Strongly exposed points in unit balls of Banach spaces of holomorphic functions

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Chapter 1
Preliminaries

In this chapter we explain the notation and develop the necessary background for this thesis. In particular, we define the sets of extreme, exposed and strongly exposed points in a Banach space (Section 1.2), and we offer an elementary introduction to Hardy space theory (Section 1.3) and uniform algebras (Section 1.4).

1.1 Notation

The following notation will be used throughout this text without explanation:

- $D$ will denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$, with boundary $\partial D = T = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.

- For $n = 2, 3, 4, \ldots$, $B_n = \{z \in \mathbb{C}^n : ||z|| < 1\}$ is the (open) unit ball in $\mathbb{C}^n$, with boundary $S = S_n$, the unit sphere.

- If $X$ is any Banach space with norm $\|\cdot\|$, we will refer to the closed unit ball $\{x \in X : \|x\| \leq 1\}$ of $X$ simply as the unit ball of $X$, denoted $\text{Ball}(X)$, unless explicitly stated otherwise. The obvious exceptions to this rule occur when we are dealing with the open unit balls $D$ in $\mathbb{C}$ and $B_n$ in $\mathbb{C}^n$.

- Normalized Lebesgue measure on the unit circle will be written as $\frac{d\theta}{2\pi}$; normalized Lebesgue measure on the unit sphere $S = S_n$ as $d\sigma = d\sigma_n$. Normalized Lebesgue (area) measure on the unit disc will be denoted by $dA(z) = \frac{1}{\pi} dxdy$.

- Let $K$ be a compact topological space. The algebra of continuous functions on $K$ will be written as $C(K)$.

- If $X$ is a Banach space, we will denote its dual space (equipped with the operator norm) by $X^*$. All functionals considered throughout this text will be (assumed) continuous (bounded).
1.2 Extreme, exposed and strongly exposed points in Banach spaces

**Definition.** Let $X$ be a Banach space, and let $C$ be a convex subset of $X$. We say that $x \in C$ is an extreme point of $C$, or simply extreme, if the set $C \setminus \{x\}$ is also convex. Equivalently, we cannot find $y, z$ in $C$ not equal to $x$ such that $x = \frac{1}{2}(y + z)$.

If $C$ is the unit ball of $X$, the definition of an extreme point $x \in \partial \text{Ball}(X)$ can be reformulated as follows: if $\|x + y\| = \|x - y\| = 1$ for some $y \in X$, then $y = 0$.

In general it is an interesting question to describe extreme points in a locally convex space, where there may be many extreme points, few or even none at all. For example, it is not difficult to show that a function $f$ is extreme in the unit ball of $C(K)$ if and only if $|f(x)| = 1$ for all $x \in K$. On the other hand, there are no extreme points in the unit ball of $L^1[0, 1]$ (equipped with Lebesgue measure for simplicity); indeed, given $f \in L^1$ with $\|f\|_1 = 1$, we find a number $\alpha \in (0, 1)$ such that $\int_0^\alpha |f(t)| \, dt = \frac{1}{2}$. Let $g = f(1_{[0, \alpha]} - 1_{[\alpha, 1]})$. Then $\|f + g\|_1 = \|f - g\|_1 = 1$, showing that $f$ is not extreme.

Let us recall that, for any given Banach space $X$, the weak* topology on the dual space $X^*$ is the smallest (weakest) topology such that the functions $x^* \in X^* \mapsto x^*(x)$ are continuous for every $x \in X$. Hence, a sequence $x_n^*$ in $X^*$ converges weak* to $x^* \in X^*$ if and only if $\lim_{n \to \infty} x_n^*(x) = x^*(x)$ for all $x \in X$. The most famous result on the existence of extreme points in a Banach space is the Krein-Milman theorem.

**Theorem 1.1.** A bounded, weak* closed convex subset of $X^*$ is the weak*-closure of the convex hull of its extreme points.

**Definition.** Let $C$ be a convex subset of the Banach space $X$. We say that $x \in C$ is an exposed point of $C$, or simply exposed, if there exists a functional $L$ on $X$ with the following property: for all $y \in C$, $y \neq x \implies \Re L(y) < \Re L(x)$. (For $\lambda \in \mathbb{C}$, $\Re \lambda$ denotes the real part of $\lambda$.) We say that the functional $L$ is an exposing functional for $x$, or simply that $L$ exposes the point $x$. This means that the real hyperplane determined by $L$ through the point $x$ touches $C$ only at $x$.

In the case where $C$ is the unit ball of $X$, it will be convenient to rephrase the notion of exposedness: there exists an (exposing) functional $L$ on $X^*$ with $L(x) = \|L\| = 1$ and $x$ is the only point in the unit ball at which $L$ attains the value 1.

It is obvious from the definitions that an exposed point is also extreme. The simplest examples demonstrate that the converse need not hold in general. However, we offer the following easy "converse," the proof of which is left to the reader.

**Lemma 1.2.** Let $X$ be a Banach space in which every point of unit norm is extreme in the unit ball of $X$. Then all points of unit norm are also exposed.

**Definition.** Let $C$ be a convex subset of the Banach space $X$. We say that $x \in C$ is strongly exposed in $C$ if there exists a functional $L$ such that for every sequence $\{x_n\}_{n=1}^\infty$ in $X$:

$$\lim_{n \to \infty} \Re L(x_n) = \Re L(x) \iff \lim_{n \to \infty} x_n = x.$$

Geometrically this expresses that the slices $\{y \in C : \Re L(y) > \Re L(x) - \varepsilon\}$ of the real hyperplane through $x$ determined by $L$ have arbitrarily small diameters as $\varepsilon \to 0$. 
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When \( C\) is the unit ball of a Banach space, we will often work with the following condition on the functional \( L\): for every sequence \((x_n)\) in the unit ball, \( L(x_n) \to 1 = L(x)\) occurs only when \( x_n\) converges to \( x\) in norm.

Of course, any such functional \( L\) also exposes \( x\), so a strongly exposed point is exposed. Exposed points that are not strongly exposed are called weakly exposed.

**Example.** Consider the Hilbert space \( l^2(\mathbb{N}) = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}\) and the convex subset \( C = \{(x_n) \in l^2(\mathbb{N}) : x_n \geq 0\}\). It is easy to see that the element \( C = (0,0,0,\ldots)\) is the only extreme point of \( C\). It is also exposed by any functional \( L(x) = \sum_{n=1}^{\infty} x_n y_n\), with \( y = (y_n) \in l^2(\mathbb{N})\) such that \( y_n < 0\) for all \( n\). However, if we set \( e_n = (\delta_{n,1})_{i=1}^{\infty}\) then for any functional \( L(e_n) \to 0 = L(0)\), so that \( C\) is in fact weakly exposed. Next, let \( C'\) be the intersection of \( C\) and the unit ball of \( l^2(\mathbb{N})\). The extreme points of \( C'\) are \( C\) and all points \( e\) in \( C'\) of unit norm. The point \( C\) is still, of course, weakly exposed, but now every \( e\) of unit norm is strongly exposed (by the functional \( L_e(x) = \langle x, e\rangle\)).

The following theorem of R.R. Phelps can be seen as an extension of the Krein-Milman theorem. It assures us that the spaces we will consider later (Hardy spaces, Bergman space) contain many strongly exposed points.

**Theorem 1.3 ([47]).** Let \( C\) be a closed bounded convex subset of a separable dual Banach space. Then \( C\) is the norm closure of the convex hull of the strongly exposed points of \( C\).

### 1.3 Introduction to Hardy spaces

In this section we briefly review the theory of Hardy spaces \( H^p\) on the unit disc \( D\). Thus we only go through well-known properties of \( H^p\)-functions and refer the reader to the literature for more details and full proofs, which can be found in a number of excellent books, e.g., [15], [20] and [32].

**Definition.** Let \( 0 < p < \infty\). The **Hardy space** \( H^p = H^p(D)\) consists of all holomorphic functions \( f\) on \( D\) for which

\[
\|f\|_{H^p} := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.
\]

For all \( 1 \leq p < \infty\), the norm \( \| \cdot \|_{H^p}\) turns \( H^p(D)\) into a Banach space. When no confusion is possible we will write \( \|f\|_p\) instead of \( \|f\|_{H^p}\). The Banach space of all bounded holomorphic functions on \( D\), equipped with the supremum norm \( \|f\|_{\infty} = \sup_{z \in D} |f(z)|\), will be denoted by \( H^\infty = H^\infty(D)\). Clearly, \( H^\infty(D) \subset H^p(D)\) for all \( 0 < p < \infty\), and \( \|f\|_p \leq \|f\|_{\infty}\) for all \( f \in H^\infty\). The disc algebra \( A = A(D)\) consists of all continuous functions on \( D\) that are holomorphic on \( D\), equipped with the supremum norm. It is the uniform closure of the set of polynomials on \( D\).

Let \( f\) be holomorphic on \( D\) and choose \( 0 < p < \infty\). Because the function \( |f|^p\) is subharmonic, the expressions \( M_p(f,r) := \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\) are increasing in \( r\). We
remark that the harmonic extension $u_r$ of the function $|f|^p$ on the circle $C(0, r)$ to the closed disc $B(0, r)$ majorizes $|f|^p$, because the latter function is subharmonic. Thus the sequence of harmonic functions $(u_r)$ is increasing. If this sequence converges at a single point of $D$, then it converges pointwise on $D$ and uniformly on compact subsets to a harmonic function $U$ on $D$ (Harnack’s theorem) which must be the least harmonic majorant of $|f|^p$ on $D$, by construction. Now the value of $u_r$ at the origin equals $M_p(f, r)$, so we conclude that the holomorphic function $f$ is contained in $H^p(D)$ if and only if the function $|f|^p$ admits a harmonic majorant on $D$.

We continue to assume that $0 < p < \infty$. Let us describe the zero sets of $H^p(D)$-functions. For every $a \in D \setminus \{0\}$, the map

$$B_a : z \mapsto \frac{|a|}{-a} \frac{z - a}{1 - az} \in A(D)$$

is an automorphism of the unit disc that has a zero at the point $a$. In particular, $|B_a(z)| = 1$ for all $z \in T$. We call this map the Blaschke function with zero at $a$. The identity function $B_0 : z \mapsto z$ is, of course, the Blaschke function with a zero at the origin. Suppose now that we are given a sequence of points $(z_n)$ in $D \setminus \{0\}$. We ask the question: is there a bounded holomorphic function on $D$ with zero set $(z_n)$ (where repetitions of points in the sequence account for the multiplicities of zero)? One’s first guess would be to consider the product of the functions $B_{z_n}$, but a priori it is not clear why it should converge. However, because all functions $|B_{z_n}|$ are bounded by 1, a normal families argument shows that $\prod_n B_{z_n}$ converges to a bounded holomorphic function with zero set $(z_n)$ as soon as the product converges at a single point of $D$. Now the product $\prod_n B_{z_n}(0)$ converges if and only if $\prod_n |z_n|$ converges, or equivalently, if $\sum_n (1 - |z_n|) < \infty$.

Conversely, suppose $f \neq 0$ is a function in $H^p$. Then it can be shown that the zeros $(z_n)$ of $f$ satisfy this so-called Blaschke condition:

$$\sum_n (1 - |z_n|) < \infty. \quad (1.1)$$

Hence, (assuming $f$ has a zero of order $m$ at the origin) the so-called Blaschke product

$$B(z) := z^m \prod_{n: z_n \neq 0} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - z_n z} \quad (z \in D).$$

is a well-defined bounded holomorphic function on $D$ whose zero set coincides with that of $f$; we say that $B$ is the Blaschke factor of $f$. We see that the Blaschke condition (1.1) characterizes the zero sets of $H^p(D)$-functions.

Now given a function $f \in H^p$, let $B$ be the Blaschke product of the zeros of $f$ and set $g := f/B$. Of course, $g$ is holomorphic on $D$ and zero-free by construction. For any $N \in \mathbb{N}$, let $B_N(z) := \prod_{n=1}^N B_{z_n}(z)$ be the product of the Blaschke functions of the first $N$ zeros of $f$. Because $|B_N| \to 1$ as $|z| \to 1$, $\|f\|_p = \|f/B_N\|_p$. In particular, for every $r < 1$: $M_p(f/B_N, r) \leq \|f\|_p$. Letting $N \to \infty$ in this inequality, we obtain
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$M_p(g, r) \leq \|f\|_p^p$, whence $\|g\|_p \leq \|f\|_p$. This shows that the function $g$ is contained in $H^p$. On the other hand, because $|B(z)| \leq 1$ at every $z \in D$, $\|g\|_p \geq \|f\|_p$. We conclude that the function $g$ is a function in $H^p$ of the same norm as $f$ ([50]). In particular we see that for all $0 < p \leq \infty$ and all Blaschke products $B$, $|B|_p = 1$. This factorization of $f$ is commonly referred to as Riesz factorization.

What we have seen so far is that we can write any $f \in H^p$ as the product of a bounded holomorphic function $B$ (a Blaschke product) and a zero-free function $g \in H^p$ of the same $H^p$-norm as $f$. Let $G := g^{1/2p} = \sum_n a_n z^n \in H^2(D)$. The fact that $G \in H^2$ is equivalent to the statement that $\sum_n |a_n|^2 < \infty$. Indeed, if we let $G_r$ be the function $z \mapsto G(rz)$ ($0 < r < 1$), then $M_2(G, r) = \sum_n |a_n|^2 r^{2n}$ is bounded in $r$ if and only if $\sum_n |a_n|^2$ converges. Let $G^*$ be the $L^2(T, \frac{d\theta}{2\pi})$-function with Fourier series $\sum_{n=0}^{\infty} a_n e^{in\theta}$. Then as functions on the unit circle $T$, $G_r \rightarrow G^*$ in $L^2$. Because every $G_r$ is harmonic on $\overline{D}$, $G_r$ equals the Poisson integral of $G_r$ (restricted to the unit circle) everywhere on $D$. By a simple limit argument then, $G(z) = P_z(G^*)$, where $P_z(G^*)$ is the Poisson integral of $G^*$:

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{1 - 2\Re(ze^{-i\theta}) + |z|^2} G^*(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Using Lebesgue's theorem on the differentiation of measures and the fact that the Poisson kernel is an approximate identity one obtains the important result that for almost every $\xi = e^{i\theta} \in T$, the radial limit $\lim_{r \uparrow 1} G(r\xi)$ exists and equals $G^*(\xi)$. In fact, one may even approach the point $\xi$ non-tangentially. Also, we see that the least harmonic majorant of $|G|^2$ equals $\lim_{r \uparrow 1} P_z(|G_r|^2) = P_z(|G^*|^2)$, so $\|G\|_2 = \|G^*\|_{L^2}$. Hence, for any Blaschke product $B$, the non-tangential limit $B^*$ satisfies $|B^*| = 1$ almost everywhere on $T$. Exponentiating the function $G$ to the power $2p$ and adding the Blaschke product $B$ to revert to $f$, we get the following result (with a bit of extra work).

**Theorem 1.4.** Let $f \neq 0$ be a function in $H^p(D)$ ($0 < p \leq \infty$) with Blaschke factor $B$ and set $g = f/B$. Then $g$ is zero-free and contained in $H^p$ and $\|f\|_{H^p} = \|g\|_{H^p}$. For almost all $\xi \in T$ the non-tangential limits

$$f^*(\xi) := \lim_{z \rightarrow \xi (n.t.)} f(z)$$

exist and define an element $f^*$ of $L^p(T)$. The functions $f_r(z) = f(rz)$ converge to $f$ in $H^p$ as $r \uparrow 1$. Furthermore, the following relations hold (for all $z \in D$):

$$\|f\|_{H^p} = \|f^*\|_{L^p},$$

$$f(z) = P_z(f^*) \quad (p \geq 1) \quad \text{(1.2)}$$

$$= \int_T f^*(\xi) \frac{d\xi}{\xi - z} \quad (p \geq 1), \quad \text{(1.3)}$$

$$|f(z)|^p \leq P_z(|f^*|^p). \quad \text{(1.4)}$$

Finally, the function $\log |f^*|$ is integrable on $T$ and

$$\log |f(z)| \leq P_z(\log |f^*|) \quad \text{(1.5)}$$

for all $z \in D$. 

for all $z \in D$. 


Remark. We will often drop the superscript in the notation of the boundary function $f^*$ of $f$ for reasons of convenience.

**Theorem 1.5 ([49]).** (F. & M. Riesz) Let $\mu$ be a finite complex Borel measure on $\mathbb{T}$ and suppose $\int_0^{2\pi} e^{in\theta} d\mu(e^{i\theta}) = 0$ for $n = 1, 2, 3, \ldots$. Then there exists a function $f \in H^1$ such that $d\mu = f^* \frac{d\theta}{2\pi}$.

**Proof.** For $z \in D$ we define 

$$f(z) := P_z[d\mu] = \int_0^{2\pi} \frac{1 - |z|^2}{1 - 2 \mathfrak{R}(ze^{-i\theta}) + |z|^2} d\mu(e^{i\theta}).$$

By the assumptions on $\mu$ we obtain that $f(z) = \sum_{n=0}^{\infty} (f_0^{2\pi} e^{-in\theta} d\mu(e^{i\theta})) z^n$, which expresses that the harmonic function $f$ is in fact holomorphic on $D$. Fubini’s theorem gives us that $M_1(f, r) \leq \|\mu\|$ (the total variation of $\mu$), so $f$ is contained in $H^1(D)$.

Therefore, if we set $dv = d\mu - f^* \frac{d\theta}{2\pi}$, then by equation (1.2), $\int_0^{2\pi} e^{in\theta} d\mu(e^{i\theta}) = 0$ for all $n \in \mathbb{Z}$. Hence the measure $\nu$ annihilates all continuous functions on $\mathbb{T}$. We conclude that $\nu$ is zero, which yields the desired representation of $\mu$. \hfill $\square$

**Corollary 1.6.** Let $A_0 = zA$ be the algebra of functions in the disc algebra vanishing at the origin, regarded as a subspace of $C(\mathbb{T})$. Then $H^1$ is isometrically isomorphic to the dual space of $C(\mathbb{T})/A_0$.

**Proof.** Given a function $f \in H^1$, let $L_f$ be the functional

$$g \in C(\mathbb{T}) \mapsto \int_0^{2\pi} fg \frac{d\theta}{2\pi}.$$ 

The norm of this functional is $\|f\|_{L^1} = \|f\|_{H^1}$. By (1.2) $L_f$ annihilates $A_0$. Consequently, $i : f \mapsto L_f$ is an isometric embedding of $H^1$ into $(A_0)^{\perp} \cong (C(\mathbb{T})/A_0)^*$.

In the other direction, suppose $L$ is a functional of $C(\mathbb{T})$ that annihilates $A_0$. By the Hahn-Banach theorem, there exists a finite complex Borel measure $\mu$ on $\mathbb{T}$ such that $L(g) = \int_{\mathbb{T}} gd\mu$, $g \in C(\mathbb{T})$. Because $\mu$ annihilates $A_0$, we can use the F. & M. Riesz theorem to conclude that $\mu$ is of the form $f^* \frac{d\theta}{2\pi}$ for some $f \in H^1$. Subsequently $L = L_f = i(f)$, so $i$ is surjective and $H^1 \cong (C(\mathbb{T})/A_0)^*$.

\hfill $\square$

Take any $f \in H^p$, $f \neq 0$. Let $u$ be the Poisson integral of $\log |f^*| \in L^1$ with harmonic conjugate $\bar{u}$ on $D$ and set $F(z) := \exp(u(z) + i\bar{u}(z))$. It follows from Jensen’s inequality and the inequality (1.5) that $F \in H^p$ and for all $z \in D$: $|f(z)| \leq |F(z)|$. Thus the function $I(z) := f(z)/F(z)$ is contained in the unit ball of $H^\infty$. Reasoning as above, it is not difficult to show that $u(z) \to \log |f^*(\xi)|$ for almost all $\xi \in \mathbb{T}$ as $D \ni z \to \xi$ non-tangentially. Hence, $|F^*| = |f^*|$ a.e. on $\mathbb{T}$. This means that for every $z \in D$, $\log |F(z)| = P_z(\log |F^*|)$ (cf. (1.5)). Also, $|I^*| = 1$ a.e. on $\mathbb{T}$. If $B$ is the Blaschke factor of $f$, then $S(z) := I(z)/B(z)$ is a zero-free function in the unit ball of $H^\infty$ with boundary values of modulus 1 a.e. on $\mathbb{T}$. 

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We come to the following definitions.

**Definition.** If $I$ is a function in the unit ball of $H^\infty$ with boundary values of modulus 1 a.e. on $\mathbb{T}$, then we say that $I$ is an **inner function**. If, in addition, $I$ is zero-free on $D$, then $I$ is called a **singular inner function**.

**Definition.** Let $F$ be a function in $H^p$ ($0 < p \leq \infty$), not vanishing identically on $D$. If for every $z \in D$ (equivalently: at least one $z \in D$): $\log |F(z)| = P_z(\log |F^*|)$, then we say that $F$ is an **outer function** in $H^p$.

Implicitly this definition contains a recipe for constructing all outer functions in $H^p$. Start with a function $\psi \geq 0$ in $L^p(\mathbb{T})$ that is also log-integrable: $\int_{\mathbb{T}} |\log \psi| \, d\theta < \infty$. Extend $\log \psi$ harmonically to $u$ on $D$. Then $F(z) = \exp(u(z) + i\tilde{u}(z))$ is an outer function (and $|F| = \psi$ a.e.).

Also, we can construct all inner functions. Blaschke products are inner functions and as we have shown, any inner function is the product of a Blaschke product and a singular inner function. Without proof we mention that a singular inner function $S$ is of the form $S(z) = \exp(-u(z) - i\tilde{u}(z))$, where $u$ is the positive harmonic function given by

$$u(z) = \int_0^{2\pi} \frac{1 - |z|^2}{1 - 2Re(z e^{-i\theta} + |z|^2)} \, d\mu(e^{i\theta}),$$

for some (any) non-negative measure $\mu$ on $\mathbb{T}$ that is singular with respect to Lebesgue measure $d\theta$ on $\mathbb{T}$.

Collecting the previous results, we state the following theorem which is commonly referred to as the **inner-outer factorization** of $H^p$-functions (the terminology is due to A. Beurling, see Theorem 1.8).

**Theorem 1.7 ([61]).** Let $f$ be any function in $H^p$, not vanishing identically on $D$. Then there exist a Blaschke product $B$, a singular inner function $S$ and an outer function $F$ in $H^p$ such that:

$$f(z) = B(z)S(z)F(z), \quad z \in D.$$  

We call $B$, $S$ and $F$ the **Blaschke factor** of $f$, the **singular inner factor** of $f$ and the **outer factor** of $f$, respectively. Up to multiplication by unimodular constants, the three factors are unique.

We state two famous results that underline the importance and nature of inner and outer function in Hardy spaces.

Let $S : H^2 \to H^2$ be the operator $f(z) \mapsto zf(z)$. The map $S$ is usually called the (forward) **shift operator** on $H^2$. This terminology is easily explained when we observe that the monomials $1, z, z^2, \ldots$ form an orthonormal basis of $H^2$ and the action of $S$ is given by $\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n z^{n+1}$. We say that a subspace $V$ of $H^2$ is invariant under $S$, or shift-invariant, if $S(V) \subseteq V$. Equivalently, for every polynomial $p$ and every $f \in V$: $p(z) f(z) \in V$. 


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Theorem 1.8 ([7]). (Beurling) Let $V$ be a closed subspace of $H^2$ invariant under $S$. Then either $V = \{0\}$ or there exists an inner function $I$ such that

$$V = I \cdot H^2 = \{I(z)f(z) : f \in H^2\}.$$  \hfill (1.6)

The inner function $I$ is unique except for a constant factor. Conversely, every subspace $V$ of the form (1.6) is closed and invariant under $S$.

Proof. Cf. [20], Theorem II.7.1.

Corollary 1.9. Let $f \in H^2$. Then $f$ is an outer function if and only if the subspace $\mathcal{P}f = \{pf : p \text{ a polynomial}\}$ is dense in $H^2$.

Proof. Suppose $f$ is an outer function. Let $V$ be the closure of $\mathcal{P}f$ in $H^2$. $V$ is a closed $S$-invariant subspace of $H^2$. By Beurling's theorem $V = I \cdot H^2$ for some inner function $I$. Because $f \in V$ is an outer function, the inner function $I$ must be constant, i.e., $V = H^2$. If $f$ is not an outer function, let's say $f = I \cdot F$, for some non-trivial inner function $I$, then the closure of $\mathcal{P}f$ is contained in $I \cdot H^2$, which is a proper subspace of $H^2$ so $\mathcal{P}f$ is not dense in $H^2$.

Next we come to another famous theorem on outer functions. The De Leeuw-Rudin theorem tells us that the extreme points of the unit ball of $H^1(D)$ are the outer functions of unit norm. As such, it is the starting point of our investigation of (strongly) exposed points in $H^1(D)$.

Theorem 1.10 ([33]). (De Leeuw-Rudin) A function $f$ is an extreme point of the unit ball of $H^1(D)$ if and only if $f$ is an outer function and $\|f\|_1 = 1$. Furthermore, if $f$ is of unit norm but not extreme, then there exist extreme points $F_1$ and $F_2$ such that $f = \frac{1}{2}(F_1 + F_2)$.

Proof. Suppose $f$ is an outer function of unit norm and suppose $g$ in $H^1$ is such that $\|f + g\|_1 = \|f - g\|_1 = 1$. Let $d\mu$ be the probability measure $|f| \frac{d\theta}{2\pi}$ on the unit circle. Then, with $k = g/f$ on $T$, the relations $\|f + g\|_1 = \|f - g\|_1 = 1$ imply that $\int_{0}^{2\pi} |1 + k| + |1 - k| d\mu = 2$. Because for all $z \in \mathbb{C}$, $|1 + z| + |1 - z| \geq 2$, with equality if and only if $z \in [-1, 1]$, we conclude that for almost $d\mu$-every $\xi \in T : g(\xi)/f(\xi) \in [-1, 1]$. Observe that this inclusion also holds for almost $d\theta$-every $\xi \in T$, because $\frac{df}{d\theta} = |f| \neq 0$ almost $d\theta$-everywhere on $T$. In particular, $|g| \leq |f| \ d\Omega - \text{ a.e. on } T$. By the fact that $f$ is outer (1.5), then also for all $z \in D: |g(z)| \leq |f(z)|$. Hence $g/f \in H^\infty$ has real boundary values on $T$. A glance at the Poisson integral representation of $g/f$ yields that $g/f$ is constant. Finally, because $\|f + g\|_1 = 1$, this constant is zero, i.e., $g \equiv 0$, so $f$ is extreme.

In the other direction, suppose $f = I \cdot F$ is of unit norm, but $I$ is a non-trivial inner function. Because for every $e^{i\theta} \in T \setminus \{\pm 1\}$:

$$\pm \left(1 \pm e^{i\theta}\right)^2 = 2 \pm 2\Re(e^{i\theta}) = 2 \pm 2\cos(\theta) > 0,$$

the functions $I$ and $(1 \pm I)^2$ have the same argument a.e. on $T$. Let $g = g_I = \frac{1 + I^2}{2}$. We have

$$\|f + g\|_1 = \int_{0}^{2\pi} |F|(1 \pm 2\Re(I)) \frac{d\theta}{2\pi} = 1 \pm 2\Re(\int_{0}^{2\pi} |F| \cdot I \frac{d\theta}{2\pi}).$$
Now if we replace $I$ by $\lambda I$ throughout the preceding, where $\lambda \in \mathbb{T}$ is arbitrary, then we obtain:

$$\|f \pm g\|_1 = 1 \pm 2\Re\left(\lambda \int_0^{2\pi} |F| \cdot I \frac{d\theta}{2\pi}\right).$$

Therefore, we can choose $\lambda$ in such a way that $\Re\left(\lambda \int_0^{2\pi} |F| \cdot I \frac{d\theta}{2\pi}\right) = 0$. Consequently, $\|f \pm g\|_1 = 1$, but $g \neq 0$, so $f$ is not extreme. Let $F_1 = f + g = (1 + I)^2F/2$ and $F_2 = f - g = (1 - I)^2F/2$. The functions $F_1$ and $F_2$ are of unit norm in $H^1$. Because the functions $1 + I$ and $1 - I$ have positive real parts, they are outer functions (see Lemma 3.3 below). Hence the functions $F_1$ and $F_2$ are extreme points and $f = \frac{1}{2}(F_1 + F_2)$. □

1.4 Introduction to uniform algebras

This section develops the necessary background on the theory of uniform algebras that is required for Chapters 2 and 3. For more details we refer to [62].

**Definition.** A commutative Banach algebra $A$ is a commutative complex algebra which is also a Banach space with a norm $\|\cdot\|$ satisfying $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$. In addition, we assume that $A$ contains a unit element 1: $1 \cdot a = a$ for all $a \in A$.

**Definition.** Suppose $A$ is a commutative Banach algebra. The maximal ideal space (or spectrum) $\mathcal{M}_A$ of $A$ is the set of multiplicative (linear) functionals (homomorphisms) $m : A \rightarrow \mathbb{C}$; that is, for all $a, b \in A$ and all $\lambda \in \mathbb{C}$: $m(a \cdot b) = m(a)m(b)$ and $m(\lambda \cdot 1) = \lambda$.

A few words about the terminology and the existence of multiplicative functionals. If $m$ is a multiplicative functional on $A$, then $I = \ker(m)$ is a maximal ideal in $A$. As a field, $A/I = \mathbb{C}$. Conversely, if $I$ is an ideal in $A$, the map $m : a \in A \mapsto [a] = a + I \in A/I$ is a homomorphism. By the Gelfand-Mazur theorem ([14], Theorem 2.31), if $I$ is maximal, then $A/I = \mathbb{C}$ as a field, which means that we may regard $m$ as a multiplicative functional on $A$ with kernel $I$. Hence, the maximal ideals in $A$ are in one-to-one correspondence with the multiplicative functionals on $A$. This settles the question of existence of multiplicative functionals.

By Zorn's lemma, an element $a \in A$ is not invertible if and only if $a$ is contained in a maximal ideal. In other words, an element $a \in A$ is invertible in $A$ if and only if $m(a) \neq 0$ for all $m \in \mathcal{M}_A$ and the set $\{m(a) \cdot m : m \in \mathcal{M}\}$ coincides with the spectrum $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \cdot 1 \text{ is not invertible in } \mathcal{M}\}$ of $a$. From this then it follows that every multiplicative functional $m$ on a commutative Banach algebra is a contraction. Thus $\mathcal{M}_A$ is a weak* closed subset of the unit ball of $A^*$. Consequently, when equipped with the weak* topology (as is customary), $\mathcal{M}_A$ becomes a compact Hausdorff space.

**Definition.** Let $A$ be a commutative Banach algebra. The Gelfand transform $G : A \rightarrow C(\mathcal{M}_A)$ is the map $a \mapsto \hat{a}$, where $\hat{a} : m \in \mathcal{M}_A \mapsto m(a)$.
Because the multiplicative functionals are contractive we see that the Gelfand transform in turn is a contraction from $\mathcal{A}$ into $C(\mathcal{M}_\mathcal{A})$. The Banach algebras for which the Gelfand transform is isometric are of special interest.

**Definition.** If $\mathcal{A}$ is a commutative Banach algebra and the Gelfand transform is isometric, we say that $\mathcal{A}$ is a **uniform algebra**.

Obviously, if $\|\hat{a}\|_\infty = \|a\|$ for all $a \in \mathcal{A}$, then $\|a^2\| = \|a\|^2$ for all $a \in \mathcal{A}$. Conversely, this latter condition also implies that the Gelfand transform is isometric by the formula for $\|\hat{a}\|$. Indeed, by the spectral theorem the norm of $\hat{a}$ equals $\lim_{n \to \infty} \sqrt{\|a^n\|}$, the spectral radius of $\sigma(a)$.

**Corollary 1.11.** A commutative Banach algebra $\mathcal{A}$ is a uniform algebra if and only if for all $a \in \mathcal{A}$: $\|a^2\| = \|a\|^2$.

**Definition.** Let $K$ be a compact Hausdorff space. A function algebra on $K$ is a subalgebra of $C(K)$ that separates the points of $K$ and contains the constant functions.

The corollary expresses that the Gelfand transform acts as an isometric isomorphism between a uniform algebra and a function algebra on its maximal ideal space.

**Example.** The disc algebra $A(D)$ is a uniform algebra. Let $m$ be a multiplicative functional on $A(D)$. Let $z_m$ be the point $m(Z) \in \overline{D}$, where $Z$ indicates the identity function on $D$. It follows from the fact that the polynomials are dense in $A(D)$ that the action of $m$ is point evaluation at $z_m$. By the identification $m \in \mathcal{M}_\mathcal{A} \leftrightarrow z_m \in \overline{D}$ we see that the maximal ideal space of $A(D)$ is nothing other than $\overline{D}$ and that $A(D)$ is a function algebra on $\overline{D}$. Of course, this is nothing new. We can go further. Namely, we can think of the disc algebra as a subalgebra of $C(T)$, where the functions already attain their norms and separate the points. In other words, $A(D)$ can be regarded as a function algebra on $T$. This motivates the following definition.

**Definition.** Let $\mathcal{A}$ be a function algebra on $K$. If $K'$ is a closed subset of $K$ such that for all $f \in \mathcal{A}$, $\|f\| = \max_{z \in K'} |f(z)|$, then we say that $K'$ is a **boundary** for $\mathcal{A}$.

The intersection of all boundaries is again a boundary (thus the smallest boundary) and is commonly referred to as the Shilov boundary of $\mathcal{A}$. Thus every function algebra is a function algebra on its Shilov boundary. For example, the unit circle is the Shilov boundary of the disc algebra. Similarly, because the pairing $f \in H^\infty \leftrightarrow f^* \in L^\infty(T)$ is isometric, we see that the maximal ideal space $\mathcal{M}_{L^\infty}$ of $L^\infty(T)$ is a boundary for $H^\infty(D)$. (It is in fact the Shilov boundary of $H^\infty(D)$.)

Let us exhibit special points that are contained in the Shilov boundary of a function algebra $\mathcal{A} \subset C(K)$.

**Definition.** A point $a \in K$ is called a **strong boundary point** for $\mathcal{A}$ if for every neighborhood $U$ of $a$ there is a function $f \in \mathcal{A}$ such that $\|f\| = f(a) = 1$ while $\max_{b \in U} |f(b)| \leq \frac{1}{2}$. We say that $a \in K$ is a **peak point** for $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ such that $\|f\| = f(a) = 1$ and $|f(b)| < 1$ for all $b \in K$, $b \neq a$. 

From the definition it is clear that a peak point is also a strong boundary point. In general, a function algebra need not contain any peak points. Strong boundary points, however, always exist in function algebras. First observe that all strong boundary points are contained in every boundary of a function algebra. In the other direction we have the following:

**Theorem 1.12.** If $\mathcal{A}$ is a function algebra, then every function in $\mathcal{A}$ attains its maximum modulus on the set of strong boundary points for $\mathcal{A}$.

**Proof.** Cf. [62], Theorem 7.21.

It follows that the closure of the set of strong boundary points coincides with the Shilov boundary for $\mathcal{A}$. Therefore, if $\mathcal{A}$ is any function algebra on $K$, through restriction of $\mathcal{A}$ to its Shilov boundary, one may assume that the set of strong boundary points is dense in $K$ without loss of generality.