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Chapter 4

Strongly exposed points in $H^1(\Omega)$

Having studied the sets of exposed and strongly exposed points in the unit ball of the classical Hardy space $H^1$ of the unit disc, we will investigate how these results hold up for the Hardy space $H^1$ of a domain of finite connectivity in $\mathbb{C}$. We define this space in Section 4.1. There are two important differences with the classical Hardy space that make our analysis very different. On domains of finite connectivity, $H^1$-functions may not allow a classical factorization using Blaschke products, (singular) inner functions and outer functions. Secondly, and in a way related, there now exist extreme points in the unit ball with (finitely many) zeros (Section 4.3). While such zero sets are somewhat generic for extreme points, their location plays a surprisingly crucial role in its being a (strongly) exposed point (Section 4.4).

4.1 Hardy spaces of planar domains

In this section we define the Hardy spaces $H^p(\Omega)$ of a domain $\Omega$ in $\mathbb{C}$ and explain how they may be seen as subsets of the classical spaces $H^p(D)$. More details can be found in [16].

**Definition.** Let $\Omega$ be a (bounded) domain in $\mathbb{C}$ and let $0 < p < \infty$. The Hardy space $H^p(\Omega)$ consist of all holomorphic functions $f$ on $\Omega$ for which $|f|^p$ admits a harmonic majorant on $\Omega$. The space $H^\infty(\Omega)$ consists of all bounded holomorphic functions on $\Omega$ and is equipped with the sup norm.

We can make the spaces $H^p(\Omega)$ ($1 \leq p < \infty$) into normed spaces, Banach spaces indeed, in the following way. Fix any point $z_0 \in \Omega$. For a function $f$ in $H^p(\Omega)$, let $u = u_f$ be the least harmonic majorant of $|f|^p$. The definition $\|f\|_{H^p(\Omega)} := (u(z_0))^{1/p}$ turns $H^p(\Omega)$ into a Banach space. When $\Omega = D$ and $z_0 = 0$ this definition coincides with our previous definition of $H^p(D)$. We will refer to this norm as the $H^p(\Omega)$-norm relative to the base point $z_0$. Of course, different base points yield different (albeit
equivalent) norms on $H^p(\Omega)$. We will shortly introduce another (equivalent) norm that is somewhat easier to work with.

It will be convenient to assume that the domain $\Omega$ has a “nice” boundary $\partial \Omega$. To be precise: we assume that $\partial \Omega$ consists of finitely many disjoint continua. We call such a domain a domain of finite connectivity, in short, a finite domain. (For a discussion of Hardy spaces on domains of infinite connectivity, we refer to [21].) By repeated application of the Riemann mapping theorem, we may without loss of generality assume that the finite domain $\Omega$ is a subset of $D$ and that $\Gamma = \partial \Omega$ consists of $m + 1$ disjoint closed smooth (analytic) curves $\Gamma_0 = \Gamma, \Gamma_1, \ldots, \Gamma_m$. Throughout the remainder of this chapter $\Omega$ will denote a finite domain of this type, unless explicitly stated otherwise. Also, because we have already dealt with $H^1(D)$ theory in the previous chapter, we will implicitly assume that $m \geq 1$, i.e., the domain $\Omega$ is multiply connected.

Let $0 < p < \infty$ and choose $f \in H^p(\Omega)$. We define new functions $f_0, f_1, \ldots, f_m$ on $\Omega$ as follows. Given a point $z \in \Omega$ we find closed (oriented) paths $\gamma_0 = C(0, R)(R > |z|), \gamma_1, \ldots, \gamma_m$ near the boundary curves $\Gamma_0 = \Gamma, \Gamma_1, \ldots, \Gamma_m$, chosen in such a way that $\gamma_i$ encircles $\Gamma_i$ but not the point $z$ ($i \geq 1$). We then set:

$$f_i(z) := \int_{\gamma_i} \frac{f(\xi)}{\xi - z} \frac{d\xi}{2\pi i}.$$

By Cauchy’s theorem, these functions are well-defined on $\Omega$. In fact, $f_0$ is holomorphic on $D$ and for $i \geq 1$, the integral definition of $f_i$ makes sense for all $z$ in the exterior of $\gamma_i$. Hence the function $f_i$ is holomorphic on the exterior of $\Gamma_i$, by shrinking the curve $\gamma_i$ to $\Gamma_i$. Also, $f(z) = f_0(z) + f_1(z) + \cdots + f_m(z)$. The next step is to regard the functions $f_i$ ($i \geq 1$) as functions $F_i$ on $D$ using a Riemann map of the exterior of $\Gamma_i$ (in the extended complex plane) onto $D$. Because the functions $f_i$ are holomorphic across $\Gamma_j$, hence bounded near $\Gamma_j$ ($i, j \geq 0, i \neq j$), it is easy to see that the function $|f|^p$ admits a harmonic majorant on $D$ if and only if the functions $|f_0|^p, |F_1|^p, \ldots, |F_m|^p$ admit harmonic majorants on $D$, that is, if and only if $f_0, F_1, \ldots, F_m \in H^p(D)$. Consequently, as a space of holomorphic functions we may identify $H^p(\Omega)$ with the product of $m + 1$ copies of $H^p(D)$. Also, from the representation of $f = \sum_i f_i$ – a sum of $H^p(D)$-functions – it is immediate that every $f \in H^p(\Omega)$ has non-tangential boundary values $f^*(.)$ almost everywhere with respect to arc length $d\sigma$ on $\partial \Omega$. Unless $f$ vanishes identically, it follows that $\log |f^*|$ is $d\sigma$-integrable on $\partial \Omega$ and that for every $z \in \Omega$:

$$f(z) = \int_{\partial \Omega} \frac{f^*(\xi)}{\xi - z} \frac{d\xi}{2\pi i} = \int_{\partial \Omega} f^*(\xi) d\omega(\xi, z),$$

where $\omega(., z)$ is harmonic measure on $\partial \Omega$ with respect to $\Omega$ and the point $z$, and

$$\log |f(z)| \leq \int_{\partial \Omega} \log |f^*(\xi)| d\omega(\xi, z), \quad |f(z)| \leq \int_{\partial \Omega} |f^*(\xi)| d\omega(\xi, z).$$

In other words, for the finite domain $\Omega$ we can do the function theory for Hardy spaces on the boundary $\partial \Omega$. 


For \(1 \leq p < \infty\) we make \(H^p(\Omega)\) into a Banach space with the following norm:

\[
\|f\|_{H^p(\Omega)} = \|f^*\|_{L^p(\partial\Omega, d\sigma)}.
\]

When no confusion is possible we will simply write \(\|f\|_p\).

Building further on these ideas, one can embed \(H^p(\Omega)\) in \(H^p(D)\) for all domains \(\Omega\) (possibly infinitely connected) in \(\mathbb{C}\). For this we use the universal covering space of \(\Omega\) that must be the unit disc \(D\) if \(H^p(\Omega)\) contains non-constant functions. Let \(\pi : D \to \Omega\) be the uniformizer. The map \(\pi\) is holomorphic and locally invertible. Associated to \(\pi\) there is a group \(\mathcal{G}\) of automorphisms (Möbius maps) \(\psi\) of the unit disc under which \(\pi\) is invariant: \(\pi \circ \psi = \pi\). We call \(\mathcal{G}\) the group of automorphisms of \(\pi\). If the function \(f\) on \(\Omega\) is of one of the following types: holomorphic, harmonic, continuous, then the function \(F = f \circ \pi\) is of the same type on \(D\) and \(\mathcal{G}\)-invariant; we call \(F\) the lift of \(f\). Conversely, if the function \(F\) on \(D\) is \(\mathcal{G}\)-invariant, then the function \(f := F \circ \pi^{-1}\) (where \(\pi^{-1}\) is any local inverse of \(\pi\)) is well-defined on \(\Omega\), and because \(\pi\) is locally biholomorphic, \(f\) inherits properties of \(F\) such as holomorphy, harmonicity and continuity.

Let the functions \(f\) on \(D\) and \(F\) on \(\Omega\) be related by \(F = f \circ \pi\). If \(|f|^p\) admits a harmonic majorant \(u\) then the harmonic function \(u \circ \pi\) majorizes \(|F|^p\). If \(|F|^p\) admits a harmonic majorant on \(D\), then there is also a least harmonic majorant \(U\) that must be \(\mathcal{G}\)-invariant because \(|F|^p\) is. By the preceding remarks there exists a harmonic function \(u\) on \(\Omega\) that lifts to \(U\) and majorizes \(|f|^p\) on \(\Omega\). Therefore, the lifting process allows us to identify \(H^p(\Omega)\) with the collection of \(H^p(D)\)-functions that are \(\mathcal{G}\)-invariant.

**Definition.** Let \(\mathcal{G}\) be a group of automorphisms of \(D\). We say that a Lebesgue measurable subset \(E\) of \(T\) is \(\mathcal{G}\)-invariant if the symmetrical difference of \(E\) and \(\psi(E)\) has Lebesgue measure zero for all \(\psi \in \mathcal{G}\). The \(\sigma\)-algebra of all Lebesgue measurable \(\mathcal{G}\)-invariant subsets of \(T\) will be denoted by \(L/\mathcal{G}\). Finally, the collection of \(L/\mathcal{G}\)-measurable \(L^p\)-integrable functions on \(T\) will be denoted by \(L^p/\mathcal{G}\) (where, of course, we identify functions that are equal almost everywhere) \((0 < p \leq \infty)\).

It should come as no surprise that \(L^p/\mathcal{G}\) consists of all \(L^p\)-functions \(f\) that are \(\mathcal{G}\)-invariant: \(f = f \circ \psi\) (a.e.), for every \(\psi \in \mathcal{G}\). We can summarize the preceding by the statement that \(H^p(\Omega)\) can be identified with the subset \(H^p/\mathcal{G} := H^p(D) \cap L^p/\mathcal{G}\) of \(H^p(D)\), where \(\mathcal{G}\) is the group of automorphisms of \(\pi : D \to \Omega\).

Let \(\mathcal{E} : L^1 \to L^1/\mathcal{G}\) be the conditional expectation operator with respect to the \(\sigma\)-subalgebra \(L/\mathcal{G}\) of \(L\) (all Lebesgue measurable subsets of \(T\)). This means that given a function \(f \in L^1\), \(\mathcal{E}(f)\) is the (unique) \(L^1/\mathcal{G}\)-function such that for all \(E \in L/\mathcal{G}\):

\[
\int_E f \, d\theta = \int_E \mathcal{E}(f) \, d\theta.
\]

Existence of \(\mathcal{E}\) follows from measure theory. The operator \(\mathcal{E}\) is a bounded map on \(L^1\), maps \(L^\infty\) onto \(L^\infty/\mathcal{G}\) and acts as the identity on \(L^1/\mathcal{G}\)-functions.
As an illustration we will show that \( H^1(\Omega) \) is a dual Banach space. The proof we give works for all domains \( \Omega \) (possibly infinitely connected) in \( \mathbb{C} \) and is inspired by the proof of the classical duality result for \( H^1(D) \) (cf. Corollary 1.6).

Let \( \overline{E(C)} \) and \( \overline{E(A_0)} \) be the \( L^\infty \)-closures of the spaces \( E(C) \) and \( E(A_0) \) respectively, where \( A_0 \) is the set of functions in the disc algebra that vanish at the origin.

**Proposition 4.1** ([2]). Let \( \Omega \) be a finite domain, with uniformizer \( \pi : D \to \Omega \). The dual space of \( \overline{E(C)}/\overline{E(A_0)} \) is \( H^1(\Omega) \) relative to the base point \( \pi(0) \).

**Proof.** Take \( f \in H^1(\Omega) \), and let \( F \) be the lift of \( f \). By the choice of the base point, the \( H^1(D) \)-norm of \( F \) equals the norm of \( f \) in \( H^1(\Omega) \). Consider the functional \( L = L_F \):

\[
L : g \in E(C) \mapsto \int_T gF \frac{d\theta}{2\pi}.
\]

Of course, \( L \) is bounded by \( \|F\|_1 \), hence \( L \) can be extended continuously to \( \overline{E(C)} \). We will show that as an operator on \( \overline{E(C)} \) \( L \) has norm equal to \( \|F\|_1 \). To this end, let \((g_n)\) be a sequence of a continuous functions on \( T \) that converges to \( \phi = \overline{F}/|F| \) in \( L^1 \) and that is bounded by 1 pointwise (a.e.). It follows that \( E(g_n) \) converges to \( E(\phi) = \phi \) in \( L^1 \). Taking a subsequence if necessary, we may also assume that \( E(g_n) \) converges to \( \phi \) pointwise almost everywhere. By dominated convergence:

\[
\|L\| \geq \lim_{n \to \infty} L(E(g_n)) = \int_T \phi F \frac{d\theta}{2\pi} = \|F\|_1.
\]

The functional \( L \) annihilates \( E(A_0) \): if \( g \in A_0 \) then

\[
L(E(g)) = \int_T \mathcal{E}(g) F \frac{d\theta}{2\pi} = \int_T \mathcal{E}(gF) \frac{d\theta}{2\pi} = \int_T gF \frac{d\theta}{2\pi} = 0.
\]

Hence the mapping \( f \in H^1(\Omega) \mapsto L_F \in \overline{E(A_0)}^\perp \cong (\overline{E(C)}/\overline{E(A_0)})^* \) is isometric.

We will finish the proof by showing that this mapping is also surjective. Let \( \ell \in \overline{E(A_0)}^\perp \). Then let \( L \) be the bounded functional \( L(g) := \ell(E(g)) \) on \( C(T) \). By construction, \( L \) annihilates \( A_0 \). Hence, by the F. & M. Riesz theorem there exists a unique function \( F \in H^1 \) that represents the functional \( L \):

\[
L(g) = \int_T gF \frac{d\theta}{2\pi}.
\]

By the uniqueness of the representing function it is not hard to show that \( F \) is \( \mathcal{G} \)-invariant, using the properties of \( \mathcal{G} \) and a simple coordinate transformation. It follows that \( \ell = L_F \) as desired. \( \square \)

### 4.2 Factorization in \( H^1(\Omega) \)

We now come to the question whether \( H^1(\Omega) \)-functions admit an *inner-outer factorization* like in \( H^1(D) \). Of course an easy “solution” would be to leave the domain \( \Omega \),
because as we have seen, we can embed $H^1(\Omega)$ in $H^1(D)$, where we have the classical factorization at our disposal. However, if we insist on working with the domain $\Omega$, we cannot circumvent certain problems intrinsic to the multiple connectivity of $\Omega$. In particular, we will see that there are extreme points in the unit ball of $H^1(\Omega)$ with a zero when $m \geq 2$.

We call a function $I \in H^\infty(\Omega)$ an inner function if $|I^*| = 1$ a.e. ($d\sigma$) on $\partial \Omega$. Consequently, $|I(z)| \leq 1$ for all $z \in \Omega$. We call a function $F \in H^1(\Omega)$ an outer function if for every $z$ in $\Omega$ (equivalently, for at least one $z \in \Omega$):

$$
\log |F(z)| = \int_{\partial \Omega} \log |F^*(\xi)| \, d\omega(\xi, z).
$$

Obviously then, an outer function is zero free on $\Omega$.

Can we find an analogue in $H^\infty(\Omega)$ of a Blaschke factor, i.e., an inner function with a single zero $z_0$, that is continuous on $\overline{\Omega}$? Suppose such a function $B$ exists. Then $G(z; z_0) := -\log |B(z)|$ equals Green's function for $\Omega$ with a pole at $z_0$, so that $B(z) = \lambda \exp(-G - i\tilde{G})$ for some $\lambda \in \mathbb{T}$, where $\tilde{G}$ denotes the harmonic conjugate of $G$. But it is well-known that unless $\Omega$ is simply connected ($m = 0$), both $G$ and $\exp(-i\tilde{G})$ are not well-defined (single valued) functions on $\Omega$. Therefore, no such $B$ exists.

Nevertheless, for $f \in H^1(\Omega)$, the zeros $z_1, z_2, \ldots$ of $f$ satisfy the so-called Blaschke condition

$$
\sum_n G(z; z_n) < \infty
$$

for all $z \in \Omega, z \neq z_i$. Given any such sequence of points $(z_n)$, one is tempted to write down the following (formal) Blaschke product

$$
B(z) = \exp(-\sum_n G(z; z_n) - i\sum_n \tilde{G}(z; z_n)). \quad (4.1)
$$

While in general, as we have pointed out, this is not a well-defined (single valued) function, it has the following properties:

- $|B(z)|$ is well-defined;
- $|B(z)|$ is bounded on $\Omega$;
- Locally on $\Omega, |B|$ coincides with the absolute value of a holomorphic function.  
  In other words, for every $z \in \Omega$ there is a neighborhood $U$ of $z$ in $\Omega$ and a holomorphic function $\psi$ defined on $U$ such that $|B(z)| = |\psi(z)|$ on $U$.

Any (multiple-valued) function $B$ that has these three properties will be referred to as a function in $MH^\infty(\Omega)$. If $|B|$ also has non-tangential limits 1 a.e. on $\Gamma$, we say that $B$ is an inner function in $MH^\infty(\Omega)$. The generalized Blaschke products (4.1) are in fact inner functions in $MH^\infty(\Omega)$. If $I$ is a zero free inner function on $\Omega$, we say that $I$ is a singular inner function on $\Omega$.

In general, we say that a multiple valued function $F$ on $\Omega$ is modulus holomorphic if
\[ |F(z)| \text{ is well-defined}; \]

- Locally on \( \Omega \), \( |F| \) coincides with the absolute value of a holomorphic function.

If in addition, \( |F(z)|^p \) admits a harmonic majorant on \( \Omega \) \((0 < p < \infty)\), then we say that \( F \in MH^p(\Omega) \). One can show that if \( F \in MH^p(\Omega) \), then \( |F| \) has non-tangential limits (denoted \( |F^*| \)) a.e. on \( \partial \Omega \), \( |F^*| \in L^p(\partial \Omega, d\sigma) \), and unless \( F \equiv 0 \), \( \log |F^*| \in L^1(\partial \Omega, d\sigma) \) and

\[
\log |F^*(z)| \leq \int_{\partial \Omega} \log |F^*(\xi)|d\omega(\xi, z), \tag{4.2}
\]

for all \( z \in \Omega \). If equality holds in (4.2) at all points \( z \in \Omega \) (equivalently, for at least one \( z \in \Omega \)), we call \( F \) an outer function in \( MH^1(\Omega) \).

Through the process of lifting and factoring in \( H^p(D) \) we can factor \( H^p(\Omega) \)-functions using a (modulus holomorphic) Blaschke product, singular inner function and an outer function on \( \Omega \). Let us pick a function \( f \in H^p(\Omega) \) and let \( g = f \circ \pi \in H^p(D) \) be its lift. In \( H^p(D) \) we factor \( g \) as \( g = B_1 \cdot S_1 \cdot F_1 \), where \( B_1 \) is a Blaschke product, \( S_1 \) a singular inner function, and \( F_1 \) an outer function. While in general the functions \( B_1, S_1 \) and \( F_1 \) need not be \( G \)-invariant (where \( G \) is the group of automorphisms of \( \pi \)), it is not difficult to see that the functions \( |B_1|, |S_1| \) and \( |F_1| \) are \( G \)-invariant. Thus there exists functions \( |B|, |S| \) and \( |F| \) on \( \Omega \) with lifts \( |B_1|, |S_1| \) and \( |F_1| \) respectively. It is claimed that the former are modulus holomorphic functions on \( \Omega \). In fact, as one would expect, they are a Blaschke product, a singular inner function and an outer function, respectively. These are the contents of the following theorem.

**Theorem 4.2** ([64]). Let \( f \in H^p(\Omega) \) and suppose \( f \) is not identically zero. Then there exist a Blaschke product \( B \in MH^{\infty}(\Omega) \), a singular inner function \( S \in MH^{\infty}(\Omega) \) and an outer function \( F \in MH^p(\Omega) \) such that

\[
|f(z)| = |B(z)| \cdot |S(z)| \cdot |F(z)|
\]

for all \( z \in \Omega \). This factorization is unique in the following sense: if \( B_1, S_1 \) and \( F_1 \) are a Blaschke product, a singular inner function and an outer function on \( \Omega \) for which \( |f(z)| = |B_1(z)| \cdot |S_1(z)| \cdot |F_1(z)| \), then \( |B| = |B_1|, |S| = |S_1| \) and \( |F| = |F_1| \).

### 4.3 Extreme points in \( H^1(\Omega) \)

The question arises which functions are extreme in the unit ball of \( H^1(\Omega) \). After the De Leeuw-Rudin theorem (1.10), the following result is elementary:

**Lemma 4.3.** If \( f \in H^1(\Omega) \) is an outer function of unit norm, then \( f \) is extreme in the unit ball of \( H^1(\Omega) \).

Any attempt to copy the proof of the DeLeeuw-Rudin theorem in the other direction will break down. For suppose \( f = I \cdot F \) where \( I \) is a non-trivial inner function (in \( MH^{\infty}(\Omega) \)). Then De Leeuw and Rudin look at the function \( g = (1 + I^2)F \) and show that \( \|f \pm g\|_1 = 1 \). However, unless \( I \) is a single valued inner function, \( 1 + I^2 \) is not
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a well-defined modulus holomorphic function. There is no remedy for this problem, because as we have already mentioned, when $m \geq 2$ there exist extreme points with a non-trivial inner part.

The following theorem of F. Forelli is crucial in understanding which inner functions can appear in the inner-outer factorizations of an extreme points.

**Theorem 4.4 ([18]).** Let $f$ be an extreme point of the unit ball of $H^1(\Omega)$. Then the codimension of the $H^1$-closure of $f \cdot H^\infty$ in $H^1$ is at most $\frac{m}{2}$, when $\Omega$ is bounded by $m + 1$ closed smooth curves.

Because all functions in the $H^1$-closure of $f \cdot H^\infty(\Omega)$ inherit the zeros of $f$, Forelli’s theorem implies the following:

**Corollary 4.5 ([18]).** If $f$ is an extreme point of the unit ball of $H^1(\Omega)$, then the Blaschke factor of $f$ is a finite Blaschke product with at most $\frac{m}{2}$ zeros.

Forelli’s theorem also expresses that an extreme point in the unit ball of $H^1(\Omega)$ has a trivial (constant) singular inner factor: Suppose $S$ is a non-trivial singular inner function in $MH^\infty(\Omega)$. Let $S^{1/2}$ be the singular inner function in $MH^\infty(\Omega)$ defined by $|S^{1/2}| := |S|^{1/2}$. There exists an outer function $H$ that is invertible in $MH^\infty(\Omega)$ and a function $h \in H^\infty(\Omega)$ such that $|S|/|HS^{1/2}| = |S^{1/2}|/|H| = |h|$. (Start with a smooth real-valued function $u$ on $\partial \Omega$ and extend $u$ harmonically to $\Omega$. One may choose $u$ on $\partial \Omega$ in such a way that the periods of the harmonic conjugate $\hat{u}$ of $u$ along each of the boundary curves are the same as the those of the harmonic conjugate of $\frac{1}{2} \log |S|$. Finally, set $H = \exp(u + i\hat{u})$.) If $f = S \cdot F$ for some $F \in MH^1(\Omega)$, then by construction $g = HS^{1/2} \cdot F$ is a well-defined function in $H^1(\Omega)$. Furthermore, the $H^1$-closure of $f \cdot H^\infty$ is contained in the $H^1$-closure of $g \cdot H^\infty$. This inclusion is proper because any function in the closure of $f \cdot H^\infty$ inherits the singular inner factor $S$ of $f$. Repeating the above, we see that the $H^1$-closure of $f \cdot H^\infty$ does not have finite codimension in $H^1$ if $f$ is any $H^1$-function with a non-trivial singular inner factor.

**Corollary 4.6 ([18]).** If $f$ is an extreme point of the unit ball of $H^1(\Omega)$, then the inner part of $f$ is a finite Blaschke product with at most $\frac{m}{2}$ zeros.

Inspection of the proof of the De Leeuw-Rudin theorem gives us the following criterion for extremity (where we identify $H^1$-functions with their boundary values):

**Lemma 4.7.** Let $f \in H^1(\Omega)$ be of unit norm. Then $f$ is not extreme if and only if there exists a non-constant real function $k \in L^\infty(\partial \Omega)$ for which $kf \in H^1(\Omega)$.

Suppose $f = I \cdot F$ is of unit norm where $I$ is a finite Blaschke product, and $F$ outer. Let us suppose that $f$ is not an extreme point of the unit ball of $H^1(\Omega)$. Then let $k$ be as in the lemma, and let $g \in H^1(\Omega)$ have boundary values $kf$. Because $F$ is an outer function, for all $z \in \Omega$: $|g(z)| \leq \|k\|_\infty \cdot |F(z)|$. Hence, because also $|I| = 1$ everywhere on $\partial \Omega$, the meromorphic function $h = g/f$ is bounded near $\partial \Omega$, real-valued (a.e.) on $\partial \Omega$, and has its poles in the zeros of $f$ (with corresponding multiplicities). Conversely, if $h$ is a meromorphic function on $\Omega$ with these three properties, then with $k := h$ on $\partial \Omega$ and $g := hf \in H^1(\Omega)$, we have $g = kf$ on $\partial \Omega$, so by the previous lemma, $f$ is not an extreme point. We come to the following definition.
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**Definition.** Let $I$ be a finite Blaschke product. We say that $I$ is an extremal Blaschke product if there exists no meromorphic function $h$ on $\Omega$ that is bounded near $\partial \Omega$, real-valued on $\partial \Omega$ and has its poles in the zeros of $I$, with no greater multiplicity than the zeros of $I$.

The conclusions of the previous paragraphs may thus be summarized as follows:

**Proposition 4.8.** Let $f \in H^1(\Omega)$ be of unit norm. Then $f$ is an extreme point of the unit ball of $H^1(\Omega)$ if and only if the inner part of $f$ is an extremal Blaschke product.

We wish to stress that Forelli’s theorem also gives us an upper bound for the number of zeros of an extremal Blaschke product on $\Omega$. Also, by the previous proposition we see that it is only the location of the zeros of a function in $H^1(\Omega)$ and not so much the outer factor that decides whether or not the function is extreme in the unit ball (after normalization).

The problem of determining the extreme points of $H^1(\Omega)$ has thus been reduced to a problem on meromorphic functions on $\Omega$ with predescribed poles, that is: a problem concerning meromorphic divisors on $\Omega$.

Let $I$ be a finite Blaschke product with zeros $z_1, z_2, \ldots, z_n$ repeated according to multiplicity. Thus $I$ has $n$ zeros on $\Omega$. Let $\delta := 1 \cdot z_1 + 1 \cdot z_2 + \cdots + 1 \cdot z_n$ be the divisor on $\Omega$ associated with $I$. If $\delta' = \sum_{z \in \Omega} d'(z) \cdot z$ is another divisor on $\Omega$ we say that $\delta' \geq \delta$ if at every $z \in \Omega$: $\delta'(z) \geq \delta(z)$. The space of all meromorphic differentials $\omega$ on $\Omega$ that are real-valued on $\partial \Omega$ and for which the associated divisor $(\omega)$ satisfies $(\omega) \geq \delta$ is a real linear space of dimension $MD(\delta)$. Using a theorem of H.L. Royden [51], based on the Riemann-Roch theorem, T.W. Gamelin & M. Voichich proved the following result:

**Theorem 4.9 ([19]).** The Blaschke product $I$ is extremal if and only if $MD(\delta) + 2n = m$.

In particular, using only the fact that an extreme point has a finite Blaschke product as its inner factor (as shown by Forelli), Gamelin & Voichich also arrived at Forelli’s upper bound $(\frac{m}{2})$ for the number of zeros of an extremal Blaschke product. They proved that this upper bound is also sharp.

**Theorem 4.10 ([19]).** The $H^1$-closure of the set of extreme points in the unit ball of $H^1(\Omega)$ is the collection of all functions in $H^1(\Omega)$ that have unit norm and no more than $\frac{m}{2}$ zeros.

There is a special type of finite domains where the zero sets of extreme points can be described explicitly, namely the so called real slit domains.

**Definition.** Let $\mathbb{C}^*$ be the extended complex plane. Any domain $\mathcal{R}$ of form $\mathbb{C}^* \setminus (I_1 \cup \cdots \cup I_{m+1})$, where $I_1, I_2, \ldots, I_{m+1}$ are disjoint, bounded and closed intervals in $\mathbb{R}$ is called a real slit domain.

**Theorem 4.11 ([19]).** Let $\mathcal{R}$ be a real slit domain and let $z_1, z_2, \ldots, z_n$ be points of $\mathcal{R}$ (not necessarily distinct). Then the Blaschke product with zero set $z_1, z_2, \ldots, z_n$ is extremal if and only if:
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- \( n \leq \frac{m}{2} \) and
- for all \( i, j \): \( z_i \neq \overline{z_j} \). (In particular, none of the \( z_i \) is real.)

**Proof.** Because the meromorphic function \( \frac{1}{z-z_i} + \frac{1}{z-z_j} \) is bounded and real-valued on \( \partial \mathcal{R} \subset \mathbb{R} \), we see that no zeros of an extremal Blaschke product are conjugated. Conversely, if the sequence \( z_1, z_2, \ldots, z_n \) satisfies the two conditions, the proof of Lemma 4.14 below readily shows that the function \( f(z) = c(z-z_1)(z-z_2) \cdots (z-z_n) \) is exposed, hence the Blaschke product with zero set \( z_1, z_2, \ldots, z_n \) is extremal. \( \square \)

4.4 Strong exposedness and the location of zeros

In this section we investigate exposed and strongly exposed points in the unit ball of \( H^1(\Omega) \). We give several examples and criteria for (strongly) exposed points. Also we show that the non-trivial properties of strongly exposed points in \( H^1(D) \) (for example: \( L^1 \)-invertibility on the boundary) do not translate to finite domains. Finally, we again look at the zero sets of extreme points and the question of divisibility of extreme functions by functions of the form \( (1+u)^2 \), where \( u \) is a non-constant inner function.

Using the Hahn-Banach theorem it is again easy to show that if \( f \) is an exposed point of the unit ball of \( H^1(\Omega) \), then the exposing functional \( L \) for \( f \) is unique and given by:

\[
L : g \in H^1 \mapsto \int_{\partial \Omega} g \overline{\frac{f}{|f|}} \, d\sigma.
\]

Hence, like \( H^1(D) \), a function \( f \) in the boundary of the unit ball of \( H^1(\Omega) \) is exposed if and only if it is rigid: apart from (positive) constant multiples of \( f \) there is no \( H^1 \)-function with the same argument a.e. \( (d\sigma) \) on \( \partial \Omega \).

We recall the following criteria for rigidity of \( H^1(D) \)-functions which carry over to finite domains word for word:

- If \( f \in H^1 \) and \( 1/f \in H^1 \), then \( f \) is rigid (Theorem 3.5).
- If there is a \( g \in H^\infty \) such that \( \text{Re}(fg) > 0 \) a.e. on \( \partial \Omega \), then \( f \) is rigid (Theorem 3.6).
- If \( u \) is a non-constant inner function such that \( f/(1+u)^2 \) is in \( H^1 \), then \( f \) is not rigid (or \( f \equiv 0 \)).

A priori the first two conditions can only be used to demonstrate rigidity of outer functions. In both cases \( |f| \) cannot be too small near the boundary of \( \Omega \): if \( f \) satisfies the second condition, then \( 1/f \in H^{1-\varepsilon}(\partial \Omega) \) for all \( \varepsilon > 0 \), so \( 1/f \) is "nearly" in \( H^1 \). We will extend the first condition to allow for exposed points with zeros on \( \Omega \).

**Proposition 4.12 ([5]).** If \( f \) is extreme in \( H^1(\Omega) \) and \( 1/|f| \in L^1(\partial \Omega) \) then \( f \) is exposed.

**Proof.** We remark that \( 1/|f| \in L^1 \) occurs precisely when the outer factor \( F \) of \( f \) is invertible in \( MH^1(\Omega) \). Suppose \( g \in H^1 \) has the same argument as \( f \) a.e. on \( \partial \Omega \).
Let $M$ be the meromorphic function $g/f$. Near $\partial \Omega$, away from the zeros of $f$ that is, $|M|^{1/2}$ admits a harmonic majorant, and the boundary values of $M$ are positive a.e. on $\partial \Omega$. By a local version of Lemma 3.4 (the result that a $H^{1/2}$-function extends holomorphically across any subarc of $T$ where it is positive; cf. [31],[20], page 95, exercise 2.13), applied to $M$ and subarcs of each of the boundary curves of $\Omega$ one obtains that $M$ extends holomorphically across the boundary of $\Omega$. In particular, $M$ is bounded near $\partial \Omega$. Because the inner part of $f$ is extremal it follows that $M$ is a positive constant. In other words, $g$ is a constant multiple of $f$ so $f$ is rigid. 

The following theorem can be proved by a minor adaptation of the proofs of Theorems 3.7 and 3.8.

**Theorem 4.13.** Let $f$ be a function in $H^1(\Omega)$. Then $f$ is strongly exposed in the unit ball of $H^1$ if and only if $f$ is exposed and $L^\infty - \text{dist}(f/|f|, H^\infty + C(\partial \Omega)) < 1$.

Below we will use this result only for functions $f$ for which $f/|f|$ is continuous, in which case the proof of strong exposedness can be simplified using the fact that $H^1(\Omega)$ has a predual of continuous functions on $\Omega$ ([2],[16],[63]; compare with Corollary 3.12).

Throughout the remainder of this chapter, the domain $R$ will be the unit disc $D$ with $m$ disjoint closed intervals $I_1, I_2, \ldots, I_m \subset (-1, 1)$ deleted. For convenience we assume that the origin is contained in $R$. The finite domain $R$ is conformally equivalent to a real slit domain. Let us recall that a Blaschke product with zero set $z_1, z_2, \ldots, z_n$ in $R$ is extremal if and only if $n \leq \frac{3}{2}$ and $z_i \neq \overline{z_j}$ for all $i, j$ (4.11).

Note that on $T \setminus \{i\}$ the function $(z + i)^2$ has the same argument as $iz$ so $(z + i)^2$ is not rigid in $H^1(D)$.

**Lemma 4.14 ([5]).** For all $m \geq 2$, the normalized function $f(z) = c(z + i)^2$ is strongly exposed in the unit ball of $H^1(R)$.

**Proof.** Suppose $g \in H^1$ has the same argument as $f$ a.e. on $\partial R$, and set $h = g/f$. We define $h(z) = \overline{h(1/\overline{z})}$ for $z \in C^*$ (the extended complex plane) and $1/\overline{z} \in R$. Because $h^* > 0$ a.e. on $T$ and because $1/f$ is locally bounded on $T \setminus \{-i\}$, Schwarz reflection across $T$ gives that $h$ extends holomorphically across $T \setminus \{-i\}$. Thus we obtain a holomorphic function $h$ on $C \setminus \{-i\}$ with $2m$ slits in $\mathbb{R}$ deleted, that is bounded at infinity. We will refer to this domain as $R_2$. Note that $g = hf$ is holomorphic on $R_2$; actually, $g$ is regular at $-i$ as well. Recall that a.e. on $T$, $(z + i)^2$ and $iz$ have the same argument. By the above reasoning $g(z)/iz$ extends across all of $T$, so the same applies to $g$. We conclude that $h$ has a pole of order 2 or less at $-i$.

Next we will show that $h$ extends across the $2m$ slits. Pick one of these and extend it to infinity from one side. Applying a square root, the resulting domain can be mapped biholomorphically onto the right half plane with a finite number of slits in $(0, \infty)$ deleted. Under composition with this nice biholomorphic map, $h$ goes over into, say, $h_2$, a function that is in $H^1$ near the imaginary axis. Almost everywhere on the image of the (half-open) slit we picked out, $h_2$ is positive. By reflection, $h_2$ extends holomorphically across this open arc in the imaginary axis. Going back to $R_2$, and varying over the slits, $h$ is easily seen to be continuous at all endpoints of
the $2m$ slits, as well as on the interior of the slits, in the sense that

$$h_+(x) = \lim_{z \to x, 3z > 0} h(z), \quad h_-(x) = \lim_{z \to x, 3z < 0} h(z)$$

are well-defined continuous functions on $\mathbb{R}$, that are nonnegative on the slits.

A priori, it is not obvious why on the interior of a slit $h_+(x) = h_-(x)$ should hold. (Indeed, the function $f$ is not exposed when $m = 1$; see also Theorem 4.15 below.)

Let $k(z) = (h(z) - \overline{h(z)})^2$. Then $k$ is meromorphic on $\mathcal{R}_2$, bounded at infinity, continuous on $\mathbb{R}$, and its only singularities are poles found in $\pm i$. By Morera’s theorem, $k$ is meromorphic on $\mathbb{C}$, and for some polynomial $p$ of degree 8 or less: $k(z) = p(z)/(z^2 + 1)^4$. But $k$ has zeros at each of the $4m$ endpoints of the slits and (double) zeros at $\pm 1$, so $k$ vanishes everywhere.

This shows in particular that $h_+ = h_-$, and another application of Morera’s theorem yields that $h$ is meromorphic on $\mathbb{C}$, bounded at infinity, and it can only have a pole at $z = -i$. By the relation $h(z) = \overline{h(\overline{z})}$, $h$ is regular at $z = -i$, so $h$ is a constant $C$.

It follows that $g = Cf$: $f$ is exposed.

Finally, it is easy to see that $\bar{f}/|f|$ is continuous on $\partial \Omega$, so by Theorem 4.13, $f$ is strongly exposed.

\[\square\]

**Remark.** The technique of the proof of the Lemma readily shows that if $m > k + 1$, then the normalized function $f_{2k}(z) = c(z + i)^{2k}$ is strongly exposed in the unit ball of $H^1(\mathcal{R})$. In particular we see that there exist strongly exposed points in the unit ball of $H^1(\mathcal{R})$ that are “small” on the boundary: $1/|f_{2k}| \notin L^{1/2k}(\partial \mathcal{R})$.

We recall that for $f = I \cdot F$ to be an extreme point the only requirement is that the inner part $I$ of $f$ is an extremal Blaschke product – a generic zero set; “most” of the properties of the function $f$ then follow from its outer factor, i.e., the size of $|f|$ on the boundary $\partial \Omega$. It is reasonable to ask whether exposedness is also essentially a property of the outer factor. We make this question precise in the following sense: if $f \in H^1$ is a rigid outer function, $I$ is an extremal Blaschke product on $\Omega$, and $g$ is invertible in $MH^{\infty}(\Omega)$ and such that $Ig \in H^{\infty}(\Omega)$ (a single valued function), is the extreme point $Ig \cdot f/|Ig \cdot f|$ also exposed? (Compare with the first example in Section 3.1.) Proposition 4.12 tells us the answer is yes if $1/f \in L^1(\partial \Omega)$. However, we will demonstrate that in general the answer is no.

Next, we recall that if $f$ is divisible in $H^1$ by $(1 + u)^2$ for some non-constant inner function $u$, then $f$ is not exposed. As we explained Section 3.1, Inoue [29] proved that the converse of this statement does not generally hold for $H^1(D)$-functions. Surprisingly, for finite domains, the existence of extreme points with zeros leads to (another) reason for failure of sufficiency of this criterion for exposedness.

We combine these two results in the following theorem.

**Theorem 4.15 ([5]).** For $m = 3$, there exists $\xi \in \mathcal{R}$ such that the function

$$f(z) = c(z - \xi)(z + i)^2$$

is extreme in the unit ball of $H^1(\mathcal{R})$, but not exposed. For all non-constant inner functions $u$ on $\mathcal{R}$, $f/(1 + u)^2$ is not contained in $H^1(\Omega)$. 
Proof. We will explain how to construct non-trivial $H^1(\mathcal{R})$-functions with the same argument as $f$ a.e. on $\partial \mathcal{R}$, for suitable $\xi \in \mathcal{R} \setminus \mathbb{R}$.

For the moment, assume we have such a function $g$, and let $h(z) = g(z)/f(z)$. Suppose the three slits are the intervals $[x_1, y_1], [x_2, y_2], [x_3, y_3]$. By the arguments in the proof of Lemma 4.14, $h$ extends meromorphically to $\mathbb{C}$ minus 6 slits in $\mathbb{R}$, its only singularities being poles of order 1 (or less) in $\xi$ and $1/\xi$, and a pole of order 2 (or less) in $-i$. Again the functions $h_+, h_-$ are continuous on $\mathbb{R}$ and coincide on $\mathbb{R} \cap \mathbb{R}$. Let $k(z) = (h(z) - \overline{h(z)})^2$. As before, $k$ extends holomorphically across the slits, and is therefore meromorphic on $\mathbb{C}$, bounded at infinity, and its only singularities are poles of order 2 (or less) in $\xi, \overline{\xi}, 1/\xi, 1/\overline{\xi}$, and poles of order 4 (or less) in $\pm i$; $k$ has zeros at the 12 points $x_i^{\pm 1}, y_i^{\pm 1}$, and double zeros at $\pm 1$. Hence, for some constant $A \leq 0$:

$$k(z) = \frac{A(z-x_1)(z-y_1) \cdots (z-1/x_3)(z-1/y_3)(1-z^2)^2}{(1+z^2)^4(z-\xi)^2(z-\overline{\xi})^2(1-\xi z)^2(1-\overline{\xi}z)^2}.$$ 

In our approach $A$ will be non-zero, and we may then assume without loss of generality that $A = -1$.

Define

$$q(z) = \pm i(1-z^2)\sqrt{\frac{z-x_1}{z-y_1}} \cdots \sqrt{\frac{z-1/x_3}{z-1/y_3}}(z-y_1) \cdots (z-1/y_3) \in H^\infty(\mathcal{R}),$$

where the sign is to be chosen to make $q(i) < 0$. The function $h$ will now satisfy the relation

$$h(z) - \overline{h(z)} = \frac{q(z)}{(1+z^2)^2(z-\xi)(z-\overline{\xi})(1-\xi z)(1-\overline{\xi}z)}. \quad (4.3)$$

Because for all $z \in \mathbb{T}$ $(z-\xi)(1-\overline{\xi}z)/z > 0$, the right hand side of $(4.3)$ is (indeed) negative on the upper half of the unit circle, and positive on the lower half of the unit circle, minus the point $-i$.

Similar considerations will show that the function $h(z) + \overline{h(z)}$ is meromorphic on $\mathbb{C}$, its only singularities being poles in $\pm i, \xi, \overline{\xi}, 1/\xi, 1/\overline{\xi}$. There exists a polynomial $p(z)$ of degree 8 (or less) for which

$$h(z) + \overline{h(z)} = \frac{p(z)}{(1+z^2)^2(z-\xi)(z-\overline{\xi})(1-\xi z)(1-\overline{\xi}z)}.$$  

Because $p$ is symmetrical with respect to the unit circle and the real axis, we find $\alpha, \beta \in \overline{\mathbb{D}}$, and a positive constant $C$ such that

$$p(z) = C(z-\alpha)(z-\overline{\alpha})(1-\overline{\alpha}z)(1-\alpha z)(z-\beta)(z-\overline{\beta})(1-\beta z)(1-\overline{\beta}z). \quad (4.4)$$

Therefore,

$$2h(z) = \frac{p(z) + q(z)}{(1+z^2)^2(z-\xi)(z-\overline{\xi})(1-\xi z)(1-\overline{\xi}z)} \quad (4.5)$$

$$= \frac{1}{(z-\xi)(z+i)^2} \left[ \frac{p(z) + q(z)}{(z-\xi)(z-\xi)^2} \cdot \frac{1}{(1-\xi z)(1-\xi z)} \right]. \quad (4.6)$$
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We are nearly done if we can find $\alpha, \beta, \xi$ for which the term in square brackets is in $H^1(\mathbb{R})$. For this we need to ensure that $p(z) + q(z)$ has a double zero at $i$, and next, a zero $(\bar{\xi})$ on $\mathbb{R} \setminus \mathbb{R}$, implying our choice of $\xi$.

The two equations

\[
p(i) + q(i) = 0 \quad (4.7) \\
p'(i) + q'(i) = 0 \quad (4.8)
\]

are reduced to the single condition

\[
\frac{p'(i)}{p(i)} = \frac{q'(i)}{q(i)}, \quad (4.9)
\]

by choosing $C > 0$ appropriately (recall that $q(i) < 0$).

Now $p(z)/q(z)$ is negative on the upper half of the unit circle, so $(p(z)/q(z))'(i)$ is real. In other words, $(p'(i)/p(i)) - (q'(i)/q(i)) \in \mathbb{R}$, which means that the equation for the imaginary part of condition (4.9) is trivially satisfied ($\Im(p'(i)/p(i)) = \Im(q'(i)/q(i)) = -4i$).

Explicit calculation shows that

\[
\Re\frac{p'(i)}{p(i)} = \frac{\Re\alpha(1 + |\alpha|^2)}{(1 - |\alpha|^2)^2 + 4(\Re\alpha)^2} + \frac{\Re\beta(1 + |\beta|^2)}{(1 - |\beta|^2)^2 + 4(\Re\beta)^2}.
\]

It is easy to see that $\alpha \mapsto \Re\alpha(1 + |\alpha|^2)/((1 - |\alpha|^2)^2 + 4(\Re\alpha)^2)$ maps $D$ onto $\mathbb{R}$. We can therefore choose $\beta$ arbitrarily, and find $\alpha \in D$ such that (4.9) holds. Now fix any such $\alpha, \beta$.

So far we have established that the function

\[
\frac{p(z) + q(z)}{(1 + z^2)(z - \xi)(z - \bar{\xi})(1 - iz)(1 + iz)}
\]

is regular at $z = i$, real-valued on $\mathbb{T} \setminus \{-i\}$, positive on the lower half of the unit circle minus $-i$, and bounded below on $\partial \mathcal{R} \cap \{3z \geq 0\}$.

Thus, for large $K > 0$, the function

\[
\frac{p(z) + q(z) + Kz^2(1 + z^2)^2}{(1 + z^2)(z - \xi)(z - \bar{\xi})(1 - iz)(1 + iz)}
\]

will be strictly positive on $\partial \mathcal{R} \setminus \{-i\}$, and regular at $z = i$. (Of course, the function $p(z) + Kz^2(1 + z^2)^2$ is again of the form (4.4), with (apparently) better suited parameters $\alpha, \beta, C$ than those we have chosen, while condition (4.9) is still satisfied.)

By the argument principle applied to the function

\[
\frac{p(z) + q(z) + Kz^2(1 + z^2)^2}{z^4},
\]

that is positive on $\partial \mathcal{R} \setminus \{i\}$, with a double zero at $i$, we conclude that $p(z) + q(z) + Kz^2(1 + z^2)^2$ has three zeros on $\mathcal{R}$. Pick any of these, and call its conjugate $\bar{\xi}$. We claim that $\bar{\xi} \notin \mathbb{R}$. For $p(z) + Kz^2(1 + z^2)^2$ is nonnegative on $\mathbb{R}$, and $q(z) \in i(\mathbb{R} \setminus \{0\})$ on $\mathcal{R} \cap \mathbb{R}$. 

Let $$g(z) = \frac{p(z) + q(z) + Kz^2(1 + z^2)^2}{(z - \xi)(z - i)^2} \cdot \frac{1}{(1 - \xi z)(1 - \xi z)} \in H^\infty(\mathcal{R}).$$

The functions $f$ and $g$ have the same argument on $\mathcal{R} \setminus \{-i\}$, which implies that $f$ is not rigid. Because the inner part of $f$ is the extremal Blaschke product with zero at $\xi$, $f$ is extreme, however.

By Lemma 4.14, $(z + i)^2$ is rigid. We conclude that if $u$ is any non-constant inner function on $\mathcal{R}$, $(z + i)^2/(1 + u(z))^2 \not\in H^1(\mathcal{R})$, hence also $(z - \xi)(z + i)^2/(1 + u(z))^2 \not\in H^1(\mathcal{R})$. This concludes the proof. \qed