Provability Logic and Admissible Rules
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Chapter 7

The admissible rules of IPC

In this chapter we give a basis for the admissible rules of intuitionistic propositional logic. We proceed as follows. In the first section we define a proof system, called AR, which derives expressions of the form $A \rightarrow B$, where $A$ and $B$ are propositional formulas. In Section 7.4 we then show that AR is a proof system for the admissible rules: AR derives $A \rightarrow B$ iff $A \vdash B$. The proof of this fact has two main ingredients: In Section 7.2 we characterize AR in terms of Kripke models. We define what an AR-model is and show that AR derives $A \rightarrow B$ if and only if $B$ is valid in all AR-models on which $A$ is valid. Note that in the light of Section 7.4 this is a semantical characterization of the admissible rules. In Section 7.3 we derive a semantical characterization (in terms of classes of finite Kripke models) of the admissible rules from results by Ghilardi (1998). In Section 7.4 we show that these two characterizations are 'the same', which leads to the result mentioned above. Finally, in the last section we show how this provides us with a basis for the admissible rules.

7.1 A proof system

As explained above, we define a system AR that is a proof system which derives expressions of the form $A \rightarrow B$ where $A$ and $B$ are propositional formulas. To keep the definition of this system readable, we will use the following abbreviation,

$$(A)(B_1, \ldots, B_n) \equiv_{def} (A \rightarrow B_1) \lor \ldots \lor (A \rightarrow B_n).$$

Furthermore, we adhere to the same reading conventions as in the case of preservativity logic (Section 3.4).
Chapter 7. The admissible rules of IPC

Axioms:

\[ V \; ((A \rightarrow B \lor C) \lor D) \vdash ((A)(E_1, \ldots, E_n, B, C) \lor D), \]

for \( A = \bigwedge_{i=1}^n (E_i \rightarrow F_i) \)

\[ I \; A \vdash B, \quad \text{where IPC} \vdash (A \rightarrow B) \]

Rules:

\[
\begin{align*}
\text{Conj} & \quad \frac{C \vdash A \quad C \vdash B}{C \vdash A \land B} \\
\text{Cut} & \quad \frac{A \vdash B \quad B \vdash C}{A \vdash C}
\end{align*}
\]

Note that \( V \) is not an axiom in the strict sense. It consists in fact of the infinitely many principles \( V_n \), which are

\[ V_n \quad ((\bigwedge_{i=1}^n (E_i \rightarrow F_i) \rightarrow (B \lor C) \lor D) \vdash ((\bigwedge_{i=1}^n (E_i \rightarrow F_i))(E_1, \ldots, E_n, B, C) \lor D). \]

De Jongh and Visser observed that the rules corresponding to \( V_n \) (Section 7.5) are admissible and conjectured them to be a basis.

As noted before, if \( A \not\vdash C \) and \( B \not\vdash C \) then also \( (A \lor B) \not\vdash C \). This property of the admissible rules is not reflected in the rules of \( AR \). That is, there is no rule

\[
\text{Disj} \quad \frac{A \vdash C \quad B \vdash C}{(A \lor B) \vdash C}
\]

However, it turns out that \( AR \) satisfies this rule. This is the next lemma, which we will need in the completeness proof for \( AR \) to come.

7.1.1. Lemma. If \( AR \vdash A \vdash C \) and \( AR \vdash B \vdash C \) then \( AR \vdash (A \lor B) \vdash C \).

Proof. It is easy to prove (with an induction to the length of derivation) that \( AR \vdash A \vdash B \) implies \( AR \vdash (A \lor C) \vdash (B \lor C) \). Hence \( AR \vdash A \vdash B \) also implies \( AR \vdash (C \lor A) \vdash (C \lor B) \).

Now assume \( AR \vdash A \vdash C \) and \( AR \vdash B \vdash C \). From the previous observation it follows that \( AR \vdash (A \lor B) \vdash (C \lor B) \) and \( AR \vdash (C \lor B) \vdash (C \lor C) \). Clearly, also \( AR \vdash (C \lor C) \vdash C \). Applying Cut (twice) gives the desired result.

7.2 Completeness of the proof system

In this section we characterize \( AR \) in terms of Kripke models. The Kripke models we use have special properties, they are the so-called \( AR \)-models defined as follows.
AR-models

We call a Kripke model \( K \) an AR-model when it is a rooted model in which every finite set of nodes \( \{u_1, \ldots, u_n\} \) has a tight predecessor \( u \), i.e., a node \( u \) such that

\[ u \leq u_1, \ldots, u_n \wedge \forall u' > u (u_i \leq u'), \text{ for some } i \in \{1, \ldots, n\}. \]

(We write '\( u \leq y_1, \ldots, y_m \)' for '\( u \leq y_1 \wedge x \leq y_2 \wedge \ldots \wedge x \leq y_m \).')

We will prove that AR derives \( A \vdash B \) if and only if \( B \) is valid in every AR-model in which \( A \) is valid. The proof uses a lemma which we present separately in advance. Before stating it, let us remind the reader that a set of formulas \( x \) is called IPC-saturated if it is a consistent set such that for all \( A \) and \( B \), if \( x \vdash A \wedge B \), then \( A \in x \) or \( B \in x \). In particular, \( x \) is closed under deduction in IPC.

7.2.1. Lemma. Let \( \Theta \) be some set of formulae. Every IPC-saturated set \( x \subseteq \Theta \) can be extended to an IPC-saturated set \( y \subseteq \Theta \) such that for no IPC-saturated set \( y' \) it holds that \( y \subset y' \subseteq \Theta \).

Proof. Let \( x \) and \( \Theta \) be as in the lemma. We construct a sequence \( y_0 \subseteq y_1 \subseteq \ldots \), such that for all \( i \), \( * (y_i) \) holds, where the property \( * (\cdot) \) is defined as

\[ *(z) \text{ for all } n, \text{ for all } A_1, \ldots, A_n: \text{ if } z \vdash A_1 \vee \ldots \vee A_n, \]

then \( A_i \in \Theta \) for some \( i = 1, \ldots, n \).

We construct the sequence of sets as follows. Let \( C_0, C_1, \ldots \) be an enumeration of all formulae in which every formula occurs infinitely often. We put \( y_0 = x \). Clearly \( * (y_0) \) holds. Suppose \( y_i \) is already defined. Then we put

\[ y_{i+1} \equiv \begin{cases} y_i \cup \{C_i\} & \text{if } * (y_i \cup \{C_i\}) \text{ does hold} \\ y_i & \text{if } * (y_i \cup \{C_i\}) \text{ does not hold} \end{cases} \]

Now we take \( y = \bigcup_i y_i \). First, we have to see that this is indeed an IPC-saturated set. And second we have to show that there are no proper supersets of \( y \) which are IPC-saturated and are contained in \( \Theta \).

To see that \( y \) is IPC-saturated, suppose \( y \vdash A \vee B \). Hence \( y_i \vdash A \vee B \), for some \( i \). There are \( i \leq j \leq k \) such that \( C_j = A \) and \( C_k = B \). If \( * (y_j \cup \{C_j\}) \) or \( * (y_k \cup \{C_k\}) \) holds, then clearly \( A \) or \( B \) is in \( y \). We show that indeed one of \( * (y_j \cup \{C_j\}) \) and \( * (y_k \cup \{C_k\}) \) has to hold. Arguing by contradiction, assume this is not the case. Thus there are \( A_1, \ldots, A_n, B_1, \ldots, B_m \) such that \( y_j, C_j \vdash \bigvee_{i=1}^n A_i \) and \( y_k, C_k \vdash \bigvee_{i=1}^m B_i \) but none of \( A_1, \ldots, A_n, B_1, \ldots, B_m \) is in \( \Theta \). Since \( y_i \subseteq y_j \subseteq y_k \) and \( y_j \vdash C_j \vee C_k \), this implies that \( y_k \vdash \bigvee_{i=1}^n A_i \vee \bigvee_{i=1}^m B_i \), which contradicts the fact that \( * (y_k) \) holds.

To see that there are no IPC-saturated proper supersets of \( y \) which are contained in \( \Theta \), consider an IPC-saturated set \( y \subseteq y' \subseteq \Theta \). We show that \( y = y' \). Consider a
formula \( A \in y' \), and suppose \( C_i = A \). It is easy to see that since \( y_i \subseteq y' \subseteq \Theta \) and the fact that \( y' \) is saturated, \( \ast (y_i \cup \{C_i\}) \) holds. Hence \( A \in y \). Therefore \( y = y' \). \( \square 

Now we are ready to prove the following lemma.

**7.2.2. Proposition.** \( \text{AR} \vdash A \rightarrow B \) iff \( B \) is valid on all AR-models on which \( A \) is valid.

**Proof.** The direction from left to right. We have to see that if \( \text{AR} \vdash A \rightarrow B \) and \( A \) is valid on an AR-model, then \( B \) is valid on this model as well. This can be shown by induction to the length of the derivation of \( A \rightarrow B \) in AR.

The case that \( A \rightarrow B \) is an instance of the axiom scheme \( I \) is easy. In the induction step we have to consider the two rules. All of them are straightforward.

Therefore, we only consider \( V \). We have to show that for any conjunct of implications \( A = \bigwedge_{i=1}^{n}(E_i \rightarrow F_i) \), if \( (A \rightarrow B \lor C) \lor D \) is valid on all AR-models, then so is \( (A)(B, C, E_1, \ldots, E_n) \lor D \). Therefore, assume that indeed for such a formula \( A \), \( (A \rightarrow B \lor C) \lor D \) is valid on an AR-model \( K \). Let \( v \) be the root of \( K \). We show that \( (A)(B, C, E_1, \ldots, E_n) \lor D \) is valid in \( K \) at \( v \), whence that \( (A)(B, C, E_1, \ldots, E_n) \lor D \) is valid in \( K \).

Arguing by contradiction, assume \( (A)(B, C, E_1, \ldots, E_n) \lor D \) is not valid at \( v \). Hence \( (A \rightarrow B \lor C) \) is valid at \( v \). Moreover, \( \neg A \) is not valid at \( v \). Therefore, there is a nonempty set \( U \) of nodes, such that

\[
\forall x (x \vdash A \text{ iff for some } u \in U, \ u \leq x).
\]

Since \( (A)(B, C, E_1, \ldots, E_n) \) is not valid at \( v \), there are, for some \( m \leq n + 2 \), nodes \( u_{i_1}, \ldots, u_{i_m} \in U \) such that

\[
\forall D \in \{B, C, E_1, \ldots, E_n\} \exists u \in \{u_{i_1}, \ldots, u_{i_m}\} \ u \nvdash D.
\]

Since we consider an AR-model the set \( \{u_{i_1}, \ldots, u_{i_m}\} \) has a tight predecessor. That means that there is a node \( u \) such that

\[
u \leq u_{i_1}, \ldots, u_{i_m} \land \forall u' \geq u(u_{i_j} \leq u', \text{ for some } j \leq m).
\]

If \( A \) is valid at \( u \) then \( B \) or \( C \) has to be valid at \( u \), which contradicts the fact that for both \( B \) and \( C \) there is a node in \( u_{i_1}, \ldots, u_{i_m} \) which does not validate the formula. On the other hand, if \( A \) is not valid at \( u \), then since \( A \) is valid at all nodes \( u' \geq u \), \( E_j \) has to be valid at \( u \), for some \( j \). But this is a contradiction as well, since for every \( j \in \{1, \ldots, n\} \) there is a node in \( u_{i_1}, \ldots, u_{i_m} \) which does not validate \( E_j \).

The direction from right to left. Assume \( \text{AR} \vdash A \rightarrow B \). We construct an AR-model \( K \) in which \( A \) is valid while \( B \) is not.

First we construct an IPC-saturated set of formulas \( v \) in such a way that

\[
A \in v, \ B \not\in v, \text{ for all } C \rightarrow D: \text{ if } \text{AR} \vdash C \rightarrow D \text{ and } C \in v, \text{ then } D \in v. \quad (7.1)
\]
This $v$ will be the root of the model $K$ we are going to construct. The existence of $v$ is proved in the following Claim.

Claim If $\text{AR} \not\vdash A \supset B$, then there is an IPC-saturated set $v$ such that $A \in v$ and $B \not\in v$, which has the property that if for some $C, D$, $\text{AR} \vdash C \supset D$ and $C \in v$, then $D \in v$ as well.

Proof of Claim. Assume $\text{AR} \not\vdash A \supset B$. We construct a sequence of finite sets $\{A\} = x_0 \subseteq x_1 \subseteq \ldots$ such that for all $i$, $\text{AR} \not\vdash (\land x_i) \supset B$, and if $\text{AR} \vdash (\land x_i) \supset C$, then $C \in x_j$ for some $j$. The set $v$ we look for will be the set $\bigcup x_i$.

Let $C_0, C_1, \ldots$ be an enumeration of all formulas in which every formula occurs infinitely often. Given the set $x_i$, we show how to construct $x_{i+1}$.

\[
x_{i+1} \overset{\text{def}}{=} \begin{cases} 
  x_i & \text{if } \text{AR} \not\vdash (\land x_i) \supset C_i \\
  x_i \cup \{C_i\} & \text{if } \text{AR} \vdash (\land x_i) \supset C_i, C_i \text{ is not a disjunction} \\
  x_i \cup \{D_j, C_i\} & \text{if } \text{AR} \vdash (\land x_i) \supset C_i, C_i = D_1 \lor D_2, j = 1, 2
\end{cases}
\]

is the least such that $\text{AR} \not\vdash (\land x_i \land D_j) \supset B$.

It is easy to see that each of these sets $x_i$ has the desired properties, assuming it is well-defined. Thus it remains to show that they are indeed well-defined, i.e. that given $x_i$, $x_{i+1}$ exists. Therefore, suppose $\text{AR} \vdash (\land x_i) \supset C_i$ and $C_i = (D_1 \lor D_2)$. We have to see that either $\text{AR} \not\vdash (\land x_i \land D_1) \supset B$ or $\text{AR} \not\vdash (\land x_i \land D_2) \supset B$. Arguing by contradiction, assume this is not the case. But then we can derive the contradiction that $\text{AR} \vdash (\land x_i) \supset B$ in the following way (we do not state all the rules used, but only the crucial ones).

\[
\begin{align*}
\text{AR} \vdash & \quad (\land x_i \land D_1) \supset B \\
& (\land x_i \land D_2) \supset B \\
& (\land x_i \land (D_1 \lor D_2)) \supset B \quad \text{(Lemma 7.1.1)} \\
& (\land x_i) \supset (\land x_i \land (D_1 \lor D_2)) \quad \text{(assumption on } x_i) \\
& (\land x_i) \supset B. \quad \text{(Cut)}
\end{align*}
\]

Now we take $v = \bigcup_i x_i$. It is easy to see that $v$ has the desired properties. This proves the Claim.

Thus we know that there exists an IPC-saturated set $v$ which satisfies (7.1). Next we construct our model $K$ as follows. Its domain consists of all IPC-saturated sets which extend $v$. Its partial order $\preceq$ is the subset relation $\subseteq$. And the forcing relation is defined via

\[w \Vdash p \text{ iff } p \in w, \text{ for propositional variables } p.\]
It is easy to see that this indeed defines a Kripke model, that the model is rooted, and that $A$ is valid in this model but $B$ is not. Thus it only remains to show that $K$ is an AR-model.

Therefore, consider nodes $u_1, \ldots, u_n \in K$. We have to show that there is a node $u$ such that

$$u \leq u_1, \ldots, u_n \land \forall u' \succ u (u_i \leq u', \text{ for some } i \leq n).$$

First note that $u_1 \cap \ldots \cap u_n$ is not saturated in general. Therefore, although $u_1 \cap \ldots \cap u_n$ contains $v$, it does not have to be a node in $K$. Let now

$$\Delta = \{E \rightarrow F \mid (E \rightarrow F) \in u_1 \cap \ldots \cap u_n \land E \not\in u_1 \cap \ldots \cap u_n\}.$$

Then we have

**Claim** The set $\{C \mid v \cup \Delta \vdash C\}$ is IPC-saturated.

**Proof of Claim.** Suppose that $v \cup \Delta \vdash C_1 \lor C_2$ holds. This implies that there is a conjunct $D = \bigwedge_{i=1}^{m} (E_i \rightarrow F_i)$ of implications in $\Delta$, such that it holds that $v \vdash (D \rightarrow C_1 \lor C_2)$. Thus $(D \rightarrow C_1 \lor C_2) \in v$, because $v$ is saturated. Since the expression $(D \rightarrow C_1 \lor C_2) \supset (D)(C_1, C_2, E_1, \ldots, E_m)$ is derivable in AR, the way $v$ is constructed implies that then also $(D)(C_1, C_2, E_1, \ldots, E_m) \in v$. And thus one of $(D \rightarrow C_1), (D \rightarrow C_2), (D \rightarrow E_1), \ldots, (D \rightarrow E_m)$ is in $v$. Since no $E_i$ is in $u_1 \cap \ldots \cap u_n$, this implies that $v$ does not contain any of $(D \rightarrow E_i)$. Therefore $v$ contains either $(D \rightarrow C_1)$ or $(D \rightarrow C_2)$. Hence $v \cup \Delta$ derives either $C_1$ or $C_2$. This proves the Claim.

By the previous claim and the fact that $v \cup \Delta \subseteq u_1 \cap \ldots \cap u_n$, it follows from Lemma 7.2.1 that $\{C \mid v \cup \Delta \vdash C\}$ can be extended to an IPC-saturated set $u \subseteq u_1 \cap \ldots \cap u_n$ such that there are no saturated sets $u'$ with $u \subseteq u' \subseteq u_1 \cap \ldots \cap u_n$. We show that this is the set we look for, i.e. if $u' \succ u$ for some saturated set $u'$, then $u_i \leq u'$, for some $i \in \{1, \ldots, n\}$.

Suppose not, that is, let $u \subseteq u'$ for some saturated set $u'$ and assume that no $u_i$ is contained in $u'$. We derive a contradiction. For all $i \leq n$, we (can) choose a formula $A_i \in u_i$ outside $u'$. Then the formula $A_1 \lor \ldots \lor A_n$ is in $u_1 \cap \ldots \cap u_n$ but not in $u'$. From the construction of $u$, and the fact that $u'$ is a superset of $u$, it follows that $u'$ is not contained in $u_1 \cap \ldots \cap u_n$. Thus there is a formula $E \in u'$ which is not in this intersection. Now $(E \rightarrow A_1 \lor \ldots \lor A_n)$ is an element of $\Delta$, thus also of $u$. Hence $A_1 \lor \ldots \lor A_n$ should be in $u'$, a contradiction. This finally proves the proposition. \hfill $\Box$

### 7.3 Results by Ghilardi

In the proof of the characterization of the admissible rules in terms of $\succ$ we will use, besides the semantical completeness of AR (Section 7.2), the following fact which follows from results proved by Ghilardi (1998).
7.3.1. Proposition. If \( A \models B \), then \( B \) is valid in every stable class of finite rooted Kripke models which has the extension property (see Section 6.3.1) and in which \( A \) is valid.

This section is devoted to the recapitulation of the results by Ghilardi which lead to the proposition above. First we have to introduce some terminology.

**Terminology**

Let \( \bar{p} \) be a sequence of propositional variables. We say that a formula \( A \) is a *formula in \( \bar{p} \)*, when all the propositional variables in \( A \) are among the variables in the sequence \( \bar{p} \). We say that a Kripke model is a Kripke model *over \( \bar{p} \)*, when the forcing relation of the model is only defined for formulas in \( \bar{p} \). If \( \bar{p} \) is the sequence of all the propositional variables that occur in \( A \), then \( \text{Mod}(A) \) denotes all *finite* models of \( A \) over \( \bar{p} \).

Following Fine (Fine 1974) (Fine 1985), Ghilardi defines equivalence relations \( \sim_n \) and preorders \( \leq_n \) between rooted Kripke models. Let \( K, K' \) be two rooted Kripke models with roots \( b \) and \( b' \) respectively.

\[
\begin{align*}
K \sim_0^b K' & \quad \equiv_{def} b \models p \text{ iff } b' \models p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \sim_{n+1}^b K' & \quad \equiv_{def} \forall k \in K \exists k' \in K' ((K)_k \sim_n (K')_{k'}) \text{ and vice versa.} \\
K \leq_0^b K' & \quad \equiv_{def} b' \models p \text{ implies } b \models p, \text{ for all atoms } p \text{ in } \bar{p}. \\
K \leq_{n+1}^b K' & \quad \equiv_{def} \forall k \in K \exists k' \in K' ((K)_k \sim_n (K')_{k'}). 
\end{align*}
\]

When it is clear from the context to which sequence of variables we refer we omit this in the notation.

Moreover Ghilardi uses a measure of complexity, \( c(\cdot) \), on propositional formulas defined as follows. Put \( c(A) = 0 \) if \( A \) is a propositional variable, \( c(A \circ B) = \max\{c(A), c(B)\} \), for \( \circ = \land, \lor, \) and \( c(A \rightarrow B) = 1 + \max\{c(A), c(B)\} \).

**The proof of Proposition 7.3.1**

In the proof of Proposition 7.3.1 we will use four results by Ghilardi which we will state below. The first three have to do with the relation \( \leq_n \).

7.3.2. Proposition. (Ghilardi 1998) For two finite rooted Kripke models \( K \) and \( K' \) over \( \bar{p} \) it holds that \( K \leq_n K' \) iff for all formulas \( A \) in \( \bar{p} \) with \( c(A) \leq n \), \( K' \models A \) implies \( K \models A \).

7.3.3. Proposition. (Ghilardi 1998) Let \( \mathcal{K} \) be a class of finite rooted Kripke models over \( \bar{p} \) for which there exists a number \( n \) such that for all Kripke models \( K \) over \( \bar{p} \) it holds that

\[
\text{if there is a } K' \in \mathcal{K} \text{ with } K \leq_n K', \text{ then } K \in \mathcal{K}.
\]

Then \( \mathcal{K} = \text{Mod}(A) \) for some formula \( A \) in \( \bar{p} \).
7.3.4. **Proposition.** (Ghilardi 1998) If a stable class \( \mathcal{K} \) of finite rooted Kripke models over \( \bar{p} \) has the extension property then so does the class of models

\[ \{ K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}(K \leq_n K') \}. \]

The heart of Proposition 7.3.1 is the following theorem.

7.3.5. **Theorem.** (Ghilardi 1998) Let \( A \) be a formula in \( \bar{p} \). If \( \text{Mod}(A) \) has the extension property then there is a substitution \( \sigma \) such that \( \vdash \sigma(A) \) and for all formulas \( D \) in \( \bar{p} \), \( A \vdash D \leftrightarrow \sigma(D) \).

Now the proof of Proposition 7.3.1 runs as follows. Suppose \( A \vdash B \) and let \( \mathcal{K} \) be a stable class of finite rooted Kripke models with the extension property in which \( A \) is valid. Assume that all the propositional variables in \( A \) and \( B \) are among \( \bar{p} \). Then let \( \mathcal{K}' \) be the class of all Kripke models of \( \mathcal{K} \), but then considered as Kripke models over \( \bar{p} \). Note that \( \mathcal{K}' \) is again a stable class of finite rooted Kripke models with the extension property in which \( A \) is valid. Let \( n \) be some number such that \( c(A) \leq n \), and let

\[ \mathcal{K}'' = \{ K \mid K \text{ is a finite rooted model over } \bar{p} \text{ and } \exists K' \in \mathcal{K}'(K \leq_n K') \}. \]

By Proposition 7.3.2, \( A \) is valid in the class \( \mathcal{K}'' \) because it is valid in \( \mathcal{K}' \). And by Proposition 7.3.3 we know that \( \mathcal{K}'' = \text{Mod}(C) \) for some formula \( C \) in \( \bar{p} \). Since, by Proposition 7.3.4, we also know that \( \mathcal{K}'' \) has the extension property, we can apply Theorem 7.3.5 to conclude that there is a substitution \( \sigma \) such that

\( \text{IPC} \vdash \sigma(C) \) and \( C \vdash B \leftrightarrow \sigma(B) \).

Clearly, the fact that \( A \) is valid in \( \text{Mod}(C) \) implies that \( C \vdash A \). Hence \( \text{IPC} \vdash \sigma(A) \).

7.4 **Characterizations of admissibility**

We are now ready to give the promised characterizations of the admissible rules of \( \text{IPC} \). One is in terms of \( \triangleright \), a proof system for the admissible rules. The other two are in terms of Kripke models. Let us state them before we consider their proofs.

7.4.1. **Theorem.** \( A \vdash B \) iff \( \text{AR} \vdash A \triangleright B \).

7.4.2. **Corollary.** \( A \vdash B \) iff \( B \) is valid in every \( \text{AR} \)-model in which \( A \) is valid.

7.4.3. **Corollary.** \( A \vdash B \) iff \( B \) is valid in every stable class of finite rooted Kripke models with the extension property in which \( A \) is valid.
The last corollary is Proposition 7.3.1. The second characterization is a corollary of the first one in combination with Proposition 7.2.2 and Lemma 7.4.4. The latter is also needed in the proof of the first characterization. Lemma 7.4.4 shows that there is a natural correspondence between AR-models and stable classes of finite rooted Kripke models with the extension property. Therefore, the two corollaries are in some sense the same. We first treat this lemma and then we prove Theorem 7.4.1.

7.4.4. Lemma. For all \( n \) and all finite sequences of propositional variables \( \bar{p} \) we have the following correspondence:

(a) For every AR-model \( K \) there is a stable class \( K \) of finite rooted Kripke models with the extension property such that

\[
\text{for all } A \text{ in } \bar{p} \text{ with } c(A) \leq n: K \models A \text{ iff } K \models A.
\]

(b) For every stable class \( K \) of finite rooted Kripke models with the extension property there is an AR-model \( K \) such that

\[
\text{for all } A: K \models A \text{ iff } K \models A.
\]

Proof. Let \( n \) be some number and let \( \bar{p} \) be some finite sequence of propositional variables. First of all, let \( A \) be the set of all formulas \( A \) in \( \bar{p} \) with \( c(A) \leq n \). This set is, modulo provable equivalence, finite.

To show part (a) of the lemma, suppose \( K \) is an AR-model. Let \( \mathcal{K} \) be the class of all Kripke models \( K' \) such that \( K' \) is a finite rooted submodel of \( K \), and such that

\[
\forall A \in A \forall x \in K'(K', x \models A) \iff K, x \models A).
\]

(7.2)

It is easy to see that \( \mathcal{K} \) is stable. We show that \( \mathcal{K} \) has the extension property.

Consider models \( K_1, \ldots, K_n \) in \( \mathcal{K} \), with roots \( u_1, \ldots, u_n \) respectively. Let \( u \) be a tight predecessor of \( u_1, \ldots, u_n \) in \( K \). That means that

\[
u \preceq u_1, \ldots, u_n \land \forall u' \succ u(u_i \preceq u', \text{ for some } i \in \{1, \ldots, n\}).
\]

Let \( K' \) be the submodel the domain of which is the union of \( \{u\} \) and the domains of \( K_1, \ldots, K_n \). It is easy to see \( K' \) satisfies (7.2). Hence \( K' \) is in \( \mathcal{K} \). This shows that \( \mathcal{K} \) has the extension property.

It remains to show that

\[
\text{for all } A \in A: K \models A \text{ iff } \mathcal{K} \models A.
\]

The direction from left to right follows from the definition of \( \mathcal{K} \). The direction from right to left is shown by contraposition, i.e. by showing that for all \( A \in A \) it holds that whenever \( K \not\models A \) there is a \( K' \in \mathcal{K} \) such that \( K' \not\models A \) (it suffices to show that \( \mathcal{K} \) is not empty, but the proof is the same). This again follows from the following standard result. We include the proof for the sake of completeness.
Claim: For every Kripke model $K$, for every node $w$ in $K$, there is a finite rooted submodel $K'$ of $K$ with root $w$, such that

$$\forall A \in \mathcal{A} \forall x \in K'(K', x \vdash A \iff K, x \vdash A).$$  \hspace{1cm} (7.3)

Proof of Claim. Let $A, K = (W, \prec, \models)$ and $w$ be as in the claim. Now we choose step by step, starting with $w$, a finite subset of $W$ a copy of which will be the domain $W_w$ of our new model $K' = (W_w, \prec_w, \models_w)$. Put $\alpha_0 = w$. Suppose $\alpha_\sigma$ is defined. We choose elements $\alpha_{\sigma\sigma = (B \rightarrow C)}$ in $W$, for all elements $(B \rightarrow C) \in \{(D \rightarrow E) \in \mathcal{A} | K, \alpha_\sigma \models D \rightarrow E\}$. The node $\alpha_{\sigma\sigma = (B \rightarrow C)}$ is an element $v \in W$ such that $\alpha_\sigma \prec v, K, v \models B$ and $K, v \not\models C$. Note that such elements can always be found.

Now define $W_w = \{\sigma | \sigma \text{ is defined}\}$, and define the partial order and the forcing relation on $K$ as

$$\sigma \prec w \tau \implies_{def} \alpha_\sigma \prec \alpha_\tau,$$

$$\sigma \models_w p \implies_{def} \alpha_\sigma \models p, \text{ for } p \in \bar{p}.$$

Clearly, $K'$ is finite, as $\mathcal{A}$ is finite too. It is also easy to infer that (7.3) is satisfied. This proves the claim, and thereby part (a) of the correspondence.

To show part (b) of the lemma, let $\mathcal{K}$ be a stable class of finite rooted Kripke models with the extension property. The model $K$ we are going to construct will consist of equivalence classes of nodes of models in $\mathcal{K}$.

Replace every model in $\mathcal{K}$ by an isomorphic copy, in such a way that the domains of distinct models are disjoint. Let us define for nodes $k \in K$ and $k' \in K'$

$$k \cong k' \implies_{def} (K)_k \text{ and } (K')_{k'} \text{ are isomorphic.}$$

(Remember that $K_k$ is the submodel of $K$ generated by $k$, see Section 6.3.1.) We write $k \models A$ when $A$ is valid at $k$ in the unique model in $\mathcal{K}$ to which $k$ belongs.

Now we define the domain of $K$ as the set of all $\cong$-equivalence classes $[k]$ of nodes $k$ of models in $\mathcal{K}$. The partial order and the forcing relation on $K$ are defined via

$$[k] \prec [k'] \implies_{def} \exists l \in [k] \exists l' \in [k'] (l, l' \text{ are nodes in the same model and } l \prec l' \text{ holds in this model.})$$

$$[k] \models p \implies_{def} k \models p.$$

Since every two $\cong$-equivalent nodes force the same propositional variables the notion of forcing is well-defined. We have to see that $K$ is in fact an AR-model and that

$$\text{for all } A: K \models A \iff \mathcal{K} \models A. \hspace{1cm} (7.4)$$

We show that $K$ is an AR-model and leave the proof of (7.4) to the reader.

Consider nodes $[k_1], \ldots, [k_n]$ in $\mathcal{K}$. Assume $k_i$ is a node in the model $K_i \in \mathcal{K}$. Since $\mathcal{K}$ has the extension property there is (an isomorphic copy of) a variant of $(\Sigma(K_i)_{k_i})$ in $\mathcal{K}$. Let $b$ be the root of this variant. It is easy to see that $[b]$ is a tight predecessor of $[k_1], \ldots, [k_n]$ in $K$. This proves part (b) of the lemma. \hfill \Box
7.4.5. Corollary. The following are equivalent

(a) $B$ is valid in every AR-model in which $A$ is valid.

(b) $B$ is valid in every stable class of finite rooted Kripke models

with the extension property in which $A$ is valid.

Now we are ready to give the

Proof of Theorem 7.4.1. First the direction from right to left. (De Jongh and Visser) We have to show that for all instances $A/B$ of $V$ and $I$, $A$ admissibly derives $B$, and we have to see that the three rules of AR preserve admissibility. That is, when reading $\vdash$ for $\vdash$, if the assumptions of a rule are valid then so is the conclusion. For the two rules this is trivial. Therefore, it remains to treat the axioms. For instances $A/B$ of $I$ it clearly is the case that $A \vdash B$. Thus all we have to show is that for every instance $A/B$ of the scheme $V$ it holds that if $A$ is derivable in IPC then so is $B$.

Therefore, consider such instance $A/B$ of $V$. Let $X = \bigwedge_{i=1}^{n}(E_i \rightarrow F_i)$ and let $A = X \rightarrow C \lor D$ and $B = (X)(C, D, E_i, \ldots, E_n)$. Arguing by contradiction, suppose $A$ is derivable but $B$ is not. This implies that none of the formulas $(X \rightarrow C), (X \rightarrow D), (X \rightarrow E_i), \ldots, (X \rightarrow E_n)$ is derivable. Thus there are Kripke models $K_1, \ldots, K_{n+2}$ at which $X$ is valid but at which respectively $C, D, E_1, \ldots, E_n$ are not valid. Consider the model $(\sum K_i)'$ and call its root $b$. Since $A$ is derivable $A$ is valid at $b$. Note furthermore that none of the formulas $C, D, E_1, \ldots, E_n$ can be valid at $b$. Therefore, the conjunction $X$ cannot be valid at $b$. But it cannot be not valid either. For if so, there is some $i \leq n$ for which there is a node above $b$ at which $E_i$ is valid while $F_i$ is not valid. As $X$ is valid at all nodes except $b$ the only possibility for this is the node $b$ itself. Thus one of the formulas $E_1, \ldots, E_n$ would be valid at $b$, which cannot be.

The direction from left to right follows immediately from Proposition 7.3.1, Corollary 7.4.5 and Proposition 7.2.2.

7.5 A basis for the admissible rules

Let $R_V$ denote the rule corresponding to $V_i$ (see Section 7.1), i.e. let

$$R_{V_i} \quad (\bigwedge_{i=1}^{n}(E_i \rightarrow F_i) \rightarrow B \lor C) \lor D/(\bigwedge_{i=1}^{n}(E_i \rightarrow F_i))(E_1, \ldots, E_n, B, C) \lor D.$$

Further, let

$$R_{\neg V_i} \quad (\bigwedge_{i=1}^{n}(E_i \rightarrow F_i) \rightarrow B \lor C)/(\bigwedge_{i=1}^{n}(E_i \rightarrow F_i))(E_1, \ldots, E_n, B, C).$$
Chapter 7. The admissible rules of IPC

Let \( \mathcal{V} \) be the set \( \{R_{V_1}, R_{V_2}, \ldots\} \) and let \( \mathcal{V}^- \) be the set \( \{R_{V_1}, R_{V_2}, \ldots\} \). We need one more lemma to establish that the sets of rules \( \mathcal{V} \) and \( \mathcal{V}^- \) are respectively a basis and a subbasis for the admissible rules of IPC.

7.5.1. Lemma. If \( \text{AR} \vdash A \rightarrow B \) then the rule \( A/B \) is derivable in IPC from the set of rules \( \mathcal{V} \).

Proof. We prove the proposition by induction on the length \( n \) of the derivation of \( A \rightarrow B \) in \( \text{AR} \). For \( n = 0 \) there is nothing to prove.

For \( n > 0 \), suppose that the last rule applied in the derivation of \( A \rightarrow B \) is the Conjunction rule. This implies that there are \( B_1, B_2 \) such that \( B = B_1 \land B_2 \), and such that \( A \rightarrow B_1 \) and \( A \rightarrow B_2 \) are derivable, and moreover have derivations of length smaller than \( n \). By the induction hypothesis, \( A/B_1 \) and \( A/B_2 \) are derivable in IPC from \( \{R_{V_1}, R_{V_2}, \ldots\} \). And thus \( A/B_1 \land B_2 \) is derivable in IPC from \( \{R_{V_1}, R_{V_2}, \ldots\} \) as well. The case that the last rule applied in the derivation of \( A \rightarrow B \) is the Cut Rule is completely similar.

7.5.2. Theorem. \( \mathcal{V} \) is a basis for the admissible rules of IPC.

Proof. Immediate from Lemma 7.5.1 and Theorem 7.4.1.

7.5.3. Corollary. \( \mathcal{V}^- \) is a subbasis for the admissible rules of IPC.

7.6 The connection with Heyting Arithmetic

In this section we explain what the results of this chapter mean for the provability and preservativity logic of \( \text{HA} \).

Visser (1999) showed that the admissible rules of IPC are the same as the propositional admissible rules of \( \text{HA} \). Therefore, Corollaries 7.5.2 and 7.5.3 give us

7.6.1. Corollary. \( \mathcal{V} \) and \( \mathcal{V}^- \) are respectively a basis and a subbasis for the propositional admissible rules of \( \text{HA} \).

In Theorem 7.4.1 we saw that

\[ A/B \text{ is a propositional admissible rule of IPC iff } \text{AR} \vdash A \rightarrow B. \]

In combination with the result in (Visser 1999) that states that the propositional admissible rules of \( \text{HA} \) and IPC are the same, this gives

\[ A/B \text{ is a propositional admissible rule of HA iff } \text{AR} \vdash A \rightarrow B. \]

It is easy to see that the logic \( \text{AR} \) is equivalent to the logic axiomatized by the preservativity principles (Section 2.2) \( P1, P2, Dp \) and all the instances \( A \rightarrow B \) of \( \forall \alpha \), where \( A \) and \( B \) are propositional formulas, characterizes the admissible rules of IPC (use Lemma 7.1.1).
From the definition of preservativity it follows that if $A \triangleright B$ is in the provability logic of HA, then $A/B$ is an admissible rule of HA (Chapter 2). Finally, in combination with the fact that $AR$ is part of the preservativity logic of HA (Visser 1994), this leads to

for propositional formulas $A, B$:

- $A/B$ is a propositional admissible rule of HA iff
- $A \triangleright B$ is in the preservativity logic of HA.

This shows that HA recognizes its propositional admissible rules.