar{v}$, $y_1, y_2 \in \bar{u}$, and redefine $\bar{v}_- = x_1$, $\bar{u}_- = y_1$ or $\bar{u}_- = y_1, y_2$ or $\bar{v}_- = x_1, x_2$ (if $x_1 \neq x_2$). Let us see that, depending on $\bar{v}$ and $\bar{u}$, nodes can be chosen such that this can be done and such that (4.13) and (4.12) hold for the new $\bar{v}, \bar{u}$. If $\bar{u}_-$ is empty and $\bar{v}_- = x$ there are two possibilities. If $\bar{v}$ contains a node $y \not\equiv x$ we redefine $\bar{v}_- = x, y$. If not, $\bar{v} = \bar{v}_- = x$. By assumption $\bar{u}$ contains at least one element $y$. If for all $y \in \bar{u}$, $x \bar{R} y$, then (4.13) cannot hold (take $u = x$). Take $y \in \bar{u}$ for which $x \bar{R} y$ does not hold and redefine $\bar{u}_- = y$. This shows that from now on we can assume that $\bar{u}_-$ contains at least one element or that $\bar{u}_-$ contains two elements.

We only treat the case that both $\bar{v}_-$ and $\bar{u}_-$ contain one element, the other cases are similar. Let $\bar{v} = v_1, \ldots, v_k$, $\bar{u} = u_1, \ldots, u_m$ and $\bar{v}_- = v_{k+1}, \bar{u}_- = u_{m+1}$. Define a model on the given frame via the following valuation:

$$x \models p_i \equiv_{\text{def}} x \not\equiv v_i$$
$$x \models q_i \equiv_{\text{def}} v_i \leq x, \text{ for some } i \leq k + 1$$
$$x \models r_i \equiv_{\text{def}} x \not\equiv u_i.$$
Let
\[ A = \bigwedge_{i=1}^{k} (p_i \rightarrow q_i) \wedge \bigwedge_{i=1}^{m} \neg \Box r_i. \]

We show that \( v \vdash A \rightarrow p_{k+1} \lor \Box r_{m+1} \), \( v \not\vdash (A)(p_1, \ldots, p_{k+1}, \Box r_1, \ldots, \Box r_{m+1}) \). To see that the second statement holds it suffices to see that \( v_i \vdash A \), \( v_i \not\vdash p_i \) and \( u_i \not\vdash r_i \), which we leave to the reader. We prove that \( v \vdash A \rightarrow p_{k+1} \lor \Box r_{m+1} \). Consider a node \( u \succ v \) such that \( u \vdash A \). Hence for all \( i \leq m \), \( u' \succ u \), \( u' \not\vdash u_i \). Furthermore, for all \( i \leq m \), \( u' \succ u \), if \( u' \not\vdash v_i \) then \( u' \vdash q_i \). In particular,
\[ \forall u' \succ u ((u' \preceq v \land u' \not\vdash u) \lor \exists x \in \bar{v} (x \preceq u')). \]

By (4.12), \( \exists x \in \bar{v} (x \preceq u) \) implies that not \( u \not\vdash u_{m+1} \). Therefore, we can conclude \( u \preceq v \) or not \( u \not\vdash u_{m+1} \). All together this leads, by (4.13), to \( u \not\vdash v_{k+1} \) or not \( u \not\vdash u_{m+1} \). Hence \( u \vdash p_{k+1} \lor \Box r_{m+1} \); what we wanted to show.

(ii) Assume that in the iPV\(_n\)-canonical model we have \( wRu \land v \preceq v_1, \ldots, v_{k+1} \) and \( vRu_1, \ldots, v_{m-j} \), for some \( k, m, i, j \) such that \( k + m \leq n \). The proof that there is a node \( u \) which is a tight predecessor of \( v_1, \ldots, v_k \) holding \( u_1, \ldots, u_m \) and such that \( u \preceq v_{k+1}, \ldots, v_{k+i} \) and \( uRu_{m+1}, \ldots, u_{m+j} \), is similar to the proof of Proposition 4.6.2. The only difference occurs in the definition of the set \( A \), which we define in this case as follows. Let \( \hat{v} = v_1 \cap \ldots \cap v_k \) and \( \hat{v}^* = v_1 \cap \ldots \cap v_{k+i} \) and let \( \hat{u} = u_1 \cap \ldots \cap u_m \) and \( U^* = u_1 \cap \ldots \cap u_{m+j} \), and let
\[ \Delta = \{(E \rightarrow F) \mid F \in v^* \land E \not\in \hat{v}\} \cup \{(\Box E \rightarrow F) \mid F \in v^* \land E \not\in \hat{u}\}. \]

Let \( \ast(\cdot) \) be the property
\[ \ast(x) x \vdash A_1 \lor \ldots \lor A_q \lor \Box B_1 \lor \ldots \lor \Box B_r \]implying \( \exists h \ (A_q \in v^* \lor B_h \in u^*) \).

In a similar way as in Lemma 3.4.3 one can show that \( \ast(\cdot) \) is an extendible property. We show that \( \ast(v \cup \Delta) \) holds. Let \( C = A_1 \lor \ldots \lor A_q \lor \Box B_1 \lor \ldots \lor \Box B_r \) and suppose \( v \cup \Delta \vdash C \). This implies that there are conjuncts \( D_1 = \bigwedge_{i=1}^{k} (E_h \rightarrow F_h) \) and \( D_2 = \bigwedge_{i=1}^{m} (E_h \rightarrow F_h) \), such that \( F_h, F_h' \in v^* \), \( E_h \not\in \hat{v} \) and \( E_h \not\in \hat{u} \). Let \( F \) be the conjunction of all \( F_h, F_h' \), and let \( H_p = (\bigvee_{E_h \not\in \hat{v}} E_h) \) and \( H'_p = (\bigvee_{E_h \not\in \hat{u}} E'_h) \). Clearly, \( F \in v^* \), \( H_p \not\in v_p \) and \( H'_p \not\in u_p \). Define a new conjunct of formulas in \( \Delta \):
\[ D = \bigwedge_{p=1}^{k} (H_p \rightarrow F') \wedge \bigwedge_{p=1}^{m} (H'_p \rightarrow F). \]

We have \( v \vdash (D \rightarrow C) \) and thus \( (D \rightarrow C) \in v \). Since
\[ (D \rightarrow C) \vdash (D)(H_1, \ldots, H_{k}, \Box H'_1, \ldots, \Box H'_{m}, A_1, \ldots, A_q, \Box B_1, \ldots, \Box B_r) \]
and \( k + m \leq n \), also
\[ (D)(H_1, \ldots, H_{k}, \Box H'_1, \ldots, \Box H'_{m}, A_1, \ldots, A_q, \Box B_1, \ldots, \Box B_r) \in v. \]
From the construction of $\Delta$ it follows that $v$ does not contain any of $(D \rightarrow H_h)$ or $\Box H_h$. Therefore $v$ contains either $(D \rightarrow A_h)$ or $\Box B_h$ for some $h$. This proves that $*(v \cup \Delta)$ holds. Let $u$ be the $*$-extension of $v \cup \Delta$. The proof that $u$ has the desired properties is similar to the corresponding part in the proof of Proposition 4.6.2 and is therefore left to the reader.

4.6.5. Corollary. For all $0 < m < n$, $\forall p \in V_p$.

4.6.2 The Formalized Markov Scheme

The logic $iMa$ is axiomatized over $iK$ by the Formalized Markov Scheme

$$Ma \quad \Box \neg (\Box A \rightarrow \bigvee \Box B_i) \rightarrow \Box (\Box A \rightarrow \bigvee \Box B_i).$$

Recall that the Formalized Markov Scheme is the partial formalization of Markov's Rule (Section 2.5). In Section 2.2 the relation between this rule and Visser's Scheme was explained. In Section 3.3 we saw that $Ma$ is derivable in $iPMV$.

In this section we show that $iMa$ is complete with respect to frames with the property

$$\text{(Ma-property)} \quad wRv \rightarrow \exists x \in Top (wR^x \land vR^x = xR).$$

Recall (Section 3.4.2) that a node in $Top$, a top node, is a node $x$ such that there is no node $y$ with $x < y$. Note the similarity between the frame property for the Formalized Markov Scheme and property (4.10) discussed in Section 4.6 on Visser's Scheme.

A top node $x$ for which $wR^x$ and $vR^x = xR$ hold will be called a markov-node for the pair $(w, v)$. On brilliant frames the $Ma$-property reads

$$wRv \rightarrow \exists x \in Top (wR^x \land vR^x = xR).$$

Note that the logic $iMa$ cannot be complete with respect to gathering frames which satisfy this stronger property. Since in such frames every top node which is a successor, satisfies $\Box \bot$, the formula $\Box \neg \neg \Box \bot$ holds on such frames.

Before we give the completeness proof for $iMa$ we need a lemma.

4.6.6. Lemma. For the logic $iMa$ we have that if $\Delta = \{ D \mid \Gamma \vdash \Box D \}$, for some set $\Gamma$, and the set of formulas $\Delta, \Box A_1, \ldots, \Box A_n, \neg \Box B_1, \ldots, \neg \Box B_m$ is inconsistent, then $\Delta$ derives $(\land \Box A_i \rightarrow \bigvee \Box B_i)$.

Proof The first derivation shows that from the inconsistency of the formulas $\Delta, \Box A_1, \ldots, \Box A_n, \neg \Box B_1, \ldots, \neg \Box B_m$, it follows that $\Delta \vdash iMa \quad \land \Box A_i \rightarrow \bigvee \Box B_i$.

$$\Delta, \Box A_1, \ldots, \Box A_n, \neg \Box B_1, \ldots, \neg \Box B_m \vdash iMa \quad \bot$$
$$\Delta, \Box \land \Box A_i, \neg \Box B_1, \ldots, \neg \Box B_m \vdash iMa \quad \bot$$
$$\Delta \vdash iMa \quad \Box \land \Box A_i \rightarrow \neg \bigvee \Box B_i$$
$$\Delta \vdash iMa \quad \neg (\Box \land \Box A_i \rightarrow \bigvee \Box B_i).$$
The following derivation shows that $\Delta \vdash_{iMa} \neg\neg(\square A \to \square B)$ implies that $\Delta \vdash_{iMa} (\square A \to \square B)$.

\[
\begin{align*}
\Delta & \vdash_{iMa} \neg\neg(\square A \to \square B) \\
\Gamma & \vdash_{iMa} \neg\neg(\square A \to \square B) \\
\Gamma & \vdash_{iMa} (\square A \to \square B) \\
\Delta & \vdash_{iMa} \neg\neg(\square A \to \square B) \\
\Delta & \vdash_{iMa} (\square A \to \square B)
\end{align*}
\]

Note that the second step of the last derivation is the only place where the Formalized Markov Scheme is used. The special form of $\Delta$ is used in the first and the third step of the last derivation. 

4.6.7. Proposition. In $\mathcal{L}\square$:

(i) On finite frames $Ma$ corresponds to the $Ma$-property.

(ii) The $iMa$-canonical model has the $Ma$-property.

**Proof** (i) Only the direction from left to right. Let $\mathcal{F}$ be a finite frame which does not have the $Ma$-property; there are $w,v,u_1,\ldots,u_n$ with $wRv$ and $v\bar{R} = \{u_1,\ldots,u_n\}$ and

$$\forall x \in \text{Top}(w\bar{R}x \land x\bar{R}x \subseteq v\bar{R}x \to v\bar{R}x \subseteq x\bar{R}x).$$

Let $T_i = \{y \mid w\bar{R}y \land y\bar{R}y \subseteq v\bar{R}y \land \neg y\bar{R}u_i\}$. Define a valuation on $\mathcal{F}$ via

$$\begin{align*}
x \vdash p & \equiv_{def} v\bar{R}x \\
x \vdash q_i & \equiv_{def} \exists y \in T_i (y\bar{R}x).
\end{align*}$$

We show that with this valuation, $w \vdash \square \neg\neg(\square p \to \square q_i)$ and $w \not\vdash (\square p \to \square q_i)$. The last part is obvious, since $wRv$ and $v \vdash \square p$. That $v \not\vdash \square q_i$ for all $i$, follows from the fact that $u_i \not\vdash q_i$, for all $i$. Therefore, consider any top node $y$ above some successor $z$ of $w$. It suffices to show that $y \vdash \square p \to \square q_i$, because this would imply $z \vdash \neg\neg(\square p \to \square q_i)$. Note that $w\bar{R}y$. If $y \vdash \square p$, then $y\bar{R}x \subseteq v\bar{R}x$. Hence, by assumption, $\neg y\bar{R}u_i$, for some $i$. Therefore, $y \not\in T_i$, and thus $y \vdash \square q_i$.

(ii) Let $(W,\ll,\mathcal{R},V)$ be the $iMa$-canonical model. Let $w,v$ be two nodes such that $wRv$. We show that there is a top node $x$ such that $wRx$ and $vR = xR$. Let $\Delta = \{D \mid \square D \in w\}$. It suffices to show that the set $\Delta, v\bar{R}, \{\neg \square E \mid \square E \not\in v\}$ is consistent, as any maximal consistent extension of this set will have the desired properties. If it is not consistent, there are $\square E_1, \square E_2, \ldots, \square E_n \not\in v$ and $\square B_1, \ldots, \square B_m \in v$ such that $\Delta, \square B_1, \ldots, \square B_m, \neg \square E_1, \ldots, \neg \square E_n$ is inconsistent. But then lemma 4.6.6 implies that $\Delta \vdash \square B \to \square E_i$. Hence $(\square E_1 \lor \ldots \lor \square E_n)$ is in $v$, and that cannot be. 

\[\square\]
4.7 Independence

In this section we explain why all principles discussed above are independent. We call two principles $A$ and $B$ independent if they do not derive one another. In particular, if $A$ is a principle of the preservativity logic then we say that $B$ is independent from $A$ if $iPA \nvdash B$, i.e. if $B$ is not derivable from $A$ in the system given by $P1$, $P2$ and $Dp$ and the rules Modus Ponens and the Preservation Rule. If $A$ is a principle of the provability logic, we say that $B$ is independent from $A$ if $iA \nvdash B$, i.e. if $B$ is not derivable from $A$ in the system given by $K$ and the rules Modus Ponens and the Necessitation Rule.

To show that $B$ is independent from $A$ it suffices to show that there is a model for $A$ on which $B$ is not valid. If $A$ is complete with respect to some class of frames, this model can be obtained by giving a valuation on such a frame such that $B$ is not valid under this valuation. Using the results in this chapter it is easy to prove the following proposition.

4.7.1. Proposition.

(i) The following principles are independent: L"ob's Preservativity Principle, Montagna's Principle and Visser's Scheme. L"ob's Preservativity Principle derives the principle $4p$.

(ii) For all $n > m$, the $m$-th Visser's Principle $Vp_m$ does not derive the $n$-th Visser's Principle $Vp_n$, and $Vp_n$ derives $Vp_m$.

(iii) The following principles are independent: L"ob's Principle, Leivant's Principle and the Formalized Markov Scheme. L"ob's Principle as well as Leivant's Principle derive the principle $4$.


Proof (i) As explained above this follows easily from the completeness results in the previous sections. That L"ob's Preservativity Principle derives the principle $4p$ is shown in Section 4.3.

(ii) This is Corollary 4.6.5.

(iii) As explained above this follows easily from the completeness results in the previous sections. That L"ob's Principle derives the principle $4$ is explained in Section 4.3.1.

(iv) These statements are proved in Section 3.3.