Strongly exposed points in unit balls of Banach spaces of holomorphic functions
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Chapter 2

Exposedness and strong exposedness in Banach algebras

In this chapter we will discuss the sets of extreme and exposed points in the function spaces $H^\infty(D)$ and $A(D)$. These sets are non-trivial yet relatively easy to describe and the classic proofs of these results offer a good illustration of the concepts from the previous chapter. Next we investigate the question of strong exposedness. Our main result states that in infinite dimensional function algebras (which include $H^\infty(D)$, $A(D)$ and their several variables counterparts), there are no strongly exposed points in the unit ball. It is here that the algebra structure of the Banach space plays a crucial role.

2.1 Extreme and exposed points in $H^\infty(D)$ and $A(D)$

Let us first describe the extreme points of the unit ball of $H^\infty(D)$. We recall that a function $f$ in the boundary is not extreme if and only if there exists a function $g \in H^\infty$, not identically zero, for which $\|f + g\|_\infty = \|f - g\|_\infty = 1$. Suppose that $f$ is not extreme and let $g$ be as above. Then at every $z \in D$:

$$|f(z)|^2 + |g(z)|^2 = \frac{1}{2}(|f(z) - g(z)|^2 + |f(z) + g(z)|^2) \leq 1.$$ 

Hence, $|g(z)|^2 \leq 1 - |f(z)|^2 \leq 2(1 - |f(z)|)$. Thus for almost every $\xi \in T$: $|g(\xi)| \leq 2(1 - |f(\xi)|)$. Because $g$ is not identically zero, the integral $\int_T \log |g| \frac{dt}{2\pi}$ converges, so by our estimate, $\int_T \log(1 - |f|) \frac{dt}{2\pi} > -\infty$. Conversely, suppose the function $\psi = \log(1 - |f|)$ is integrable on $T$, for some $f$ in the unit ball of $H^\infty$. If we let $u = P[\psi] \leq 0$ be the Poisson integral of $u$ on $D$, with harmonic conjugate $\hat{u}$, the function

$$g(z) = e^{(u + i\hat{u})(z)}$$
belongs to $H^\infty$ and satisfies $|g(z)| \leq 1 - |f(z)|$ (cf. 1.5) at every point $z \in D$. We conclude that $\|f \pm g\|_{\infty} \leq 1$, so $f$ is not extreme in the unit ball of $H^\infty$.

One can easily adapt the latter argument to apply to the disc algebra. Indeed, given a function $f$ in $A(D)$ such that $\int_T \log(1 - |f|) \frac{d\theta}{2\pi} > -\infty$, one constructs a continuous function $\psi : T \rightarrow [0, 1]$ with the following properties:

- $\psi \leq 1 - |f|$;
- $\psi$ is smooth on the relative open set $\{|f| < 1\} = \{ e^{i\theta} : |f(e^{i\theta})| \neq 1 \}$;
- $\int_T \log \psi \frac{d\theta}{2\pi} > -\infty$.

Then with $u = P[\log \psi]$ as before, $g = e^{u+i\hat{u}} \in A(D)$ and again, $\|f \pm g\|_{\infty} \leq 1$, which shows that $f$ is not extreme in $A(D)$.

Let us sum up the results on extreme points in $H^\infty(D)$ and $A(D)$:

**Theorem 2.1 ([33]).** Let $f$ be an element of the unit ball of $H^\infty(D)$ or $A(D)$, respectively. Then $f$ is an extreme point of the unit ball if and only if 

$$\int_T \log(1 - |f|) \frac{d\theta}{2\pi} = -\infty.$$ 

Subsequently, the exposed points of the unit balls of $A(D)$ and $H^\infty$ were described by S. D. Fisher (1969) and E. Amar & A. Lederer (1971) respectively. Let us follow their proofs and briefly describe their results. Suppose $f \in \partial \text{Ball}(H^\infty)$ has absolute value 1 on some subset $E$ of $T$ of positive Lebesgue measure. We claim that the functional 

$$L = L_E : g \in H^\infty \mapsto \frac{1}{|E|} \int_E g(f) \frac{d\theta}{2\pi}$$

exposes $f$. Indeed, suppose $g \in \partial \text{Ball}(H^\infty)$ is such that $L(g) = L(f) = 1$. Because 

$$\frac{1}{|E|} \int_E |g| \frac{d\theta}{2\pi} \leq 1,$$

we conclude that $g \bar{f} = |g| = 1$, and hence $g = f$ almost everywhere on $E$. By the fact that $E$ has positive measure, $f$ and $g$ coincide everywhere.

Conversely, if $f$ is exposed in $A(D)$ by the functional $L$, then by the Hahn-Banach theorem, $L$ must be of the form $L(g) = \int_T g \bar{f} d\mu$ for all $g \in A(D)$ and some probability measure $\mu$ with support in the set $E = \{|f| = 1\}$. We claim that $E$ has positive Lebesgue measure. For otherwise, being a closed subset of $T$ of measure zero, $E$ would be a peak set for the disc algebra. Thus, there would exist a function $p \in A(D)$ such that $p = 1$ on $E$ and $|p(z)| < 1$ for all $z \in \bar{D} \setminus E$. But then $L(pf) = L(f) = 1$, contradicting the exposedness of $f$.

**Theorem 2.2 ([17]).** Let $f$ be an element of the unit ball of $A(D)$. Then $f$ is exposed if and only if $|f| = 1$ on a set of positive measure.

Amar & Lederer used the maximal ideal space of $H^\infty$ to extend Fisher's result to $H^\infty$:

**Theorem 2.3 ([1]).** Let $f$ be an element of the unit ball of $H^\infty$. Then $f$ is exposed if and only if $|f| = 1$ on a set of positive measure.
2.2 Strongly exposed points in function algebras

In the previous section we saw that there exist many exposed points in the unit balls of the function algebras \( A(D) \) and \( H^\infty(D) \). In fact, the exposed points are dense in the boundaries of the respective unit balls. For strongly exposed points, however, the situation is quite the opposite: unless a function algebra is trivial, i.e., finite dimensional, there are no strongly exposed points.

**Theorem 2.4 ([4])**. There are no strongly exposed points in the unit ball of an infinite dimensional function algebra \( A \subset C(K) \).

We can make the idea behind this statement intuitively clear. All functionals on \( A \) are given by integration against some regular Borel measure \( \mu \) on \( K \). Given any such functional, and a function \( f \) in the unit ball of \( A \), one can modify the size of the function \( f \) on a subset of \( K \) of small \( \mu \)-measure without essentially changing the integral of \( f \) with respect to \( \mu \). Specifically, we can find a function \( g \) in the unit ball of \( A \) such that \( |L(f) - L(g)| \) is arbitrarily small, yet \( \|f - g\|_\infty \) can be arbitrarily close to 1.

A word of warning though. It is the algebra structure of \( A \) that is used in an essential way in the modification process: by a famous theorem of Banach, any Banach space \( X \) is isometrically isomorphic to a closed subspace of \( C(K) \) for some compact Hausdorff space \( K \) ([14], Theorem 1.29) and by Theorem 1.3, there are Banach spaces with "many" strongly exposed points.

Before we proceed with the proof of Theorem 2.4 we need a lemma.

**Lemma 2.5.** Let \( \Omega \subset \mathbb{C} \) be the interior of the Jordan curve

\[
\{ z \in \mathbb{C} : |z| \leq 1, |\arg z| = (1 - |z|)^2 \} \cup \{0\}.
\]

Then

\[
\lim_{n \to \infty} \sup_{z \in \Omega} |1 - z^n| = 1.
\]

**Proof.** By the maximum modulus theorem we need to show that

\[
\lim_{n \to \infty} \sup_{z \in \partial \Omega} |1 - z^n| = 1.
\]

For \( z \in \partial \Omega \):

\[
|1 - z^n|^2 = 1 + |z|^{2n} - 2|z|^n \cos(n(1 - |z|)^2).
\]

If \( n(1 - |z|)^2 \leq \frac{1}{\sqrt{n}} \), then the cosine is at least \( \frac{1}{2} \) and \( |1 - z^n|^2 \leq 1 \). On the other hand, if \( n(1 - |z|)^2 \geq \frac{1}{\sqrt{n}} \), then \( |z|^n \) is bounded above by \( (1 - \frac{1}{n^{1/4}})^n \) which behaves like \( e^{-n^{1/4}} \). \( \square \)

We are now ready for the proof of Theorem 2.4. The proof we give here has been simplified based on an extension of the result to so called denting points due to O. Nygaard & D. Werner ([44]).
Proof of Theorem 2.4. Let $F$ be an exposed point of the unit ball and suppose $L$ is a functional on $A$ that exposes $F$. Reasoning as in the paragraph preceding Theorem 2.2, we see that there exists a probability measure $\mu$ on $K$, supported in the set $E = \{|/| = 1\}$, such that

$$L : g \in A \mapsto \int_E gF d\mu.$$ 

Choose $\varepsilon > 0$. Because there are infinitely many strong boundary points, we can find a strong boundary point $x$, an open set $V \ni x$ in $K$ such that $\mu(V) < \varepsilon$, and, subsequently, a function $p \in A$ for which $p(x) = \|p\| = 1$, and $|p| < \frac{1}{2}$ on $K \setminus V$.

Next, let $\phi : D \to \Omega$ be a Riemann map of the unit disc to the domain $\Omega$ of Lemma 2.5, with $\phi(1) = 1$. Because $\phi$ is an element of the disc algebra, the function $\phi \circ p$ is contained in $A$. We set $f_n := (1 - (\phi \circ p)^n)F \in A$. By Lemma 2.5, $\limsup \|f_n\| \leq 1$. Because $(\phi \circ p)^n \to 0$ uniformly on $K \setminus V$,

$$\int_{E \setminus V} f_n F d\mu \to \mu(E \setminus V) = 1 - \mu(V)$$

as $n \to \infty$. By the choice of $V$, $|L(f_n) - 1| < 2\varepsilon$ for all sufficiently large $n$. We fix a large number $n$ such that $|L(f_n) - 1| < 2\varepsilon$ and $\|f_n\| < 1 + \varepsilon$.

Because $f_n(x) = 0$, we can find a small neighborhood $W \subset V$ of $x$ such that $|f_n| < \varepsilon$ on $W$. Let $q \in A$ be such that $\|q\| \leq 1 = |F(x) - q(x)|$ and $|q| < \varepsilon$ on $K \setminus W$. Consequently, $|L(q)| < 2\varepsilon$. Let $G = f_n + q$. We then have $\|G\| < 1 + 3\varepsilon$ and $|L(G) - 1| < 4\varepsilon$, but $\|F - G\| \geq 1$, whence $F$ cannot be strongly exposed. \hfill \square

The following corollary follows easily from the previous theorem and Theorem 1.3. We see in particular that the disc algebra is not a dual space.

**Corollary 2.6 ([30]).** No infinite dimensional separable uniform algebra is a dual space.