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The short-wave model and waves in two directions

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In the region where a sinusoidal wave in the cochlea reaches its maximum amplitude, the long-wave (or one-dimensional) model of the cochlea is deficient. In this region a short-wave model is more appropriate. However, in its current form, the short-wave model supports only waves in one direction. Therefore, it cannot cope with reflection effects associated with, e.g., inhomogeneities. Theoretical explorations of creation and internal reflection of otoacoustic emissions have almost exclusively been based on the long-wave model. In this article the road is paved for future explorations on a generalized form of the short-wave model, one that supports forward as well as backward waves, and thus can include internal reflections. © 2001 Acoustical Society of America. [DOI: 10.1121/1.1329623]

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I. INTRODUCTION

In recent times several publications have appeared related to the problem of internal reflection in models of the cochlea. These explorations were inspired by the somewhat puzzling properties of otoacoustic emissions from the cochlea, in particular, distortion-product emissions. Two general classes of emission sources were distinguished, wave-fixed and location-fixed (starting with Kemp and Brown, 1983, more recent papers Schneider *et al.*, 1999; Talmadge *et al.*, 1999; Knight and Brass, 2000; Prijs *et al.*, 2000; theoretical foundation: Shera and Guinan, 1999). These classes of emissions are also different in that the former is typically associated with nonlinear phenomena, and the latter operates in linear as well as nonlinear models. Most if not all modeling work was done on the *long-wave* model, also known as the *one-dimensional* model. A few relevant publications are Talmadge and Tubis (1993), Shera and Zweig (1993), Talmadge *et al.* (1997, 1998, 1999), Shera and Guinan (1999), and the basic notion of “coherent reflection” was defined by Zweig and Shera (1995). The advantages of this approach are obvious: the long-wave model is described by a second-order differential equation with only two terms, and this type of equation has been thoroughly studied by mathematicians and physicists.

The long-wave model can adequately explain many properties of the cochlea, notably those related to variations of the basilar membrane impedance. It can also explain reflection phenomena, at the stapes as well as elsewhere, e.g., reflections resulting from irregularities in cochlear structure. However, the long-wave model is deficient in the “peak region.” To illustrate this property, data are taken from a number of experiments executed in collaboration with A. L. Nuttall (see, e.g., de Boer and Nuttall, 2000). In these experiments the velocity of the basilar membrane was measured at a location in the guinea pig cochlea that is tuned to a frequency near 17 kHz. The measured frequency response curves are converted to the place domain (x), using a

standard frequency-to-place map, and the wave number $k = d\varphi/dx$, where φ is the response phase, was determined at the location of the response peak. The wave number k relates to the wavelength λ by $k = 2\pi/\lambda$. According to a very conservative condition for validity of the long-wave model the wavelength λ should everywhere be larger than 2π times the height h of one cochlear channel (de Boer, 1996, Eq. 4.2.9). In terms of the wave number, the product kh should be smaller than 1. Averaged over 12 experiments in sensitive animals, the average value of the product kh at the response peak is 13.9 ± 0.9 (the height h is taken equal to 1 mm). Over the same experiments, the value of kh at the peak is 2.82 ± 0.6 for post-mortem responses. Clearly, in the live animal the condition for long waves is not met by far, and for the post-mortem condition it is not met either.

To solve a more general case, and to include long as well as short waves, two- and three-dimensional models have been developed. This development started with Lesser and Berkley (1972), and accelerated with Allen (1977), Allen and Sondhi (1979), Steele and Taber (1979, 1981), and Neely (1985). For a review of this development see de Boer (1996). In most cases solutions to these models were obtained in digital form or via the LG or Wentzel–Kramers–Brillouin (WKB) approximation. At first sight, trying to develop generalized *analytical* relations seems an impossible task in a three-dimensional setting.

In the region of the response peak the physics of the model can better be described by the *short-wave* model than by the long-wave model. In the original formulation the differential equation for the short-wave model is of the first order, even simpler than that for the long-wave model. However, this equation allows only a solution representing a wave propagating in one direction (Siebert, 1974; de Boer, 1979). In the present article it is shown that the short-wave model can be reformulated so that *it allows waves in the forward as well as the backward direction*. The result is a second-order differential equation.¹ This does not mean that all results obtained for long-wave models can be directly transposed to the short-wave model. In the physical sense the

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two models are simply not equivalent. Mathematically, the two second-order equations are not equivalent either. However, it is felt that more widely applicable relations can be based on the generalized approach illustrated here. Moreover, the path to analytical solutions or useful approximations for the short-wave region is opened. It should be remembered, though, that no form of the short-wave model is adequate for the basal region.

II. DERIVATION

The model to be used has two channels, the complex variable $p(x)$ is the pressure in the upper channel (the one in which the stapes is located) and $-p(x)$ is the pressure in the lower channel. The (real) variable x denotes the coordinate in the longitudinal direction of the model, starting with zero at the stapes and round-window location. The velocity $v_{BM}(x)$ is counted positive when the basilar membrane (BM) moves from the lower to the upper channel. The relation between pressure $p(x)$ and BM velocity $v_{BM}(x)$ is then

$$v_{BM}(x) = \frac{-2p(x)}{Z_{BM}(x)}, \quad (1)$$

where $Z_{BM}(x)$ the impedance of the BM. The *long-wave equation* for the pressure $p(x)$ reads

$$\frac{d^2}{dx^2}p(x) - \frac{2i\omega\rho}{h_e Z_{BM}(x)}p(x) = 0. \quad (2)$$

The parameters are as follows: ω is the radian frequency, ρ is the density of the fluid, and h_e is the “effective” height of the model.² Equation (2) is the equation for the *pressure*. The long-wave equation for the BM *velocity* $v_{BM}(x)$ is somewhat more complicated:

$$\begin{aligned} \frac{d^2}{dx^2}v_{BM}(x) + 2\frac{d}{dx}\ln[Z_{BM}(x)]\frac{d}{dx}v_{BM}(x) \\ + \left[U_{BM}(x) - \frac{2i\omega\rho}{h_e Z_{BM}(x)} \right] v_{BM}(x) = 0, \end{aligned} \quad (3a)$$

with $U_{BM}(x)$ given by

$$U_{BM}(x) = \frac{1}{Z_{BM}(x)} \frac{d^2}{dx^2}Z_{BM}(x). \quad (3b)$$

The long-wave equation is valid when the wavelength of the wave in the model is “large” compared to the height h of the model (see Sec. I).

The other extreme occurs when the wavelength is small compared to h , this leads to the short-wave model. The *short-wave equation* for waves traveling to the “right,” i.e., in the direction of increasing x , reads (de Boer, 1979):

$$\frac{d}{dx}p(x) + \frac{2\omega\rho}{Z_{BM}(x)}p(x) = 0. \quad (4)$$

Similarly, the equation for waves to the “left” is

$$\frac{d}{dx}p(x) - \frac{2\omega\rho}{Z_{BM}(x)}p(x) = 0. \quad (5)$$

Note that each of these two equations has an analytical solution.

Let the solution to Eq. (4) be $p_1(x)$, and that to Eq. (5) $p_2(x)$. The general solution to the short-wave model equation in which waves in *both* directions are possible should be of the form

$$p(x) = \alpha p_1(x) + \beta p_2(x), \quad (6)$$

in which α and β are arbitrary constants. We desire to know the equation of which this is the general solution. To find it, first differentiate Eq. (4) once with respect to x . Use Eq. (4) again for reducing the term with $dp_1(x)/dx$ to $p_1(x)$ and the term with $p_1(x)$ to $dp_1(x)/dx$. The result is a three-term second-order differential equation with one plus and two minus signs. Use Eq. (5) in the same way for $p_2(x)$. Again, an equation results with one plus and two minus signs. Next, combine these two equations with coefficients α and β and rewrite the result in terms of $p(x)$ just as it appears in Eq. (6). It is found that the following equation in $p(x)$ fulfills the requirement (de Boer, 1983):

$$\frac{d^2}{dx^2}p(x) + \frac{d}{dx}\ln[Z_{BM}(x)]\frac{d}{dx}p(x) - \left[\frac{2\omega\rho}{Z_{BM}(x)} \right]^2 p(x) = 0. \quad (7)$$

The solution of this equation, with the proper boundary conditions imposed at the two ends, supports forward as well as backward waves both having the character of short waves (deep-water waves).³ It therefore also supports internal reflections, e.g., from irregularities. Equation (7) is similar in form to Eq. (3a) with two differences, (1) the factor of 2 in the second term of Eq. (3a) is missing in Eq. (7); and (2) the square of the impedance appears in Eq. (7).

About item (1), see the next section. Item (2) illustrates the well-known property of short waves in that they depend in a much stronger way on the BM impedance than long waves. In spite of these obstacles, Eq. (7) is sufficiently simple to allow analytical treatment, for instance, by a modification of the LG method. Therefore, the author feels that it is useful to reopen this formulation to theorists. It is stressed again that *in the peak region* the short-wave model describes the cochlear wave better than the long-wave model.

III. ALTERNATIVE FORMULATION

To illustrate item (1), write the pressure $p(x)$ as a product of a new unknown $q(x)$ and a coefficient function $W(x)$:

$$p(x) = W(x)q(x). \quad (8)$$

For Eq. (7) to have only two terms, $W(x)$ must satisfy

$$W(x) = \left(\frac{2\omega\rho}{Z_{BM}(x)} \right)^{1/2}. \quad (9)$$

Then the two-term differential equation for $q(x)$ is

$$\begin{aligned} \frac{d^2}{dx^2}q(x) + \left[\frac{W''(x)}{W(x)} + \frac{Z'(x)}{Z_{BM}(x)} \frac{W'(x)}{W(x)} - \left(\frac{2\omega\rho}{Z_{BM}(x)} \right)^2 \right] q(x) \\ = 0, \end{aligned} \quad (10)$$

where $W'(x)$ stands for $dW(x)/dx$, $W''(x)$ for $d^2W(x)/dx^2$, and $Z'(x)$ for $dZ_{\text{BM}}(x)/dx$. This method is one of the standard conversion methods for differential equations. In this formulation $q(x)$ does not have physical meaning. Apart from the aforementioned term $(2\omega\rho/Z(x))^2$, two extra terms appear in Eq. (10) that are related to variations of $Z_{\text{BM}}(x)$ with x . In the main response region (the response peak and the regions next to it, on both sides) these terms turn out to be relatively small so that the main properties of the solution to Eq. (10) follow from those of $(2\omega\rho/Z_{\text{BM}}(x))^2$.

Note added. Christopher Shera showed the author that at least one promising variation of the derivation in Sec. III on “alternative formulation” is possible. The author hopes that this and other variations will be developed further to give a wider scope to the theory on micro-reflections in the cochlea.

¹This second-order differential equation, Eq. (7) of this note, has been published—without derivation—earlier, in a rather inaccessible place (de Boer, 1983).

²The effective height is the area of the cross section of one channel divided by the width of the BM. It is generally larger than the height h of one channel.

³A program set that runs in Matlab® (any Windows version from 4.0 on) can be obtained from the author. One program, SW01, shows the equivalence of solutions of Eqs. (4) and (7) for a realistic impedance function. If desired, it also shows the relative sizes of the extra terms in the bracketed part of Eq. (10). Another program, SW03, illustrates the deficiency of the long-wave model and the approximate adequacy of the short-wave model in the region of the response peak. Apply via e-mail.

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