Strongly exposed points in unit balls of Banach spaces of holomorphic functions
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Chapter 5

Strongly exposed points in Bergman space

In this chapter, rather than the natural generalization to the Hardy space $H^1$ of the unit ball in $\mathbb{C}^n$, we study the strongly exposed points in the Bergman space $A^1$ of the unit disc $D$. At the same time, however, one can in fact embed the Bergman space as a relatively small subspace of $H^1$ of the unit ball of $\mathbb{C}^2$. However, we attempt to frame our questions in the Bergman space as much as possible. The results in this chapter are largely based on [6].

5.1 The Bergman space $A^1$

In this section we introduce Bergman spaces and discuss their basic properties. Our treatment is quite elementary. As such, it may be skipped by anyone familiar with Bergman spaces. For a more thorough overview of Bergman space theory we refer the reader to [23] and [12].

**Definition.** Let $0 < p < \infty$. The (unweighted) **Bergman space** $A^p = A^p(D)$ consists of all (area-integrable) holomorphic functions $f$ for which

$$\|f\|_{A^p} := \left( \int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = \frac{1}{\pi} dx dy$ denotes normalized Lebesgue area measure on $D$. When no confusion is possible we shall write $\|f\|_p$ instead of $\|f\|_{A^p}$. When $1 \leq p < \infty$, the Bergman space $A^p$ is a Banach space under the norm $\|\cdot\|_p$. That is to say, with the inherited $L^p$-norm, the space $A^p$ is closed in $L^p(D, dA(z))$ (cf. [23], Proposition 1.2).

We briefly mention that certain weighted Bergman spaces are also of special interest. Suppose $\alpha > 0$. The measure $dA_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha dA(z)$ is normalized on the unit disc. For $0 < p < \infty$, the weighted Bergman space $A^p_\alpha$ is the set of all holomorphic
functions \( f \) on \( D \) for which
\[
\|f\|_p := \left( \int_D |f(z)|^p dA(z) \right)^{1/p} < \infty.
\]

It is easy to see that for every \( 0 < p < \infty \), the Hardy space \( H^p \) is contained in the Bergman space \( A^p \) and that \( \|f\|_{A^p} \leq \|f\|_{H^p} \) for all \( f \in H^p \). Thus the spaces \( A^p \) have a sufficiently rich structure to make them suited for our purposes.

Next, we will point out some similarities between the Hardy spaces and the Bergman spaces.

**Proposition 5.1.** Let \( 0 < p < \infty \) and let \( z \in D \) be arbitrary. Then for all \( f \in A^p \),
\[
|f(z)| \leq \frac{\|f\|_p}{(1 - |z|)^{\frac{2}{p}}}.
\]

In particular, the map \( f \in A^p \mapsto f(z) \) is bounded.

**Proof.** Let \( r = 1 - |z| > 0 \). Since \( |f|^p \) is subharmonic,
\[
(1 - |z|)^2 \cdot |f(z)|^p \leq \int_{B(z,r)} |f(w)|^p dA(w) \leq \|f\|_p.
\]

The estimate now follows. \( \square \)

As before, for \( f \in H(D) \), \( f_r(z) \) is the function \( f(rz) \) on \( D \), where \( 0 < r < 1 \).

**Proposition 5.2.** Let \( 0 < p < \infty \) and suppose \( f \in A^p \). Then the functions \( f_r \) converge to \( f \) in the Bergman space \( A^p \) as \( r \uparrow 1 \).

**Proof.** This result is actually much easier to prove in Bergman spaces than in Hardy spaces. Given \( \varepsilon > 0 \), one finds a number \( \eta \) close to 1 such that \( \int_{|z|<\eta} |f(z)|^p dA(z) < \varepsilon \). Integrating over circles and using that \( |f|^p \) is subharmonic, we obtain that for all \( 0 < r < 1 \) also,
\[
\int_{|z|<1} |f_r(z)|^p dA(z) \leq \int_{|z|<\eta} |f(z)|^p dA(z) < \varepsilon.
\]

Consequently,
\[
\|f - f_r\|_p^p < 2^{p+1} \varepsilon + \int_{|z|<\eta} |f(z) - f_r(z)|^p dA(z).
\]

Because the functions \( f_r \) converge uniformly to \( f \) on the disc \( |z| \leq \eta \) as \( r \uparrow 1 \), the last integral can be made arbitrarily small by choosing \( r \) close enough to 1. This finishes the proof. \( \square \)

It follows from the Proposition 5.2 that the polynomials are dense in the Bergman space \( A^p \). (Indeed, the Bergman space \( A^p \) may also be defined as the \( L^p(D, dA(z)) \)-closure of the polynomials.) Also, one can strengthen Proposition 5.1 in the sense that for all \( f \in A^p \),
\[
f(z) = o\left( \frac{1}{(1 - |z|)^{\frac{2}{p}}} \right).
\]
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as $|z| \to 1$. Conversely, if $f \in H(D)$ satisfies the estimate $|f(z)| \leq \frac{C}{(1-|z|)^p}$ for some $C > 0, p > 0$ and all $z \in D$, then $f$ is contained in the Bergman space $A^{p'}$ for all $p' < \frac{p}{2}$. Thus $\cup_{p>0} A^p$ consists of all holomorphic functions $f$ on $D$ of boundary growth at most $\frac{1}{(1-|z|)^\beta}$, for some $\beta > 0$. Nevertheless, despite their moderate growth towards the boundary, functions in the Bergman spaces sometimes exhibit a striking difference with functions in Hardy spaces:

**Theorem 5.3.** There exists a function $f$ in $\cap_{p>0} A^p$ for which at all $\xi \in \mathbb{T}$ the radial limit $\lim_{r \uparrow \xi} f(r\xi)$ fails to exist.

The Bergman space $A^2$ is a Hilbert space under the $L^2(D, dA(z))$-inner product $\langle f, g \rangle = \int_D fg \, dA$. By Proposition 5.1, the functional $L_z : f \in A^2 \mapsto f(z)$ is continuous for every $z \in D$. By the Riesz representation theorem there exists a function $k_z(.) \in A^2$ (the so-called Bergman kernel) that represents the action of $L_z$: $L_z(f) = \langle f(.), k_z(.) \rangle$. (Thus $A^2$ is a reproducing kernel Hilbert space.) It is not difficult to calculate $k_z(w)$ explicitly, because the functions $(w^n)_{n=0}^\infty$ form an orthogonal basis for $A^2$. Namely:

$$k_z(w) = \sum_{n=0}^\infty \frac{\langle k_z(w), w^n \rangle}{\langle w^n, w^n \rangle} \cdot w^n = \sum_{n=0}^\infty (n+1)\overline{z}^n w^n = \frac{1}{(1-\overline{z}w)^2}.$$

Let $P : L^2 \to A^2$ be the orthogonal projection of $L^2(D, dA(z))$ onto $A^2$. We shall refer to $P$ as the Bergman projection. We conclude that for $f \in L^2$ and $z \in D$:

$$Pf(z) = \langle f, k_z \rangle = \int_D \frac{f(w)}{(1-\overline{z}w)^2} \, dA(w) \quad (5.1)$$

$$= \sum_{n=0}^\infty ((n+1) \cdot \int_D f(w)\overline{w}^n \, dA(w)) \, z^n. \quad (5.2)$$

Immediately we see that the Bergman projection can be extended to $L^1(D)$ by means of the integral formula (5.1). As such, it reproduces $A^1$. The Bergman projection plays an important role in our study of strongly exposed points in the Bergman space $A^1$.

5.2 A criterion for strong exposedness in $A^1$

In this section we derive a criterion for strong exposedness in $A^1$ that resembles the one we have encountered previously in Hardy space. We then use it to study which polynomials are strongly exposed in the Bergman space.

**Proposition 5.4.** In the Bergman space $A^1$ all elements of unit norm are exposed in the unit ball.

**Proof.** It is not difficult to show that all elements of unit norm are extreme in the unit ball of $A^1$. The proposition then follows from Lemma 1.2. Nevertheless, we will give a direct proof. Let $f \in A^1$ be of unit norm. Let $L$ be the functional

$$L : g \in A^1 \mapsto \int_D g \overline{f} \, dA.$$
We claim that \( L \) exposes the function \( f \) in the unit ball. For suppose \( g \in \text{Ball}(A^1) \) is such that \( L(g) = 1 \). It follows that almost everywhere on \( D \): \( g \overline{f}/|f| = |g| \), so the meromorphic function \( g/f \) is positive (almost everywhere) on \( D \), hence a constant. Because also \( |g| = ||f|| = 1 \), \( g \) must equal \( f \).

The proof of Proposition 5.4 that we gave accounts to the trivial observation that holomorphic functions on \( D \) are uniquely determined by their argument on \( D \) up to multiplication by constants. Incidentally, one can use the Hahn-Banach theorem to see that the exposing functional \( L \) for \( f \) is unique.

Now let

\[ (A^1)^\perp = \{ \psi \in L^\infty : \int_D f \overline{\psi} dA = 0 \text{ for all } f \in A^1 \} \]

denote the annihilator of \( A^1 \) contained in \( L^\infty \). Thus \( (A^1)^\perp = (A^2)^\perp \cap L^\infty \). We are now ready to give an abstract characterization of the strongly exposed points in the unit ball of \( A^1 \).

**Theorem 5.5.** Let \( f \in A^1 \) be of unit norm. Then \( f \) is strongly exposed in \( \text{Ball}(A^1) \) if and only if the \( L^\infty \)-distance of \( f/|f| \) to the space \( (A^1)^\perp + C(\overline{D}) \) is less than one.

**Proof.** We argue as in the proof of Theorem 3.8. Suppose the \( L^\infty \)-distance of \( f/|f| \) to the space \( (A^1)^\perp + C(\overline{D}) \) is 1. We will show that \( f \) is not strongly exposed. Pick a point \( z_0 \) such that \( f(z_0) \neq 0 \). Let \( A^1_{z_0} \) denote the subspace of all Bergman space functions vanishing at \( z_0 \). We let \( L' \) be the restriction of the (exposing) functional \( L : g \in A^1 \mapsto \int_D g \overline{f}/|f| dA \) to \( A^1_{z_0} \). By the Hahn-Banach theorem the operator norm of \( L' \) equals the \( L^\infty \)-distance of \( f/|f| \) to \( (A^1_{z_0})^\perp \). Now if \( \psi \) is any function in \( (A^1_{z_0})^\perp \), then with the choice \( c = \int_D \psi dA \), the function \( \psi(w) - (\overline{c} \overline{z}_0 w)^{-1} \) annihilates all \( A^1 \)-functions. This shows that \( (A^1_{z_0})^\perp \) is contained in \( (A^1)^\perp + C(\overline{D}) \). By our assumption on \( f/|f| \), \( L' \) has operator norm 1. Hence we find a sequence of functions \( f_n \) in the unit ball of \( A^1_{z_0} \), for which \( L(f_n) = L'(f_n) \to 1 \). However, the functions \( f_n \) do not converge to \( f \) in norm, because norm convergence implies pointwise convergence which fails at the point \( z_0 \).

Next we show that the distance condition is sufficient. In [66] it is shown that in the unit ball of \( H^1 \) of the unit ball \( B_n \) of \( \mathbb{C}^n \), a function \( f \) is strongly exposed if and only if \( f \) is exposed and the \( L^\infty(S_n) \)-distance of \( f/|f| \) to \( (H^1)^\perp + C(S_n) \) is less than 1. Here, of course, \( (H^1)^\perp \) is the annihilator of \( H^1(B_n) \) in \( L^\infty(S_n) \). (Observe that \( (H^1(D))^\perp = H^\infty(\overline{D}) \) and compare with Theorem 3.7.) The Bergman space \( A^1 \) is isometrically contained in the Hardy space \( H^1(B_2) \). Namely, consider all holomorphic functions \( F(z,w) \) on \( B_2 \) which depend only on \( z \), that is: \( F(z,w) = F(z,0) \). By Theorem 7.2.4 in [53], the function \( F \) is in \( H^1(B_2) \) if and only if \( f(z) := F(z,0) \) is in \( A^1 \) and the corresponding norms are then the same: \( \int_S |F(z,w)| d\sigma(z,w) = \int_D |F(z,0)| dA(z) \). Similarly, \( (A^1)^\perp \) can be seen as a subspace of \( (H^1)^\perp \) and \( C(\overline{D}) \) as a subspace of \( C(S) \).

Let our function \( f \in A^1 \) correspond with \( F \in H^1(B_2) \). We claim that the function \( F \) is exposed in the unit ball of \( H^1(B_2) \). For this we observe that all slices of the
function $F$ that stay away from the point $(1,0) \in S_2$ yield invertible functions in $A(D)$. Then applying Theorem 3.5 to all of these slices, one arrives at the desired conclusion. Furthermore, because of the inclusion $(A^1)^\perp + C(\overline{D}) \subset (H^1)^\perp + C(S)$, the function $F/|F|$ has $L^\infty$-distance less than one to $(H^1)^\perp + C(S)$, whence the function $F$ is strongly exposed in $H^1$. Thus $F$, or rather $f$, is strongly exposed in $A^1 \subset H^1$. This concludes the proof. 

Henceforth we will simply write $(A^1)^\perp + C$ instead of $(A^1)^\perp + C(\overline{D})$.

The question now is: how can we estimate the distance in $L^\infty$ of $\varphi = f/|f|$ to $(A^1)^\perp + C$, where $f$ is a given function in $A^1$? (Clearly the distance cannot exceed one.) Throughout the remainder of this chapter we will use various techniques to estimate such distances.

Let us first look at polynomials, of a particularly simple form: $f(z) = c(z - \alpha)^n$, where $c$ is normalizing. In Section 5.5 (Corollary 5.22) we will show that strong exposedness of a general polynomial can be reduced to strong exposedness of these special polynomials. We will assume $n \geq 1$ because the constant functions are strongly exposed (Theorem 5.5). We distinguish three cases in order of increasing difficulty: $|\alpha| > 1$, $|\alpha| < 1$ and $|\alpha| = 1$.

The case where $|\alpha| > 1$ is very easy: $f/|f|$ is continuous on $\overline{D}$, so $f$ is strongly exposed. In fact, we may even take non-integer powers $n$ and products of such functions and we always obtain strongly exposed points after normalization.

When $|\alpha| < 1$, the proof that $f$ is strongly exposed is a little more involved. The basic idea is the following. Let us write $\varphi = f/|f|$. If we can show that the Bergman projection $P\varphi$ is continuous on $\overline{D}$ (thus bounded), then $\varphi - P\varphi$ is also bounded, so $\varphi = (\varphi - P\varphi) + P\varphi$ is contained in $(A^1)^\perp + C$ (cf. Lemma 5.15 for another proof of this statement). By Theorem 5.5 then, $f$ is strongly exposed.

Write $\varphi = \psi_0 + \psi_1$, where $\psi_1$ is compactly supported in $D$ and $\varphi \equiv \psi_1$ on a neighborhood of $\alpha$, and $\psi_0$ is smooth on $\overline{D}$. From (5.1) we see that $P\psi_1$ is holomorphic across the unit circle because $\psi_1$ is compactly supported in $D$. Next, because $\psi_0$ is smooth on $\overline{D}$, partial integration in the series expansion (5.2) for $P\psi_0$ gives us that $P\psi_0$ is smooth on $\overline{D}$, thus continuous on $\overline{D}$. This proves that $P\varphi$ is continuous on $\overline{D}$ and we conclude that $f$ is strongly exposed.

Again, our reasoning readily shows that the normalized product $f$ of functions $(z - \alpha_i)^{n_i}$, for all $n_i$ and all $\alpha_i \notin T$, is strongly exposed. One simply writes $\varphi = f/|f| = \psi_0 + \psi_1 + \cdots + \psi_m$, where $\psi_0$ is smooth on $\overline{D}$ and $\psi_1, \psi_2, \ldots$ are compactly supported in $D$ and account for the singularities of $\varphi$ at the zeros of $f$ on $D$; calculation of the Bergman projection then gives $\varphi \in (A^1)^\perp + C$.

The following observation is now immediate:

**Lemma 5.6.** The strongly exposed points in the unit ball of $A^1$ are dense in the boundary of the unit ball.

**Proof.** We recall that the polynomials are dense in the Bergman space. Combined with Proposition 5.2 it follows that the normalized polynomials that are zero free on
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\[T \text{ are dense in the unit ball of } A^1. \text{ By the previous remark, all such polynomials are strongly exposed.} \]

Let us now suppose \(|\alpha| = 1\). For simplicity we may take \(\alpha = 1\). Let us write \(f_n(z) = c_n(1 - z)^n\) and \(\varphi_n = f_n/|f_n|\). The corresponding exposing functional \(L\) for \(f_2\) is given by

\[L(g) = \int_D g(z) \frac{(1 - \overline{z})^2}{|1 - z|^2} dA(z) = \int_D \frac{g(z)}{1 - z} (1 - \overline{z}) dA(z).\]

Integrating first over circles we see that there exist constants \(C_0\) and \(C_1\) (independent of \(g\)) such that \(L(g) = C_0 g(0) + C_1 g'(0)\). There exists a polynomial \(p_2\) such that \(L(g) = \int_D g p_2 dA\). (Alternatively, show that \(p_2 := P g_2\) is a polynomial.) But this says that \(\varphi_2 - p_2\) is contained in the annihilator of \(A^1\), hence that \(\varphi_2 \in (A^1)^\perp + C\) so \(f_2\) is strongly exposed.

Quite similarly one shows that for all even \(n\), \(\varphi_n\) is contained in \((A^1)^\perp + C\) and that \(f_n\) is strongly exposed in \(A^1\). Again, we may introduce non-integer exponents. Let \(f_\beta = c_\beta (1 - z)^\beta\), where \(\beta > -2\) is needed to ensure that \(f_\beta \in A^1\); the constant \(c_\beta > 0\) is chosen such that \(f_\beta\) has norm 1 in \(A^1\). Finally, set \(\varphi_\beta = f_\beta/|f_\beta|\).

**Proposition 5.7.** For all \(\beta > -1\), the \(L^\infty\)-distance of \(\varphi_\beta\) to \((A^1)^\perp + C\) is at most \(|\sin(\pi \beta/2)|\). In particular, for all \(\beta > -1\), \(\beta \neq 1, 3, 5, \ldots\), the function \(f_\beta\) is strongly exposed in the unit ball of \(A^1\).

**Proof.** Of course, there is nothing to prove for odd \(\beta\), so we take \(\beta > -1\) not odd. We will exploit the fact that the functions \(\varphi_0, \varphi_2, \varphi_4, \ldots\) are contained in the space \((A^1)^\perp + C\). We find an integer \(n \geq 0\) such that \(\beta \in (2n - 1, 2n + 1)\). Let \(\theta = |\beta - 2n| < 1\). Because \(\varphi_a + b = \varphi_a \varphi_b\),

\[\|\varphi_\beta - \cos(\pi \theta/2) \varphi_{2n}\|_\infty = \|\varphi_\beta - \cos(\pi \theta/2)\|_\infty = \sup_{|t| < \frac{\pi \beta}{2}} |e^{it} - \cos(\pi \theta/2)| = \sin(\pi \theta/2) = |\sin(\pi \beta/2)|.\]

By Theorem 5.5, \(f_\beta\) is strongly exposed. \[\square\]

We will come back to odd exponents in Section 5.5, once we have obtained a better understanding of the Bergman projection. In light of the previous Lemma and Proposition 5.5 (cf. Lemma 1.2), however, it seems reasonable to ask whether there are any points in the unit ball of \(A^1\) that are not strongly exposed.

**Proposition 5.8.** The normalized function \(f(z) = \frac{e_2 z^2}{(1 - z)^2 \log^2(1 - z)}\) is not strongly exposed in the unit ball of \(A^1\).

**Proof.** Recall the definitions of \(f_\beta\) and \(\varphi_\beta\) for \(\beta > -2\). By construction \(\int_D f_\beta \varphi_\beta dA = 1\) for all \(\beta\). Let \(\varphi_{-2} = \frac{1 - z}{1 - \overline{z}}\). Because \(\|\varphi_\beta - \varphi_{-2}\|_\infty \to 0\) as \(\beta \downarrow -2\), it follows that \(\lim_{\beta \to -2} \int_D f_\beta \varphi_\beta dA = 1\). Let \(\varphi = f/|f| = \varphi_{-2} \cdot \frac{z \log(1 - \overline{z})}{\frac{2}{3} \log(1 - z)}\). Because \(\frac{z \log(1 - \overline{z})}{\frac{2}{3} \log(1 - z)} \to 1\) as \(D \ni z \to 1\), the bounded function \(\varphi - \varphi_{-2}\) is continuous on \(D \setminus \{0\}\) and vanishes
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At \( z = 1 \). Now, the normalizing constants \( c_\beta \) that appear in the definition of the functions \( f_\beta \) make the functions \( f_\beta \) tend to 0 uniformly on \( \overline{D} \setminus B(1, \varepsilon) \) for every \( \varepsilon > 0 \), as \( \beta \downarrow -2 \). Hence, \( \lim_{\beta \downarrow -2} \int_D f_\beta (\overline{\varphi} - \overline{\varphi}_0) \, dA = 0 \), or \( \lim_{\beta \downarrow -2} \int_D f_\beta \overline{\varphi} \, dA = 1 \). Because the functions \( f_\beta \) tend to zero pointwise, they do not converge to \( f \) in norm as \( \beta \downarrow -2 \). We conclude that \( f \) is not strongly exposed. \( \square \)

### 5.3 The Bloch space \( B \)

Recall the Bergman projection \( P : L^2 \to A^2 \),

\[
Pf(z) = \int_D \frac{f(w)}{(1 - z \overline{w})^2} \, dA(w).
\]

We have already used the Bergman projection \( P \) to prove strong exposedness, namely in those cases where \( P \) projects the bounded function \( \varphi = f/|f| \) to a continuous function on \( \overline{D} \). However, a priori we cannot even expect \( P \) to project bounded functions to bounded functions. Clearly we would like to understand better how \( P \) acts on bounded functions. For this we need to discuss the Bloch space.

**Definition.** The Bloch space \( B \) consists of all holomorphic functions \( f \) on \( D \) with the property that \( (1 - |z|^2)|f'(z)| \) is bounded on \( D \). Equipped with the norm

\[
\|f\|_B := |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)|,
\]

\( B \) becomes a Banach space. The set of all functions \( f \) in \( B \) for which the expression \( (1 - |z|^2)|f'(z)| \to 0 \) as \( |z| \to 1 \) is a closed subspace of \( B \), called the *little Bloch space* \( B_0 \).

Let \( C_0 \) denote the continuous functions on \( \overline{D} \) that are zero on \( \partial D \). We have the following theorem of R. Coifman, R. Rochberg and G. Weiss:

**Theorem 5.9 ([10]).** The Bergman projection \( P \) maps \( L^\infty(D) \) boundedly onto \( B \). Furthermore, \( P \) maps both \( C(\overline{D}) \) and \( C_0 \) boundedly onto \( B_0 \).

Before we prove this beautiful result, we recall a well-known lemma.

**Lemma 5.10.** For all \( z \in D \),

\[
\int_D \frac{1}{|1 - z \overline{w}|^3} \, dA(w) \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.
\]

**Proof.** ([53], Proposition 1.4.10) Using the identity

\[
\frac{1}{(1 - z \overline{w})^{\frac{3}{2}}} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{n!} (z \overline{w})^n,
\]

and the orthogonality of the functions \( \overline{w}^n \) \((n = 0, 1, 2, \ldots)\) with respect to the measure \( dA(w) \), one obtains the following equality:

\[
\int_D \frac{1}{|1 - z \overline{w}|^3} \, dA(w) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})^2}{n!(n + 1)!} |z|^{2n}.
\]
One readily checks that the expressions \( \frac{\Gamma(n + \frac{1}{2})^2}{n!(n+1)!} \) are increasing in \( n \) and that their limit is 1 by Stirling's formula:

\[
\lim_{x \to \infty} \frac{\Gamma(x + 1)}{\sqrt{2\pi x}} \left(\frac{x}{e}\right)^x = 1.
\] (5.4)

Thus,

\[
\int_{D} \frac{1}{|1 - zw|^3} dA(w) \leq \frac{4}{\pi} \sum_{n=0}^{\infty} |z|^{2n} = \frac{4}{\pi} \frac{1}{(1 - |z|^2)}.
\]

\[\square\]

**Proof of Theorem 5.9.** (cf. [23], Theorem 1.12)

First take \( \varphi \in L^\infty(D) \) and let \( f = P\varphi \). Differentiating under the integral sign one obtains:

\[
f'(z) = 2 \int_{D} \frac{\overline{w}\varphi(w)}{(1 - zw)^3} dA(w).
\]

From the previous lemma it follows that \( (1 - |z|^2)|f'(z)| \leq \frac{8}{\pi} \|\varphi\|_\infty \). Also, one trivially sees that \( |f(0)| \leq \|\varphi\|_\infty \). We conclude that \( P \) maps \( L^\infty(D) \) boundedly into \( B \).

Next, it is easy to see that if \( \varphi \) is a trigonometric polynomial on \( \overline{D} \), then \( P(\varphi) \) is a polynomial and hence is contained in \( B_0 \). Because the trigonometric polynomials are uniformly dense in \( C(\overline{D}) \) and because \( B_0 \) is closed in \( B \), the boundedness of \( P \) implies that \( P \) maps \( C(\overline{D}) \) into \( B_0 \).

We finish the proof by demonstrating the surjectivity. Suppose \( f \) is contained in \( B \). It will be convenient to assume \( f(0) = f'(0) = 0 \) which is no restriction because the polynomials are contained in the image of \( C_0 \) under the Bergman projection. The fact that \( f \) is contained in \( B \) means precisely that the function \( f' \) is contained in the weighted Bergman space \( A_1^1 \). Reasoning as before, one easily shows that the ('weighted') Bergman projection \( P_1 : L^2(D, A_1(z)) \to A_1^1 \) is of the form:

\[
P_1 g(z) = \int_{D} \frac{g(w)}{(1 - zw)^3} dA_1(w) = 2 \int_{D} \frac{(1 - |w|^2)g(w)}{(1 - zw)^3} dA_1(w),
\]

and that this integral operator extends to \( L^1(D, A_1(z)) \)-functions and reproduces the weighted Bergman space \( A_1^1 \). Consequently,

\[
f'(z) = P_1(f') = 2 \int_{D} \frac{(1 - |w|^2)f'(w)}{(1 - zw)^3} dA_1(w).
\]

Let \( \varphi = \frac{(1 - |w|^2)f'(w)}{w^3} \). By our assumptions on \( f \), we see that \( \varphi \in L^\infty(D) \). Also, if \( f \in B_0 \), then actually \( \varphi \in C_0 \). Through differentiation under the integral sign, one easily verifies that \( f' = (P(\varphi))' \), which proves that \( P \) maps \( L^\infty(D) \) onto \( B \) and \( C_0 \) onto \( B_0 \). By the previous results, \( P \) then also maps \( C(\overline{D}) \) onto \( B_0 \). \[\square\]

**Theorem 5.11.** The dual space of \( A^1 \) is the Bloch space \( B \) under the following pairing:

\[
g \in B : f \in A^1 \mapsto \lim_{r \to 1} \int_{D} f_r(z)\overline{g(z)}dA(z).
\] (5.5)
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Proof. Cf. Theorem 1.21 ([23]). Note that equation (5.5) indeed defines a bounded functional on \(A^1\) because there exists \(\psi \in L^\infty\) such that \(P\psi = g\), so \(\int_D f_r(z)\overline{g(z)}\,dA(z)\) tends to \(\int_D f(z)\overline{\psi(z)}\,dA(z)\) as \(r \uparrow 1\).

In a similar vein, one has the following:

**Theorem 5.12.** The dual space of the little Bloch space \(B_0\) is the Bergman space \(A^1\) under the pairing:

\[
f \in A^1 : g \in B_0 \mapsto \lim_{r \uparrow 1} \int_D f(z)\overline{g_r(z)}\,dA(z).\]

Proof. Cf. □

We wish to stress that in Theorem 5.11, when one identifies \((A^1)^*\) with the Bloch space \(B\), the operator norm on \(B\) yields a norm that is equivalent with but not equal to the norm \(\|\|_B\) that we have previously defined on \(B\). Hence, there exists a norm \(\|\|_\bullet\) on the Bergman space \(A^1\) that is equivalent to \(\|\|_1\) and is such that the operator norm of \(g \in B = (A^1)^*\) equals \(\|g\|_B\). The strongly exposed points in the unit ball of \(A^1\) with the norm \(\|\|_\bullet\) have been described by C. Nara ([38]), who also showed that up to isometrical isomorphisms, \(A^1\) with the \(\|\|_\bullet\) norm is the unique pre-dual of \(B\).

5.4 **The space \((A^1)^ot + C\)**

We recall that \((A^1)^ot + C\) plays the same role in Theorem 5.5 with respect to the Bergman space as \(H^1) + C(T) = H^\infty + C(T)\) with respect to the Hardy space \(H^1(D)\) (Theorem 3.7). The space \(H^\infty + C(T)\) has been studied extensively. We recall from Theorem 3.9 that \(H^\infty + C\) is closed in \(L^\infty\). From this then it followed relatively easily that \(H^\infty + C(T)\) is in fact an algebra. We will now discuss how these results extend to the space \((A^1)^ot + C\).

**Theorem 5.13.** The space \((A^1)^ot + C\) is a proper, closed subspace of \(L^\infty\).

Proof. We mimic the proof of Theorem 3.9. The kernel of the map \(P : L^\infty \to B\) is \((A^1)^ot\). Because \(B_0\) is closed in \(B\), \(P^{-1}(B_0)\) is closed in \(L^\infty\) by the continuity of \(P\). By Theorem 5.9, \(L^\infty \neq P^{-1}(B_0) = (A^1)^ot + C\) and we are done. □

**Theorem 5.14.** The space \((A^1)^ot + C\) is a \(C\)-module.

Before we give the proof we need a lemma.

Let \(L_0^\infty\) be the subspace of \(L^\infty\) consisting of all \(L^\infty\)-functions that satisfy

\[
\lim_{r \to 1} \text{ess sup}_{r < |z| < 1} |f(z)| = 0.
\]

**Lemma 5.15.** The space \(L_0^\infty\) is a closed subspace of \((A^1)^ot + C\) and \((A^1)^ot + C\) is closed under multiplication by functions in \(L_0^\infty\).
Proof. The space \( L_0^\infty \) is closed in \( L^\infty \). Also, the product of a function in \( L_0^\infty \) and a bounded function will again be in \( L_0^\infty \) so what remains is to show that \( L_0^\infty \) is contained in \((A^1)^\perp + C\). Take \( \psi \in L_0^\infty \). We write \( \psi = \psi_1 + \psi_2 \), where \( \psi_1 \) is the restriction of \( \psi \) to the disc around zero with radius \( r \). If \( r \) is close enough to 1, then \( \|\psi_2\|_\infty \) will be arbitrarily small by the assumption on \( \psi \). Hence the \( B \)-norm of \( P\psi_2 \) will be arbitrarily small by the continuity of \( P \). On the other hand, \( P\psi_1 \) is holomorphic across \( T \), so \( P\psi_1 \in B_0 \). It follows that the \( B \)-distance of \( P\psi \) to \( B_0 \) will be at most the \( B \)-norm of \( P\psi_2 \), i.e., arbitrarily small. Because \( B_0 \) is closed in \( B \), we conclude that \( P\psi \in B_0 \). By the proof of Theorem 5.13, \( \psi \in (A^1)^\perp + C \) and we are done. \( \square \)

Proof of Theorem 5.14. The annihilator \((A^1)^\perp \) is closed under multiplication by \( z \), because \( A^1 \) is closed under multiplication by \( z \). By the closedness of \((A^1)^\perp + C \) and the Stone-Weierstrass theorem we need only show that \( zg(z) \in (A^1)^\perp + C \) when \( g \) is in \((A^1)^\perp \). Take \( f \in A_0^1 = zA^1 \), say \( f(z) = zF(z) \). Then, with the \( L^2 \)-inner product \((.,.)\), \( (f,zg) = \langle F, |z|^2 g \rangle \). Because \((1 - |z|^2)g(z) \in L_0^\infty \), \(|z|^2 g \in (A^1)^\perp + C \) by Lemma 5.15. Put \(|z|^2 g = g_1 + \varphi_1\), where \( g_1 \in (A^1)^\perp \) and \( \varphi_1 \in C \). So \( (f,zg) = \langle F, \varphi_1 \rangle \). Next we approximate \( \varphi_1 \) with a trigonometric polynomial \( p_1 = p_1(\varphi_1) \) such that \( \|\varphi_1 - p_1\|_\infty < \epsilon \). The integral \( \langle F, p_1 \rangle \) then depends on the Taylor coefficients of \( F \) in a finite fixed set of places. Because \( f = zF \) has the same coefficients, albeit shifted, we can find a trigonometric polynomial \( p_2 \) such that \( \langle F, p_1 \rangle = \langle f, p_2 \rangle \), for all \( f = zF \in A_0^1 \).

Now,

\[
\langle f, zg \rangle = \langle F, \varphi_1 \rangle = \langle F, p_1 \rangle + \langle F, \varphi_1 - p_1 \rangle = \langle f, p_2 \rangle + \langle F, \varphi_1 - p_1 \rangle,
\]

so that \( |\langle f, zg - p_2 \rangle| \leq \epsilon \|F\|_{A^1} \). Next, we remark that the \( A^1 \)-norms of \( f \) and \( F \) are equivalent in the sense that for all \( F \in A^1 \):

\[
\|zF\|_{A^1} \leq \|F\|_{A^1} \leq 4 \|zF\|_{A^1}.
\]

Hence, \( |\langle f, zg - p_2 \rangle| \leq 4\epsilon \|f\|_{A^1} \), for all \( f \in A_0^1 \). By the Hahn-Banach theorem, the \( L^\infty \)-distance of \( zg - p_2 \) to the annihilator of \( A_0^1 \) is at most \( 4\epsilon \). And since \( p_2 \) is continuous, and \((A_0^1)^\perp + C = (A^1)^\perp + C \), the \( L^\infty \)-distance of \( zg \) to \((A^1)^\perp + C \) is at most \( 4\epsilon \), thus zero. By Theorem 5.13, \( zg \in (A^1)^\perp + C \) and the proof is complete. \( \square \)

It is well-known that the space \( H^\infty + C(\overline{D}) \) is closed in \( L^\infty(D) \) ([53], Theorem 6.5.5). Let us write \( A := L_0^\infty + \overline{H^\infty} + C(\overline{D}) \), where the bar denotes complex conjugation. By the preceding remarks, the space \( A \) is a non-trivial closed algebra contained in \((A^1)^\perp + C \), and the space \((A^1)^\perp + C \) is an \( A \)-module. It should be stressed however that \((A^1)^\perp + C \) is not an algebra.

Lemma 5.16. Let \( f_\beta = (1 - z)^\beta \) and let \( \varphi_\beta = f_\beta/|f_\beta| \) for \( \beta \in \mathbb{R} \). Then \( \varphi_{-4} \in (A^1)^\perp \), but \( \varphi_{-2} \) is not contained in the space \((A^1)^\perp + C \).

Proof. Using the Stokes theorem one obtains that, at least formally, for every poly-
nominal $F$:

$$\int_D F \overline{\varphi - \frac{z}{2}} \, dA = \int_D -\overline{\varphi} (F(z)(1 - z) \log(1 - \overline{z})) \, dA(z)$$

$$= \int_S -F(z)(1 - z) \log(1 - \overline{z}) \, \frac{dz}{2\pi i}$$

$$= \int_S F(z) \left[ \frac{(1 - \overline{z}) \log(1 - \overline{z})}{\overline{z}} \right] \, \frac{dz}{2\pi i}$$

$$= \int_D F(z) \overline{\varphi} \left[ \frac{(1 - \overline{z}) \log(1 - \overline{z})}{\overline{z}} \right] \, dA(z)$$

$$= \int_D F(z) \left( \frac{-\overline{z} - \log(1 - \overline{z})}{\overline{z}^2} \right) \, dA(z),$$

and

$$\int_D F \overline{\varphi - \frac{z}{4}} \, dA = \int_D \overline{\varphi} \left[ F(z)(1 - z)^2 \right] \, dA(z)$$

$$= \int_S F(z) (1 - z)^2 \, \frac{dz}{2\pi i}$$

$$= \int_S -F(z)(1 - z)z \, \frac{dz}{2\pi i} = 0.$$

(In fact, by the same argument, $\int_D F(z) \overline{\varphi - \frac{z}{2k}} \, dA = 0$ for all $k = 2, 3, 4, \ldots$) Here we have used the identity $z\overline{z} = 1$ on $S$ to simplify the integrals over the circle. We conclude that (formally) $P_{\varphi - 2}(z) = \frac{-z - \log(1 - z)}{z^2}$, and $P_{\varphi - 4} = 0$, that is, $\varphi - 4 \in (A^1)^\perp$. These arguments can be made precise by a standard limit argument involving integration over the unit disc with a small disc around the point $z = 1$ punched out. Alternatively, one can directly calculate the Bergman projections. The lemma now follows because $P_{\varphi - 2}$ is not contained in the little Bloch space $B_0$. (In fact, it follows from our calculations in the proof of Proposition 5.8 that the $L^\infty$-distance of $\varphi - 2$ to $(A^1)^\perp + C$ is equal to 1.)

**Corollary 5.17.** The space $(A^1)^\perp + C$ is not an algebra.

Indeed, $\varphi - 4$ and $\varphi 2$ are both contained in $(A^1)^\perp + C$, but their product $\varphi - 2$ is not.

Next, let $u$ be an automorphism (Möbius map) of $D$. If $\psi$ is an element of $L^\infty(D)$ one can define the composition $\psi \circ u$ in $L^\infty$ of $\psi$ and $u$ as (represented by) the composition of $\Psi$ with $u$, where $\Psi$ is any representative of $\psi$. That this yields a well-defined element of $L^\infty$ follows from the fact that $u$ and its inverse map sets of Lebesgue measure zero to sets of Lebesgue measure zero. It is easily seen that the map $\psi \mapsto \psi \circ u$ is an isometric isomorphism of $L^\infty$.

**Proposition 5.18.** The space $(A^1)^\perp + C$ is invariant under composition with automorphisms of $D$. 

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Proof. There is nothing to prove for the composition of a continuous function with an automorphism. Take an element \( g \in (A^1)^\perp \), and let \( u \) be an automorphism of \( D \). We will show that \( g \circ u \) is contained in \( (A^1)^\perp + C \). Let \( f \) be an element of \( A^1 \). Then \( \int_D f g \circ u \, dA = \int_D (f \circ u^{-1}) g J_R(u^{-1}) \, dA \), where \( J_R(u^{-1}) \) is the real Jacobian of \( u^{-1} \), an element of \( C \). By Theorem 5.14 there exist \( g^* \in (A^1)^\perp \) and \( h \in C \) such that \( g J_R(u^{-1}) = g^* + h \). Thus, because \( f \circ u^{-1} \) is contained in \( A^1 \),

\[
\int_D f g \circ u \, dA = \int_D (f \circ u^{-1}) h \, dA = \int_D f (h \circ u) J_R(u) \, dA.
\]

We conclude that \( g \circ u = (h \circ u) J_R(u) \) annihilates the Bergman space, hence \( g \circ u \in (A^1)^\perp + C \).

Proposition 5.19. Let \( f \) be a strongly exposed point in \( A^1 \). Then

(a) if \( u \) is an automorphism of \( D \), then the normalized function \( f_1 = C_1 (f \circ u) \) is strongly exposed;

(b) if \( v \in A(D) \) is zero-free on the circle, then the normalized function \( f_2 = C_2 f v \) is strongly exposed.

Furthermore, the functions \( \varphi = f / |f|, \varphi_1 = f_1 / |f_1| \) and \( \varphi_2 = f_2 / |f_2| \) have the same \( L^\infty \)-distance to \((A^1)^\perp + C\).

Proof. (a) There exist \( g \in (A^1)^\perp, h \in C \) such that \( \| \varphi - g - h \|_\infty < 1 \). By Proposition 5.18 the function \( g \circ u \) is again contained in \( (A^1)^\perp + C \). Because \( \varphi_1 = \varphi \circ u \), \( \| \varphi_1 - g \circ u - h \circ u \|_\infty < 1 \), so we conclude that \( f_1 \) is strongly exposed. Also, the \( L^\infty \)-distance of \( \varphi_1 \) to \((A^1)^\perp + C\) does not exceed that of \( \varphi \). Replacing \( u \) by its inverse, the reverse inequality follows.

(b) With \( g \) and \( h \) as above and \( \varphi_2 = \varphi (w) = C_2 f_2 (w) \), \( \| \varphi_2 - g \|_\infty < 1 \). One finishes the proof as before, using Lemma 5.15 and the fact that \( f / |f| \) is invertible in \( L_0^\infty + C \).

5.5 Strong exposedness of \((1 - z)^\beta\)

We saw in Section 5.2 that the functions \( f_\beta = c_\beta (1 - z)^\beta \) are strongly exposed in the unit ball of \( A^1 \) for all \( \beta > -1 \) except possibly when \( \beta = 1, 3, 5, \ldots \). This was deduced from rather straightforward estimates of the \( L^\infty \)-distances of the functions \( \varphi = f_\beta / |f_\beta| \) to the space \((A^1)^\perp + C \) (Proposition 5.7). In this section we will sharpen these estimates and answer the question of strong exposedness for odd exponents.

Theorem 5.20. For all \( \beta \geq 0 \), the Bloch distance of the function \( P \varphi_\beta \) to \( B_0 \) equals

\[
\frac{4}{\pi} \frac{|\sin(\beta \pi)|}{\beta + 2}.
\]

Proof. We have seen in Section 5.2 that the functions \( P \varphi_{2n} \) are contained in \( B_0 \) so henceforth we will assume that \( \beta \) is not even. It is convenient to rewrite \( \varphi_\beta \) as \( \varphi_\beta(w) = (1 - w)^{\beta/2} / (1 - \bar{w})^{\beta/2} \). Using the series expansions for the Bergman kernel
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\[1/(1 - z\bar{w})^2\ (\text{see (5.2)})\), as well as for \((1 - u)^\beta/2\), and \(1/(1 - \bar{w})^\beta/2\), we evaluate the Bergman projection \(P\varphi_\beta\). One obtains \(P\varphi_\beta = \sum_{n=0}^{\infty} c_{\beta,n} z^n\), where

\[c_{\beta,n} = \frac{n + 1}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\beta}{2})} \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} \cdot \frac{1}{n\beta(\beta + 2)}(1 + o(1)), (5.6)\]

we claim that for fixed \(\beta > 0\):

\[\sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} = \frac{4}{n^2\beta(\beta + 2)}(1 + o(1)),\]

where the \(o(1)\)-term tends to zero as \(n \to \infty\). This implies that

\[c_{\beta,n} = \frac{4}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\beta}{2})} \frac{1}{n\beta(\beta + 2)}(1 + o(1)) = -\frac{2\sin\left(\frac{\beta\pi}{2}\right)}{\pi(\beta + 2)n},\]

where the \(o(1)\)-term vanishes as \(n \to \infty\). (Here we have used the functional equations \(\Gamma(z + 1) = z\Gamma(z)\) and \(\Gamma(z)\Gamma(1 - z)\sin(\pi z) = \pi\).) But then,

\[\lim_{x \to 1} |(1 - x^2)(P\varphi)'(x)| = \frac{4|\sin\left(\frac{\beta\pi}{2}\right)|}{\pi(\beta + 2)},\]

so the Bloch distance of \(P\varphi_\beta\) to \(B_0\) is at least \(\frac{4|\sin\left(\frac{\beta\pi}{2}\right)|}{\pi(\beta + 2)}\). On the other hand, for large \(N\),

\[|(\sum_{n=N}^{\infty} c_{\beta,n} z^n)'| \leq \sum_{n=N}^{\infty} n|c_{\beta,n}| \cdot |z|^{n-1} \leq \frac{2\sin\left(\frac{\beta\pi}{2}\right)}{\pi(\beta + 2)} \frac{1 + o(1)}{1 - |z|},\]

where the \(o(1)\)-term tends to zero as \(N\) increases. Using the fact that the polynomials are contained in \(B_0\) it follows that the Bloch distance of \(P\varphi_\beta\) to \(B_0\) is at most \(\frac{4|\sin\left(\frac{\beta\pi}{2}\right)|}{\pi(\beta + 2)}\). This then proves the theorem.

We turn to the claim (5.6). Let us first assume \(\beta > 2\). Given any large \(n \in \mathbb{N}\), let \(M = M_n\) be the integer nearest to \(\sqrt{n}\). We write

\[\sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} = \sum_{m=0}^{M-1} + \sum_{m=M}^{\infty} \cdot \]

Because \(\beta > 2\), \(\frac{\Gamma(m + \frac{\beta}{2})}{m!}\) is increasing in \(m\). On the other hand, \(\frac{\Gamma(n + m - \frac{\beta}{2})}{(n + m + 1)!}\) is decreasing in \(m + n\). The first sum can thus be estimated by

\[\sum_{m=0}^{M-1} \leq M \frac{\Gamma(M + \frac{\beta}{2})\Gamma(n - \frac{\beta}{2})}{(M)!}\frac{1}{(n + 1)!} \cdot \]

Thus, by Stirling's formula (5.4), there exists a constant \(A\), independent of \(n\), such that

\[\sum_{m=0}^{M-1} \leq A \frac{M \cdot M^{\frac{\beta}{2}-1}}{n^{2+\frac{\beta}{2}}} = A \left(\frac{M}{n}\right)^{\frac{\beta}{2}}.\]
Hence
\[
\sum_{m=0}^{M-1} = \frac{o(1)}{n^2},
\]
as \( n \to \infty \). In the remaining sum, \( \sum_{m=M}^{\infty} \), all the arguments in the Gamma functions and factorials tend to infinity as \( n \to \infty \). Another application of Stirling's formula seems in place. One obtains that, given any \( \varepsilon > 0 \), for all sufficiently large \( n \) and all \( m \geq M \),
\[
\left| \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} / \frac{m^{\frac{\beta}{2} - 1}}{(n + m)^{\frac{\beta}{2} + 2}} - 1 \right| < \varepsilon.
\]
In particular,
\[
\left| \sum_{m=M}^{\infty} \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} / \frac{m^{\frac{\beta}{2} - 1}}{(n + m)^{\frac{\beta}{2} + 2}} - 1 \right| < \varepsilon,
\]
as \( n \to \infty \). Therefore, by (5.7), the claim (5.6) follows once we show that
\[
\sum_{m=M}^{\infty} \frac{m^{\frac{\beta}{2} - 1}}{(n + m)^{\frac{\beta}{2} + 2}} / \frac{4}{n^2\beta(\beta + 2)} \to 1,
\]
as \( n \to \infty \). Let us investigate the functions \( g_n(x) = \frac{x^{\frac{\beta}{2} - 1}}{(n + x)^{\frac{\beta}{2} + 2}} \). For all \( x \geq 1 \), \( g_n(x) \leq \frac{1}{x(n+x)^2} \), so \( g_n(x) \leq \frac{1}{n^{\beta/2}} \) when \( x \geq M \). There is a number \( x_{\beta, n} > 0 \) such that \( g_n(x) \) is increasing on the interval \( (0, x_{\beta, n}] \) and decreasing on the interval \( [x_{\beta, n}, \infty) \).

Hence, the sum \( \sum_{m=M}^{\infty} \frac{m^{\frac{\beta}{2} - 1}}{(n + m)^{\frac{\beta}{2} + 2}} \) and the integral \( \int_{M}^{\infty} \frac{x^{\frac{\beta}{2} - 1}}{(n + x)^{\frac{\beta}{2} + 2}} \ dx \) differ at most
\[
\frac{4}{n^{\beta/2}} = \frac{o(1)}{n^2}.
\]
By a change of variables,
\[
\int_{M}^{\infty} \frac{x^{\frac{\beta}{2} - 1}}{(n + x)^{\frac{\beta}{2} + 2}} \ dx = \frac{1}{n^2} \int_{M}^{\infty} \frac{x^{\frac{\beta}{2} - 1}}{(1 + x)^{\frac{\beta}{2} + 2}} \ dx.
\]
Now, with \( B(\ldots) \) the standard Beta-function,
\[
\int_{0}^{\infty} \frac{x^{\frac{\beta}{2} - 1}}{(1 + x)^{\frac{\beta}{2} + 2}} \ dx = B\left(\frac{\beta}{2}, 2\right) = \frac{4}{\beta(\beta + 2)}.
\]
On the other hand, as \( n \to \infty \),
\[
\int_{0}^{M} \frac{x^{\frac{\beta}{2} - 1}}{(1 + x)^{\frac{\beta}{2} + 2}} \ dx = o(1).
\]
By the preceding estimates, the claim (5.6) now follows for all \( \beta > 2 \).

When \( 0 < \beta < 2 \) we proceed as follows. Given a large \( n \in \mathbb{N} \), we let \( M = M_n \) be the integer nearest to \( n^{\frac{\beta}{2}} \). Now the terms in the sum
\[
\sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{\beta}{2})\Gamma(m + n - \frac{\beta}{2})}{m!(m + n + 1)!} = \sum_{m=0}^{M-1} + \sum_{m=M}^{\infty}.
\]
are decreasing. The first sum can be estimated by
\[ \sum_{m=0}^{M-1} \leq M \Gamma(\beta) \Gamma(n - \frac{\beta}{2}) \frac{M(\beta)}{(n+1)!} \leq \frac{A_\beta}{n^{2+\frac{\beta}{4}}} = \frac{o(1)}{n^2}. \]
The second sum can be dealt with as before. (Now the functions \( g_n(x) \) are decreasing on \((0, \infty)\), which makes the analysis even simpler.) We omit the details. This finishes the proof of equation (5.6) for all \( \beta > 0 \).

**Corollary 5.21.** Let \( d(\varphi_\beta, (A^1)^\perp + C) \) denote the \( L^\infty \)-distance of \( \varphi_\beta \) to \((A^1)^\perp + C \). Then for all \( \beta \geq 0 \),
\[
\frac{1}{2} \left| \frac{\sin \frac{\beta \pi}{2}}{\beta + 2} \right| \leq d(\varphi_\beta, (A^1)^\perp + C) \leq \frac{4}{\pi} \left| \frac{\sin \frac{\beta \pi}{2}}{\beta + 2} \right| \leq \frac{2}{\pi},
\]
(5.8)
In particular, all \( f_\beta \) are strongly exposed for \( \beta \geq 0 \).

**Proof.** Let \( q : B \to B/B_0 \) be the quotient map. By Theorem 5.9, the map \( q \circ P : L^\infty \to B/B_0 \) is continuous and surjective. In the proof of Theorem 5.13 it was shown that the kernel of the map \( q \circ P \) is the space \((A^1)^\perp + C \). It follows that the derived map
\[ P^* : L^\infty /((A^1)^\perp + C) \to B/B_0 \]
is bijective and bounded (by \( \frac{8}{\pi} \) upon inspection, cf. Lemma 5.10 and Theorem 5.9). This gives the lower bound for \( d(\varphi_\beta, (A^1)^\perp + C) \), because \( \|P^* \varphi_\beta\| = \frac{4}{\pi} \left| \frac{\sin \frac{\beta \pi}{2}}{\beta + 2} \right| \).
By the closed graph theorem, the inverse \( P^{*-1} \) of \( P^* \) is also bounded. Actually, we will show directly that \( \|P^{*-1}\| \leq 1 \), which in turn yields the upper bound for \( d(\varphi_\beta, (A^1)^\perp + C) \). Let us suppose that \( F \in B/B_0 \) has norm 1. We need to show that \( P^{*-1}(F) \) has norm at most 1 in \( L^\infty /((A^1)^\perp + C) \). For any \( \varepsilon > 0 \), we can find a representative \( f \in B \) of the coset \( F \) such that \( \|f\|_B < 1 + \varepsilon \). We recall from the proof of Theorem 5.9 that
\[ \psi(w) = (1 - |w|^2) \cdot \frac{f'(w) - f'(0)}{\bar{w}} \in L^\infty \]
satisfies \( f(z) - P\psi(z) = f(0) + f'(0)z \in B_0 \). Thus \( \psi \) is a representative of \( P^{*-1}(F) \) in \( L^\infty \). Hence, by Lemma 5.15,
\[
\|P^{*-1}(F)\|_{L^\infty /((A^1)^\perp + C)} \leq d(\psi, (A^1)^\perp + C) \leq \lim \text{ess sup}_{r < |w| < 1} |\psi(w)| = \lim \sup_{|w| \to 1} |(1 - |w|^2)f'(w)| \leq ||f||_B < 1 + \varepsilon.
\]

**Corollary 5.22.** Suppose \( g \in A(D) \) vanishes nowhere on \( \mathbb{T} \). Let \( z_1, z_2, \ldots, z_n \in \mathbb{T} \) be distinct and let \( \beta_1, \beta_2, \ldots, \beta_n \) be real numbers greater than \(-2 \). Then the normalized function \( f(z) = cg(z) \prod_{i=1}^{n} (1 - z z_i)^{\beta_i} \) is strongly exposed in the unit ball of \( A^1 \) if and only if all functions \( f_{\beta_i} = c\beta_i (1 - z)^{\beta_i} \) are strongly exposed. In particular, all choices of \( \beta_i > -1 \) yield strongly exposed points and all normalized polynomials are strongly exposed in the unit ball of \( A^1 \).
Proof. By part (b) of Proposition 5.19, the factor \( g(z) \) has no effect on strong exposedness of the function \( f \). Let \( d_i = d(\varphi, (A^1)_{\perp} + C) \) and let \( \varphi = f/|f| \). We will show that \( d(\varphi, (A^1)_{\perp} + C) = \max_i d_i \), which will give the desired result.

We find small pairwise disjoint neighborhoods \( U_i \) of the \( z_i \) and a partition \( \chi_i \) of the unity relative to the \( U_i \)'s and \( D \). That is to say, we find continuous functions \( \chi_i \geq 0 \) on \( D \) such that \( \chi_i \equiv 1 \) on \( U_i \) and \( \sum_i \chi_i \equiv 1 \) on \( D \). Then \( \varphi = \sum_i \chi_i \varphi = \sum_i \varphi(i) \). For every \( i \), there exists a unimodular constant \( \lambda = \lambda_i \) for which \( \varphi(i)(z) - \lambda \varphi(i)(z; \bar{z}) \in C \).

Consequently, \( d_i = d(\varphi(i), (A^1)_{\perp} + C) \). Using the \( C \)-module structure of \( (A^1)_{\perp} + C \) and the fact that \( \chi_i \leq 1 \), it is easily seen that \( d(\varphi(i), (A^1)_{\perp} + C) \leq d(\varphi, (A^1)_{\perp} + C) \), hence \( \max_i d_i \leq d(\varphi, (A^1)_{\perp} + C) \). Conversely, if the functions \( g_i \in (A^1)_{\perp} + C \) are such that \( \|\varphi(i) - g_i\|_{\infty} < d_i + \varepsilon \), then

\[
\left\| \sum_i \chi_i \varphi(i) - \sum_i \chi_i g_i \right\|_{\infty} < \max_i d_i + \varepsilon.
\]

Because \( \varphi - \sum_i \chi_i \varphi(i) = \sum_i \chi_i (1 - \chi_i) \varphi \in C \) and \( \chi_i g_i \in (A^1)_{\perp} + C \), it follows that \( d(\varphi, (A^1)_{\perp} + C) < \max_i d_i + \varepsilon \). \( \Box \)

We will now show that an estimate analogous to inequality (5.8) also holds for \( \beta \in (-2,0) \).

Lemma 5.23. Let \( g_n = \frac{1}{n} (1 - z)^{-2+\frac{1}{n}} \). Then \( \lim_{n \to \infty} \|g_n\|_1 = 1 \).

Proof. Reasoning as in the proof of Lemma 5.10 we perform the calculation

\[
\int_D |1 - z|^{-2+\frac{1}{n}} dA(z) = \sum_{k=0}^{\infty} \frac{\Gamma^2(k + 1 - \frac{1}{2n})}{\Gamma^2(1 - \frac{1}{2n})k!(k + 1)!}.
\]

The terms \( g_{n,k} = \frac{\Gamma^2(k + 1 - \frac{1}{2n})}{k!(k + 1)!} \) are decreasing in \( k \). For large \( n \), let \( K = K_n = [\sqrt{n}] \).

Then \( \sum_{k=1}^{K} g_{n,k} \leq \sqrt{n} \). In the remaining sum, we can approximate the terms using Stirling's formula (5.4): \( g_{n,k} \sim \frac{1}{k^{1+\frac{1}{n}}} \). Therefore, \( \sum_{k=K+1}^{\infty} g_{n,k} \sim \int_{\sqrt{n}}^{\infty} \frac{1}{x^{1+\frac{1}{n}}} \sim n \).

This proves the claim. \( \Box \)

Proposition 5.24. For all \(-2 < \beta < 0 \),

\[
\frac{2}{\pi} \frac{|\sin\left(\frac{3\pi}{2}\right)|}{\beta + 2} \leq d(\varphi, (A^1)_{\perp} + C),
\]

where again \( d(\varphi, (A^1)_{\perp} + C) \) denotes the \( L^\infty \)-distance of \( \varphi \) to \( (A^1)_{\perp} + C \).

Proof. Fix any \( \beta \in (-2,0) \) and let \( L = L_\beta \) be the functional \( L : g \in A^1 \mapsto \int_D g \overline{\varphi} \, dA \).

We will show that for the sequence \( g_n \) from Lemma 5.23,

\[
\lim_{n \to \infty} L(g_n) = \frac{2}{\pi} \frac{|\sin\left(\frac{3\pi}{2}\right)|}{\beta + 2}.
\]

From this we will get the desired lower bound as follows. The functions \( g_n \) and all their derivatives tend to zero uniformly on compact subsets of \( D \). Given any
(fixed) integer \( N \) we define functions \( g_n^*(z) := g_n(z) - \sum_{k=0}^{N-1} \frac{g_n^{(k)}(0)}{k!} z^k \). It follows that 
\[
\lim_{n \to \infty} \| g_n^* \|_1 = 1 \text{ and} 
\lim_{n \to \infty} L(g_n^*) = \frac{2 |\sin(\frac{\beta \pi}{2})|}{\beta + 2}.
\tag{5.11}
\]
Furthermore, by construction, the first \( N \) derivatives of the \( g_n^* \) vanish at the origin. If we let \( z^N A^1 \subset A^1 \) denote the closed subspace of all functions in \( A^1 \) whose first \( N \) derivatives vanish at the origin, then by equation (5.11), the norm of the functional \( L \) restricted to \( z^N A^1 \) is at least \( \frac{2}{\pi} |\sin(\frac{\beta \pi}{2})|/(\beta + 2) \). By the Hahn-Banach theorem, the \( L^\infty \)-distance of \( \varphi_\beta \) to \( (z^N A^1)^\perp \) is at least \( \frac{2}{\pi} |\sin(\frac{\beta \pi}{2})|/(\beta + 2) \). Consequently, the \( L^\infty \)-distance of \( \varphi \) to \( \mathcal{P} = \bigcup_{N=1}^\infty (z^N A^1)^\perp \) is at least \( \frac{2}{\pi} |\sin(\frac{\beta \pi}{2})|/(\beta + 2) \). Observe that \( \mathcal{P} \) is uniformly dense in \( (A^1)^\perp + C \), because it contains \( (A^1)^\perp \) and all trigonometric polynomials. Therefore, the \( L^\infty \)-distance of \( \varphi_\beta \) to \( (A^1)^\perp + C \) is at least \( \frac{2}{\pi} |\sin(\frac{\beta \pi}{2})|/(\beta + 2) \).

Let us turn to formula (5.10). We calculate
\[
nL(g_n) = \int_D \frac{1}{(1-z)^{2+\frac{\beta}{2}-\frac{1}{n}}} \frac{1}{(1-z)^{-\frac{\beta}{2}}} \, dA(z)
\]
using the series expansions for \((1-z)^\alpha\) and \((1-z)^\alpha\). After a routine calculation, one obtains the following expression:
\[
nL(g_n) = \frac{1}{\Gamma(2 + \frac{\beta}{2} - \frac{1}{n}) \Gamma(-\frac{\beta}{2})} \sum_{k=0}^\infty \frac{\Gamma(2 + \frac{\beta}{2} - \frac{1}{n} + k) \Gamma(k - \frac{\beta}{2})}{(k+1)! k!}.
\]
Now
\[
\frac{1}{\Gamma(2 + \frac{\beta}{2} - \frac{1}{n}) \Gamma(-\frac{\beta}{2})} \to \frac{1}{\Gamma(2 + \frac{\beta}{2}) \Gamma(-\frac{\beta}{2})} = \frac{2 |\sin(\frac{\beta \pi}{2})|}{\pi \beta + 2},
\]
as \( n \to \infty \). So what’s left to do is to show that
\[
\frac{1}{n} \sum_{k=0}^\infty \frac{\Gamma(2 + \frac{\beta}{2} - \frac{1}{n} + k) \Gamma(k - \frac{\beta}{2})}{(k+1)! k!} \to 1,
\]
as \( n \to \infty \). This can easily be done by following the proof of Lemma 5.23. \( \Box \)

We end this chapter with a conjecture on the functions \( f_\beta \) for \(-2 < \beta < 0\), for which strong exposedness is already implied by Proposition 5.7 when \(-1 < \beta < 0\). Note that inequality (5.9) is "asymptotically sharp" for \( \beta \downarrow -2 \):
\[
\lim_{\beta \downarrow -2} \frac{2 |\sin(\frac{\beta \pi}{2})|}{\beta + 2} = 1 = d(\varphi_{-2}, (A^1)^\perp + C).
\]

**Conjecture 5.25.** For all \(-2 < \beta < 0\), \( d(\varphi_\beta, (A^1)^\perp + C) = \frac{2 |\sin(\frac{\beta \pi}{2})|}{\pi \beta + 2} \). In particular, the functions \( f_\beta \) are strongly exposed for all said \( \beta \).
CHAPTER 5. STRONGLY EXPOSED POINTS IN BERGMAN SPACE