Provability Logic and Admissible Rules
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In this chapter we show that IPC is characterized by its admissible rules: In Chapter 7 we gave a countable basis $\mathcal{V}$ for the admissible rules of IPC. Here we show (Section 8.1) that the only intermediate logic with the Disjunction Property for which all rules in this basis are admissible, is IPC. In Section 8.2 we prove that the characterization is optimal. We show that for any finite subset $X$ of $\mathcal{V}$ there is a proper intermediate logic for which $X$ is admissible. In Section 8.3 we show that the characterization is effective.

### 8.1 The characterization

In Chapter 7 we gave a simple c.e. description of the admissible rules (Theorem 7.5.2) which implied (Corollary 7.5.3) that the set $\mathcal{V}$ is a subbasis for the admissible rules of IPC. Let us recall the definition of this subbasis (Section 7.5): $\mathcal{V}$ is the collection of rules

$$R_{\mathcal{V}_n} \left( \bigwedge_{i=1}^{n} (E_i \rightarrow F_i) \rightarrow B \vee C \right) / \left( \bigwedge_{i=1}^{n} (E_i \rightarrow F_i) \right)(B, C, E_1, \ldots, E_n).$$

where we use the abbreviation,

$$(A)(B_1, \ldots, B_m) \equiv_{def} (A \rightarrow B_1) \lor \ldots \lor (A \rightarrow B_m).$$

The rest of this section is devoted to the proof that these admissible rules together with the Disjunction Property characterize IPC, i.e. we will show that for any intermediate logic which is not equal to IPC either the Disjunction Property does not hold or one of the rules $R_{\mathcal{V}_1}, R_{\mathcal{V}_2}, \ldots$ is not admissible. It is convenient to have the Disjunction Property built-in into the admissible rules. Therefore, we need the following definition.
Chapter 8. A characterization of IPC

Definition of the rules $P_n$

A theory $T$ has the property $P_n$ if for all substitutions $\sigma$,

\[
\text{if } \vdash_T \sigma(\bigwedge_{i=1}^n (p_i \rightarrow q_i) \rightarrow r \lor s) \text{ then } \\
\vdash_T \sigma(\bigwedge_{i=1}^n (p_i \rightarrow q_i) \rightarrow r) \text{ or } \vdash_T \sigma(\bigwedge_{i=1}^n (p_i \rightarrow q_i) \rightarrow s) \text{ or } \\
\vdash_T \sigma(\bigwedge_{i=1}^n (p_i \rightarrow q_i) \rightarrow p_i) \text{ or \ldots or } \vdash_T \sigma(\bigwedge_{i=1}^n (p_i \rightarrow q_i) \rightarrow p_n). 
\]

We will show that an intermediate logic is equal to IPC iff it has the property $P_n$, for all $n \geq 0$. The characterization mentioned above is an immediate corollary of this.

Note that a logic has $P_0$ if and only if it has the Disjunction Property. A logic has $P_n$ for all $n \geq 0$ if and only if it has the Disjunction Property and for all $n \geq 1$ the rule $V_n$ is admissible.

We need the following fact by Smoryński.

8.1.1. Fact. (Smoryński 1973) IPC is complete with respect to Jaskowski models.

8.1.2. Lemma. If an intermediate logic has the extension property it is the logic IPC.

Proof The lemma follows from the following two claims.

Claim If $T$ is an intermediate logic with the extension property, then every basic Jaskowski model is a model of $T$.

Proof of the Claim Let $T$ be an intermediate logic with the extension property (Subsection 6.3). Let $K$ be a basic Jaskowski model (Section 6.3.1). We show that $K_x$ is a model of $T$ by induction to the depth of the node $x$. The maximal nodes of $K$ clearly are models of $T$ since every classical model is a model of $T$. Suppose $x$ is another node in $K$ and let $x_1, \ldots, x_n$ be the immediate successors of $x$, i.e. the nodes $y$ such that $x < y$ and such that there is no node $x < z < y$. By the induction hypothesis the models $K_{x_1}, \ldots, K_{x_n}$ are models of $T$. Observe that $K_x$ is the model $(\sum K_{x_i})'$ (Section 6.3.1). Because every propositional variable is valid at at most one node in $K$ there is no other variant of $(\sum K_{x_i})'$ then the model itself. Since $T$ has the extension property this implies that $K_x$ is a model of $T$. This proves the Claim.

Claim If $T$ is an intermediate logic such that every basic Jaskowski model is a model of $T$, then $T = IPC$.

Proof of the Claim We show that $T \subseteq IPC$ by proving that if $\models_{IPC} A$ holds, then $\models_T A$ holds as well. If $\models_{IPC} A$ then there is a Jaskowski model $K$ in which $A$ is not valid (Fact 8.1.1). Let $K'$ be a basic model based on the frame of $K$. By assumption $K'$ is a model of $T$.

Now we define a substitution $\sigma$ via $\sigma(p) = \bigvee_{K, x \models p} A_x$, where the formulas $A_x$ are given by Fact 6.3.1. To see that $\sigma(A)$ is not valid at $K'$, observe that for
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every node $x$ and for every formula $B$ we have that $K, x 
ot \vdash B$ iff $K', x 
ot \vdash \sigma(B)$. Therefore, $\not \vdash_T \sigma(A)$. Hence $\not \vdash_T A$.

In the following lemma we need the notion of a saturated set. A $T$-saturated set $x$ is a set of formulas such that $A \in x$ or $B \in x$ whenever $x \vdash_T A \lor B$. In particular, a $T$-saturated set is closed under deduction in $T$.

8.1.3. Lemma. If an intermediate logic has the property $P_n$ for every $n \geq 0$, then it has the extension property.

Proof Let $T$ be an intermediate logic with the Disjunction Property, for which, for all $n$, $R_{V_n}$ is admissible. Consider models $K_1, \ldots, K_n$ of $T$ with roots $x_1, \ldots, x_n$ respectively. From now on we confuse a node with the set of formulas it forces.

Claim There exists a $T$-saturated set $x \subseteq x_1 \cap \ldots \cap x_n$ such that for all $T$-saturated sets $x \subset y$ there is some $i \leq n$ such that $x_i \subseteq y$.

Proof of the Claim Consider

$$\Delta = \{(E \rightarrow F) \mid E \not \in x_1 \cap \ldots \cap x_n \text{ and } F \in x_1 \cap \ldots \cap x_n\}.$$ 

Clearly, $\Delta \subseteq x_1 \cap \ldots \cap x_n$. Observe that the set $x_0 = \{A \mid \Delta \vdash_T A\}$ is $T$-saturated because for all $m$, the property $P_m$ holds. Now we construct a sequence of sets $x_0 = z_0 \subseteq z_1, \ldots$ as follows. Let $C_0, C_1, \ldots$ enumerate all formulas, with infinite repetition. Define the property $*(\cdot)$ on sets via

$$* (y) \quad \text{for all } m, \text{ for all } A_1, \ldots, A_m: \text{ if } y \vdash_T A_1 \lor \ldots \lor A_m,$$

then $A_i \in x_1 \cap \ldots \cap x_n$, for some $i = 1, \ldots, m$.

Note that $*(z_0)$ holds. If $*(z_i \cup \{C_i\})$ does not hold then put $z_{i+1} = z_i$. If $*(z_i \cup \{C_i\})$ holds do the following: if $C_i$ is no disjunction, put $z_{i+1} = z_i \cup \{C_i\}$; if $C_i = D \lor E$, let $z_{i+1}$ be $z_i \cup \{D\}$ if $*(z_i \cup \{D\})$ holds and $z_i \cup \{E\}$ otherwise. It is easy to see that at least one of $*(z_i \cup \{D\})$ and $*(z_i \cup \{E\})$ has to hold. Therefore, $*(z_i)$ holds for all $i$. Let $x = \bigcup_i z_i$. Clearly, $x$ is $T$-saturated and $x \subseteq x_1 \cap \ldots \cap x_n$.

Finally, we have to see that for all $T$-saturated sets $x \subset y$ there is some $i \leq n$ for which $x_i \subseteq y$. Arguing by contradiction assume $y \supset x$ and $x_i \not \subseteq y$ for all $i \leq n$. From the construction of $x$ it is easy to see that $y \not \subseteq x_1 \cap \ldots \cap x_n$. Thus there are formulas $E \in y$, $E \not \in x_1 \cap \ldots \cap x_n$ and $A_i \in x_1, A_i \not \in y$, for all $i \leq n$. Hence $(E \rightarrow A_1 \lor \ldots \lor A_n) \in \Delta$. Thus $A_1 \lor \ldots \lor A_n \in y$, quod non. This proves the Claim.

Now we define a variant of ($\sum K_i$) by requiring ($b \not \vdash p$ iff $p \in x$) at the root $b$ of ($\sum K_i$), for propositional variables $p$.

Claim For all formulas $B$: $b \not \vdash B$ iff $B \in x$.

Proof of the Claim We prove this by formula-induction. The case of the propositional variables and the connectives $\land$ and $\lor$ is trivial. Consider a formula $B = (C \rightarrow D)$. If $(C \rightarrow D) \in x$ then it is easy to see that indeed $b \not \vdash (C \rightarrow D)$. We prove that $x \not \vdash B$ implies $B \in x$ by contraposition. Therefore, assume
(C → D) ∉ x. It is not difficult to see that this implies the existence of a T-saturated set y ⊇ x such that C ∈ y and D ∉ y. From the construction of x it follows that x = y or xi ⊆ y for some i = 1, ..., n. In the first case the induction hypothesis gives b ⊨ C and b ∤ D, thus b ∤ (C → D). In the other case it follows that for some i, xi ∤ (C → D). Thus again we can conclude that b ∤ (C → D). This proves the claim.

By the last claim the defined extension is a model of T. This proves that T has the extension property.

These two lemmas lead to the following characterization of IPC:

8.1.4. Theorem. For any intermediate logic T it holds that T = IPC iff T has the property Pn for every n ≥ 0.

8.1.5. Corollary. For any intermediate logic T it holds that T = IPC iff T has the Disjunction Property and all the rules Rνn are admissible. Thus IPC is maximal with respect to V and hence maximal.

8.2 Optimality of the characterization

In the previous section we saw that the properties P1, P2, ... characterize IPC. In this section we show that no finite subset of P1, P2, ... characterizes IPC. This proves that our characterization is optimal. Note that it is not interesting to consider infinite subsets of P0, P1, P1, ..., since having the property Pm+1 implies having the property Pm.

We use logics Dn (n ≥ 1) given by Gabbay and de Jongh (1974). The logic Dn is axiomatized by

\[ D_n \models \bigwedge_{i=0}^{n+1} ((A_i \rightarrow \bigvee_{j \neq i} A_j) \rightarrow \bigvee_{j \neq i} A_j) \rightarrow \bigvee_{i=0}^{n+1} A_i. \]

We need the following theorem.

8.2.1. Theorem. (Gabbay and de Jongh 1974) The intermediate logic Dn is a proper extension of IPC with the Disjunction Property. Dn is complete with respect to the class of finite trees in which every point has at most (n + 1) immediate successors.

Knowing this, it is easy to prove the following lemma.

8.2.2. Lemma. The logic Dn has the property Pn+1 and it does not have the property Pn+2.
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**Proof** To see that $D_n$ has the property $P_{n+1}$, suppose $D_n$ derives the formula $(A \to D \lor E)$, where $A = \bigwedge_{i=1}^{n+1}(B_i \to C_i)$. Suppose also that $D_n$ does not derive $(A)(B_1, \ldots, B_{n+1}, D, E)$. By the Disjunction Property and the completeness of $D_n$ this implies that there are models $K_i$, such that $K_i \models A$ and, for $i \leq n+1$, $K_i \not\models B_i$ and $K_{n+2} \not\models D$ and $K_{n+3} \not\models E$. Furthermore, the frame of every $K_i$ is a finite tree in which every node does not have more than $(n+1)$ immediate successors. Consider $(((\sum_{i=1}^{n+1} K_i)' + K_{n+2}') + K_{n+3}')$. Clearly, the frame of this model is again a finite tree in which every node does not have more than $(n+1)$ immediate successors. In this model $A$ is valid while $(D \lor E)$ is not, contradicting the assumption that $D_n$ derives $(A \to D \lor E)$.

To see that $D_n$ does not have the property $P_{n+2}$, consider the axiomatization of $D_n$. It is easy to see, using the completeness of $D_n$, that $D_n$ does not derive

\[ (\bigwedge_{i=0}^{n+1} ((A_i \to \bigvee A_j) \to \bigvee A_j))((A_0 \to \bigvee A_j), \ldots, (A_{n+1} \to \bigvee A_j)). \]

This completes the proof of the lemma. $\Box$

8.2.3. Corollary. No finite subset of $P_1, P_2, \ldots$ characterizes $\text{IPC}$.

In fact, $D_n$ is characterized by $P_{n+1}$ in the same way as $\text{IPC}$ is characterized by all the $P_0, P_1, \ldots$, see Corollary 8.2.7. The proof of this proposition is analogous to the one of Theorem 8.1.4: the next lemma is the analogue of Lemma 8.1.2 and the following one is the analogue of Lemma 8.1.3.

8.2.4. Lemma. If an intermediate logic has the extension property up to $(n+1)$, then it is contained in $D_n$.

**Proof** Let $T$ be an intermediate logic that has the extension property up to $(n+1)$. Suppose $D_n \not\vdash A$. It easily follows from Theorem 8.2.1 that $D_n$ is complete with respect to the class of the finite trees in which every point has at most $(n+1)$ immediate successors, and in which no two nodes have exactly the same maximal nodes above them. To be precise, the last property reads:

\[ \forall x \forall y \exists z(x \neq y \to \neg \exists z'(z < z') \land ((x \leq z \land y \neq z) \lor (y \leq z \land x \neq z))). \]

Let $M$ be a model based on such a frame $F$ in which $A$ is not valid. Let $M'$ be a basic model on $F$ (see Section 6.3.1). By the same reasoning as before it follows that $M'$ is a model of $T$. Define the substitution $\sigma$ via $\sigma(p) = \bigvee_{M, x \vdash p} A_x$, where the formulas $A_x$ are given by Fact 6.3.1. Clearly,

$M, x \models B$ iff $M', x \models \sigma(B)$.

Thus $\vdash_T \sigma(A)$. Hence $\vdash_T A$. This shows that the logic $T$ is contained in the logic $D_n$. $\Box$
8.2.5. **Lemma.** If an intermediate logic has the property $P_n$ it has the extension property up to $n$.

**Proof** Let $T$ be an intermediate logic that has the property $P_n$. The proof that $T$ has the extension property up to $n$ is completely similar to the proof of Lemma 8.1.3, except for one point, which we will explain. The rest of the proof we leave to the reader.

In the first Claim of Lemma 8.1.3 we define a set $\Delta$ and observe that, in the notation of this lemma, the set $x_0 = \{ A \mid \Delta \vdash_T A \}$ is $T$-saturated because for all $m$, $P_m$ holds. In this case, having only $P_n$, this is the only place in the proof where we have to be careful. Assume $x_0 \vdash_T A \lor B$. Hence there are $E_1, \ldots, E_m \not\in x_1 \cap \ldots \cap x_n$ and $F_1, \ldots, F_m \in x_1 \cap \ldots \cap x_n$ such that

$$\vdash_T \bigwedge_{i=1}^{m} (E_i \rightarrow F_i) \rightarrow A \lor B.$$ 

For $i \leq n$, let $G_i = \bigvee \{ E_j \mid j \leq m, E_j \not\in x_i \}$ and let $F = \bigwedge_{i=1}^{m} F_i$. Observe that $G_i \not\in x_i$ and that $(G_i \rightarrow F) \in \Delta$. Clearly,

$$\vdash_T \bigwedge_{i=1}^{n} (G_i \rightarrow F) \rightarrow A \lor B.$$ 

And thus, since $T$ has $P_n$, we can conclude

$$\vdash_T \bigwedge_{i=1}^{n} (G_i \rightarrow F)(G_1, \ldots, G_n, A, B).$$ 

Since $\bigwedge_{i=1}^{n} (G_i \rightarrow F) \in x_1 \cap \ldots \cap x_n$ while $G_i \not\in x_1 \cap \ldots \cap x_n$, we have either

$$\vdash_T \bigwedge_{i=1}^{n} (G_i \rightarrow F) \rightarrow A \text{ or } \vdash_T \bigwedge_{i=1}^{n} (G_i \rightarrow F) \rightarrow B.$$ 

And because $x_0 \vdash_T \bigwedge_{i=1}^{n} (G_i \rightarrow F)$ either $x_0 \vdash_T A$ or $x_0 \vdash_T B$. And this proves that $x_0$ is $T$-saturated. \qed

8.2.6. **Proposition.** Any intermediate logic $T$ which has $P_{n+1}$ is contained in $D_n$.

8.2.7. **Corollary.** For any intermediate logic $T \supseteq D_n$ it holds that $T = D_n$ iff $T$ has $P_{n+1}$. Thus $D_n$ is maximal with respect to $R_{n+1}$ and hence maximal.

Since the union of the $D_n$ is equivalent to IPC, Theorem 8.1.4 follows from the previous proposition. However, we preferred to give a separate proof of the theorem in advance.
8.3. Effectiveness

In (de Jongh 1970) the following characterization of IPC in terms of the Kleene slash | (Kleene 1962) is given: IPC is the only intermediate logic $T$ satisfying

$$\text{if } A \vdash_T A \text{ and } \vdash_T (A \rightarrow B \lor C), \text{ then } \vdash_T (A \rightarrow B) \text{ or } \vdash_T (A \rightarrow C).$$

We remind the reader that the Kleene slash is defined as follows. (We use the abbreviation $\Gamma \vdash_T A \equiv_{df} (\Gamma |_T A \text{ and } \Gamma \vdash_T A).$)

- $\Gamma |_T p \equiv_{df} \Gamma \vdash_T p \text{ for } p \text{ a propositional variable or } \bot$
- $\Gamma |_T A \land B \equiv_{df} \Gamma |_T A \text{ and } \Gamma |_T B$
- $\Gamma |_T A \lor B \equiv_{df} \Gamma \vdash_T A \text{ or } \Gamma \vdash_T B$
- $\Gamma |_T A \rightarrow B \equiv_{df} \Gamma \vdash_T A \implies \Gamma |_T B.$

De Jongh (1970) also proved that the characterization in terms of the Kleene slash is an effective one: given any intermediate logic $T \neq \text{IPC}$ we can obtain formulae $A, B, C$ such that $A |_T A, \vdash_T (A \rightarrow B \lor C)$ but $\vdash_T (A \rightarrow B), \vdash_T (A \rightarrow C)$ in an effective way. We show that the characterization in terms of the admissibles rules treated in this chapter, is effective as well, by giving an effective reduction from the characterization in terms of the Kleene slash to the one in terms of the admissible rules.

Let us call a triple of formulas $A, B, C$ a $J$-example or an $I$-example of $T \neq \text{IPC}$ if respectively

$$A |_T A, \vdash_T (A \rightarrow B \lor C), \vdash_T (A \rightarrow B), \vdash_T (A \rightarrow C),$$

or for $A = \bigwedge (D_i \rightarrow E_i),$

$$\vdash_T (A \rightarrow B \lor C), \vdash_T (A \rightarrow B), \vdash_T (A \rightarrow C), \vdash_T (A \rightarrow D_i).$$

The following proposition shows that there exist effective reductions from one characterization to the other.

8.3.1. Proposition. For any intermediate logic $T \neq \text{IPC}$ there is an effective way of creating an $I$-example from a $J$-example, and vice versa.

Proof During the proof $\vdash, |$ stand for $\vdash_T, |_T$ respectively. The second part of the proposition is easy: any $I$-example $A = \bigwedge (D_i \rightarrow E_i), B, C$ of $T \neq \text{IPC}$ is a $J$-example because $\vdash (A \rightarrow D_i)$ for all $i$, implies $A | A$.

For the other part, suppose $A, F, G$ is an $I$-example of $T \neq \text{IPC}$. We are going to construct, in an inductive way, formulas $A_1, A_2, \ldots$ which are all equivalent to $A$ in $T$. Every $A_i$ is a conjunction of propositional variables, disjunctions and implications such that for the implications $(B \rightarrow C)$ either $A_i | (B \rightarrow C)$ or
$A_i \not\vdash B$, and for the disjunctions $B$, $A_i \vdash B$. Note that $A$ is such a formula. Let $A_1 = A$. During the construction we will often use, without mentioning, the fact that if $E \vdash F$ and $\vdash E \leftrightarrow E'$ then $E' \vdash F$.

If $A_i$ is a conjunction in which one of the conjuncts is a disjunction (note that this captures the case that $A_i$ is a disjunction), let $(B \lor C)$ be the first such reading from left to right. Thus $A_i = D \land (B \lor C) \land E$ for some $D, E$. By assumption $A_i \vdash (B \lor C)$. Hence $A_i \vdash B$ or $A_i \vdash C$. In the first case put $A_{i+1} = D \land B \land E$, in the second case $A_{i+1} = D \land C \land E$. Now consider the case that $A_i$ is a conjunction of implications and propositional variables. If every conjunct either is a propositional variable or an implication $(B \to C)$ such that $A_i \not\vdash B$, put $A_{i+1} = A_i$. If not, let $(B \to C)$ be the first implication, reading from left to right, such that $A_i \vdash B$. Thus $A_i = D \land (B \to C) \land E$ for some $D, E$. By assumption $A_i \vdash (B \to C)$. We inductively define $A_{i+1}$.

* If $B = p$, put $A_{i+1} = D \land C \land E$. Note that $A_{i+1} \vdash C$ since $A_i \vdash C$ which again follows from $A_i \vdash (B \to C)$ and $A_i \vdash B$.

* If $B = B_1 \land B_2$ observe that $A_i \vdash B$ implies $\vdash A_i \leftrightarrow D \land (B_j \to C) \land E \leftrightarrow D \land C \land E$. Hence $D \land (B_j \to C) \land E \vdash B_j$. If for some $j = 1, 2$, $D \land (B_j \to C) \land E \vdash B_j$, let $A_{i+1} = D \land (B_j \to C) \land E$. It cannot be that for no $j$, $D \land (B_j \to C) \land E \vdash B_j$. For if so, then $D \land (B_j \to C) \land E \vdash B_j$. Hence $D \land (B \to C) \land E \vdash B$, and so $D \land (B \to C) \land E \vdash C$. Whence $D \land C \land E \vdash C$ and thus $D \land (B_j \to C) \land E \vdash (B_j \to C)$, a contradiction.

* If $B = B_1 \lor B_2$ observe that $\vdash A_i \leftrightarrow D \land (B_1 \to C) \land (B_2 \to C) \land E$ and that $A_i \vdash (B_1 \to C)$. Put $A_{i+1} = D \land (B_1 \to C) \land (B_2 \to C) \land E$.

* Finally $B = (B_1 \to B_2)$. If $A_i \not\vdash B_1$ or $A_i \not\vdash B_2$ then $A_i \vdash B$ and therefore $A_i \vdash C$. Put $A_{i+1} = D \land C \land E$. If $A_i \vdash B_1$ and not $A_i \vdash B_2$ then $\vdash A_i \leftrightarrow D \land B_1 \land (B_2 \to C) \land E$ and clearly $A_i \vdash B_1$ and $A_i \vdash (B_2 \to C)$. Put $A_{i+1} = D \land B_1 \land (B_2 \to C) \land E$. This ends the construction of the $A_i$.

It is easy to check that the $A_i$ have the desired properties. Moreover, the construction shows that eventually $A_i = A_{i+1}$. Hence $A_i$ is a conjunction of propositional variables and implications $\land_{i=1}^n p_i \land \land_{i=1}^m (B_i \to C_i)$ such that $A_i \not\vdash B_i$. Let $A' = \land_{i=1}^n (B_i \to C_i)$ and let $\sigma$ be the substitution which is the identity on all variables except $p_1, \ldots, p_n$, on which it is $T$. Hence $\sigma(A_i)$ is equivalent to $\sigma(A')$. Since $A_i$ is equivalent with $A$ in $T$,

\[ \vdash (A_i \to F \lor G), \not\vdash (A_i \to F), \not\vdash (A_i \to G). \]

Clearly, we have

\[ \vdash (\sigma(A') \to \sigma(F) \lor \sigma(G)), \]

In general, nonderivability is not preserved under substitution but this particular choice of $\sigma$ leads to

\[ \not\vdash (\sigma(A') \to \sigma(F)), \not\vdash (\sigma(A') \to \sigma(G)), \not\vdash (\sigma(A') \to \sigma(B_i)). \]

Hence $\sigma(A'), \sigma(F), \sigma(G)$ is an $I$-example of $T \not\vdash IPC$. \qed