Logic for social software
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When Coalition Logic is used to reason about coalitional ability in extensive games, for instance, the formula $[C] \varphi$ holds in case coalition $C$ can bring about $\varphi$ in one move. Thus, Coalition Logic is a logic for reasoning about local coalitional ability in games. Naturally, many essential properties of such games cannot be expressed in terms of local effectivity alone, most notably the property of a coalition having a winning strategy in the extensive game as a whole. For this reason, the present chapter extends the language of Coalition Logic with an additional modality to talk about ability in the long run, i.e., about what coalitions can bring about eventually.

We will start by discussing various kinds of effectivity in the long run and their interrelationship. In most cases, Extended Coalition Logic, formally defined in section 4.2, will be able to express all of these different notions through one single modality. The complexity of the model checking problem and the expressive power of the richer language of Extended Coalition Logic will be investigated in section 4.3. Furthermore, we will discuss the relationship between Extended Coalition Logic and Alternating Temporal Logic (section 4.4) and work in distributed artificial intelligence (section 4.5).

### 4.1 Ability in the Long Run

Given a coalition frame $F = (S, \{E_C | C \subseteq N\})$ which contains information about effectivity at every state, we can also investigate effectiveness in the long run. The two basic notions of long-term effectivity we shall use are goal maintenance and (eventual) goal achievement. A coalition $C$ can eventually achieve a set of states $X$ provided that it has a strategy which establishes $X$ after some finite number of moves which does not need to be fixed before the game starts. Using a fixpoint construction (see appendix A for background material on fixpoints), we can formally define eventual goal achievement.
Chapter 4. Extended Coalition Logic

**Definition 4.1 (Eventual Goal Achievement).** Given a coalition frame $\mathcal{F} = (S, \{E_C|C \subseteq N\})$, the *eventual goal achievement* effectivity function $E^*$ is defined as

$$E^*_C(X) = \mu Y. X \cup (E^*_\emptyset(\emptyset) \cap E_C(Y)).$$

Note that we give fixpoint operators lowest precedence, so that the scope of a fixpoint operator extends as far to the right as possible.

Intuitively, $sE^*_C X$ holds precisely when at state $s$ coalition $C$ can bring about $X$ in the long run. The term $E^*_\emptyset(\emptyset)$ guarantees that states will not be included simply because they are terminal states where all members of $C$ win. As a consequence of this definition of $E^*$, $E_C(X) \not\subseteq E^*_C(X)$, for at a terminal state $s \not\in X$ where $C$ wins, $sE_CX$ but not $sE^*_C X$. What we do have is $E_C(X) \cap E^*_\emptyset(\emptyset) \subseteq E^*_C(X)$ and $E_C(X) \subseteq E^*_C(X \cup E_C(\emptyset))$.

Turning now toward goal maintenance, coalition $C$ can maintain a set of states $X$ provided that it lias a strategy which will guarantee that every future position of the game play will be in $X$.

**Definition 4.2 (Goal Maintenance).** Given a coalition frame $\mathcal{F} = (S, \{E_C|C \subseteq N\})$, the *goal maintenance* effectivity function $E^\times$ is defined as

$$E^\times_C(X) = \nu Y. X \cap (E^*_\emptyset(\emptyset) \cup E_C(Y)).$$

Intuitively, $sE^\times_C X$ holds precisely when at state $s$ coalition $C$ can maintain $X$ indefinitely. Now the additional union with $E^*_\emptyset(\emptyset)$ is necessary to guarantee that states will not be excluded simply because they are terminal states where some members of $C$ lose.

The definitions of effectivity in the long run are general in that they do not presuppose the coalition frame to satisfy any additional properties such as weak playability. On the other hand, the example given to motivate certain aspects of the definition assumed an application to extensive games, and this is in fact the class of models we shall look at when dealing with some applications in chapter 5.

Recall from section 2.4.2 that coalition frames can be used to model extensive games as well as game forms. For extensive game forms, $sE^*_C X$ will hold in case coalition $C$ can eventually achieve $X$, no matter how the empty games at terminal states are defined (thanks to the $E^*_\emptyset(\emptyset)$ term in the definition of $E^*$). For extensive games (with significant payoff information), we can additionally consider whether coalition $C$ has a winning strategy, expressed by $E^*_C(E_C(\emptyset))$, or whether it can achieve a win or $X$, expressed by $E^*_C(X \cup E_C(\emptyset))$. Note, however, that due to the possibility of infinite runs, having a winning strategy can be interpreted in two ways: In the (for $C$) best case, coalition $C$ has a strategy which guarantees that game play will terminate in a state which is a win for all members of $C$. This is the interpretation corresponding to $E^*_C(E_C(\emptyset))$. If $C$ is somewhat less fortunate, however, it may only have a strategy which guarantees
that if game play terminates, then all members of $C$ win. As the example of a game where all plays are infinite illustrates, the two kinds of strategies are not equivalent. This distinction between strong and weak winning strategies is not usually made in game theory, where it is assumed that also infinite runs generate payoffs to the players [93]. Note also that in order to distinguish strong from weak winning strategies, it is crucial that our semantic model allows for terminal states (compare this with Alternating Temporal Logic, discussed in section 4.4).

**Total vs. Partial Terminal Effectivity**

We shall formalize the distinction between strong and weak strategies as the difference between total and partial terminal effectivity. While the notion of eventual goal achievement formalizes what it means to bring about a goal at some point in the future, we are often more interested in what terminal outcomes or outcome states a coalition can achieve. If we are dealing with an extensive game, this question asks for whether a coalition has a strategy which will yield a win for all its members. For an extensive game form on the other hand, we want to know which sets of terminal states can be achieved.

Given a coalition frame $\mathcal{F} = (S, \{E_C|C \subseteq N\})$, we define total terminal effectivity as

$$E^t_C(X) = E^*_C(E_\emptyset(\emptyset) \cap X) = \mu Y. (E_\emptyset(\emptyset) \cap X) \cup (\overline{E_\emptyset(\emptyset)} \cap E_C(Y)).$$

The set $E^t_C(X)$ includes the set $E_\emptyset(\emptyset) \cap X$ which denotes the set of terminal states which are in $X$. Furthermore, it includes the nonterminal states from which $C$ can achieve one of those terminal states in $X$, and so on. The weaker version of total terminal effectivity is partial terminal effectivity defined as

$$E^p_C(X) = E^*_C(E_\emptyset(\emptyset) \cup X) = \nu Y. (E_\emptyset(\emptyset) \cup X) \cap (E_\emptyset(\emptyset) \cup E_C(Y)).$$

The goal to be maintained here is that “if the present state is terminal then it is in $X$”. If we are dealing with an extensive game which has infinite runs, we can then distinguish a strong winning strategy $E^t_C(E_C(\emptyset))$ from a weak winning strategy $E^p_C(E_C(\emptyset))$.

For extensive games and game forms, the four possible instantiations of the scheme “some/all plays of the game are finite/infinite” are of particular interest. All of them can be defined in terms of the notions just introduced. For weakly playable frames $\mathcal{F} = (S, \{E_C|C \subseteq N\})$, we have

$$E^t_\emptyset(S) = \mu Y. E_\emptyset(Y) \text{ holds iff all plays of the game are finite.}$$

$$E^p_N(S) = \mu Y. E_\emptyset(\emptyset) \cup E_N(Y) \text{ holds iff at least one play of the game is finite.}$$

and their negations for the other two instantiations of the scheme.
Theorem 4.3. For every \( C \)-regular and \( C \)-maximal coalition frame, eventual goal achievement and goal maintenance are duals, as are total and partial terminal effectivity. Formally, \( E^*_C(X) = E^*_C(\bar{X}) \) and \( E^t_C(X) = E^t_C(\bar{X}) \).

Proof. For the first duality,
\[
E^*_C(\bar{X}) = \mu Y.\bar{X} \cap (E_\emptyset(\emptyset) \cup \overline{E_C(Y)}) = \nu Y.X \cap (E_\emptyset(\emptyset) \cup \overline{E_C(Y)}) = E^*_C(X)
\]
by \( C \)-maximality and \( C \)-regularity. The second claim is then an easy corollary.

The relationship between partial and total terminal effectivity is further described by the following result which should be seen as a generalization of Dijkstra’s work, establishing links between partial and total program correctness.

Theorem 4.4. For every coalition frame, total implies partial terminal effectivity, i.e., \( E^t_C(X) \subseteq E^p_C(X) \). For superadditive coalition frames, at states where no infinite play is possible, the converse also holds, i.e., \( E^t_\emptyset(S) \cap E^p_C(X) \subseteq E^t_C(X) \).

Proof. Let \( \mathcal{F} = (S, \{E_C|C \subseteq N\}) \) be any coalition frame.

(1) Given \( Y \subseteq S \), let \( F_t(Y) = (E_\emptyset(\emptyset) \cap X) \cup (E_\emptyset(\emptyset) \cap E_C(Y)) \) and \( F_p(Y) = (E_\emptyset(\emptyset) \cap X) \cap (E_\emptyset(\emptyset) \cap E_C(Y)) \). We show by transfinite induction that for all ordinals \( \kappa \), \( F_t^{\kappa+1} \subseteq F_p^{\kappa} \). For the inductive step,
\[
F_t^{\kappa+1} = (E_\emptyset(\emptyset) \cap X) \cup (E_\emptyset(\emptyset) \cap E_C(F_t^{\kappa})) \subseteq (E_\emptyset(\emptyset) \cap X) \cup (E_\emptyset(\emptyset) \cup E_C(F_p^{\kappa})) = F_p^{\kappa+1}
\]
using the induction hypothesis and monotonicity. For limit ordinals, the inductive step follows from the fact that for all \( \kappa_0 \leq \kappa \), \( F_t^{\kappa_0} \subseteq F_p^{\kappa_0} \) and \( F_t^{\kappa} \subseteq F_p^{\kappa} \). Now to prove the first claim, if \( s \in E^t_C(X) \), there is some closure ordinal \( \alpha \) such that for all \( \beta \geq \alpha \), \( s \in F_t^{\beta} \) and hence \( s \in F_p^{\beta} \). Thus, for all ordinals \( \gamma, s \in F_p^{\gamma} \), in particular for the closure ordinal \( \gamma_0 \) for which \( F_p^{\gamma_0} = F_p(F_p^{\gamma_0}) = E^p_C(X) \).

(2) Given \( Y \subseteq S \), let \( F_f(Y) = E_\emptyset(Y) \), and assume that \( \mathcal{F} \) is superadditive. We show by induction that for all ordinals \( \kappa \), \( F_f^{\kappa} \cap F_p^{\kappa} \subseteq F_t^{\kappa} \). For the inductive step \( \kappa + 1 \), we must show that
\[
E_\emptyset(F_f^{\kappa}) \cap (E_\emptyset(\emptyset) \cup X) \cap (E_\emptyset(\emptyset) \cup E_C(F_p^{\kappa})) \subseteq (E_\emptyset(\emptyset) \cap X) \cup (E_\emptyset(\emptyset) \cap E_C(F_t^{\kappa}))
\]
which follows from superadditivity and the induction hypothesis. The rest of the proof is analogous to the proof of (1).

4.2 Syntax & Semantics

As the name suggests, Extended Coalition Logic (ECL) extends Coalition Logic with an extra modality for expressing effectivity in the long run. As we have seen in the previous section, all of the notions discussed can be expressed in terms of eventual goal achievement \( E^* \) and goal maintenance \( E^x \).
Definition 4.5 (Extended Coalition Logic Syntax). Given a finite non-empty set of agents/players $N$, and a set of atomic propositions $\Phi_0$, formulas $\varphi$ of Extended Coalition Logic can have the following syntactic form:

$$\varphi ::= \perp | p | \neg \varphi | \varphi \lor \varphi | [C][\varphi] | [C^*][\varphi] | [C^x][\varphi]$$

where $p \in \Phi_0$ and $C \subseteq N$.

As before, the other boolean connectives are defined in the standard way. Like in basic Coalition Logic, ECL formulas are interpreted over coalition models $M = ((S, \{E_C | C \subseteq N\}), V)$ just as before, the only difference being the additional modalities:

Definition 4.6 (Extended Coalition Logic Semantics). Given a coalition model $M = ((S, \{E_C | C \subseteq N\}), V)$, the truth of a formula $\varphi$ in a model $M$ at a state $s$ is defined as follows:

$$M, s \not\models \perp$$
$$M, s \models p \iff p \in \Phi_0 \text{ and } s \in V(p)$$
$$M, s \models \neg \varphi \iff M, s \not\models \varphi$$
$$M, s \models \varphi \lor \psi \iff M, s \models \varphi \text{ or } M, s \models \psi$$
$$M, s \models [C][\varphi] \iff sE_C[\varphi]^M$$
$$M, s \models [C^*][\varphi] \iff sE^*_C[\varphi]^M$$
$$M, s \models [C^x][\varphi] \iff sE^x_C[\varphi]^M$$

By theorem 4.3, we know that for all $C$-regular and $C$-maximal coalition models, $[C^*][\varphi] \leftrightarrow \neg[C^x][\neg \varphi]$ is valid. In fact, the proof of theorem 4.3 shows that it is sufficient that $M$ is $C$-maximal and $C$-regular at all non-terminal states, i.e., at all states not in $E_0(\emptyset)$. Finally, to simplify notation even more, we introduce natural abbreviations for total and partial terminal effectivity, writing $[C^*](\emptyset \lor \varphi)$ as $[C^t][\varphi]$ and $[C^x](\emptyset \land \varphi)$ as $[C^p][\varphi]$.

Theorem 3.4 which showed that formulas of Coalition Logic are invariant for bisimulation can be generalized to Extended Coalition Logic.

Theorem 4.7. Bisimilarity implies ECL-equivalence.

Proof. Two additional cases need to be added to the proof of theorem 3.4. Let $M = ((S, \{E_C | C \subseteq N\}), V)$ and $M' = ((S', \{F_C | C \subseteq N\}), V')$ be two coalition models such that $s \equiv s'$. We show that if $M, s \models [C^*][\varphi]$ then $M', s' \models [C^*][\varphi]$. The case for $[C^x][\varphi]$ is dealt with analogously.

Let $X = \varphi^M$, $X' := \{x' \in S' | \exists x \in X : x \equiv x'\}$ and

$$Z := \{z \in S | \forall z' : z \equiv z' \Rightarrow z'F^*_CX'\}.$$
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Now it is sufficient to show that $E^*_C(X) \subseteq Z$, and given the definition of $E^*_C(X)$ as a least fixpoint, it suffices to show that $Z$ is a fixpoint, i.e. that

$$X \cup (E^*_\emptyset(\emptyset) \cap E^*_C(Z)) \subseteq Z.$$  

Supposing that $x \in X$ and for some $x'$ we have $x \equiv x'$, we have $x' \in X' \subseteq F^*_C(X')$. On the other hand, suppose that $x \in (E^*_\emptyset(\emptyset) \cap E^*_C(Z))$ and $x \equiv x'$. Then by bisimulation, there is some $Z'$ such that $x' \in (F^*_\emptyset(\emptyset) \cap F^*_C(Z'))$ and for all $z' \in Z'$ there is some $z \in Z$ such that $z \equiv z'$. But then $Z' \subseteq F^*_C(X')$, and so by monotonicity, $x' \in F^*_\emptyset(\emptyset) \cap F^*_C(F^*_C(X')) \subseteq F^*_C(X')$. \hfill \qed

Axiomatically, we can define extensions of the coalition logics defined in section 3.4. For each of the 2 new modalities, one axiom and one inference rule needs to be added.

**Definition 4.8 (Extended Coalition Logic Axiomatics).** Given the set of players $N$, an extended coalition logic for $N$ is a set of ECL-formulas $\Lambda$ which is a coalition logic and which additionally is closed under the inference rules of figure 4.1 below.

<table>
<thead>
<tr>
<th>Axioms:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\varphi \lor \neg[\emptyset] \bot \land [C][C^<em>]\varphi) \rightarrow [C^</em>]\varphi$</td>
</tr>
<tr>
<td>$[C^<em>]\varphi \rightarrow (\varphi \land ([\emptyset] \bot \lor [C][C^</em>]\varphi))$</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Inference Rules:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\varphi \lor \neg[\emptyset] \bot \land [C]\psi) \rightarrow \psi$</td>
</tr>
<tr>
<td>$[C^*]\varphi \rightarrow \psi$</td>
</tr>
<tr>
<td>$\psi \rightarrow (\varphi \land ([\emptyset] \bot \lor [C]\psi))$</td>
</tr>
<tr>
<td>$\psi \rightarrow [C^*]\varphi$</td>
</tr>
</tbody>
</table>

Figure 4.1: Inference rules for Extended Coalition Logic.

Intuitively, the axiom for $^*$ states that $[C^*]\varphi$ is a fixpoint of the operation $\varphi \lor (\neg[\emptyset] \bot \land [C]X)$ and the fixpoint rule states that $[C^*]\varphi$ is the least such fixpoint. Similarly for $^x$ and the greatest fixpoint.

Let Play*, MaxPlay*, and Ind* be the extended coalition logics with the additional axioms for playability, maximal playability and individualism, respectively. We conjecture that these logics are complete with respect to Play, MaxPlay, and Ind, respectively, but at present we only have the following:

**Theorem 4.9.** Play*, MaxPlay*, and Ind* are sound with respect to Play, MaxPlay, and Ind, respectively.
4.3 Some Meta-Theory

The additional modalities of Extended Coalition Logic do not only allow for a range of applications, they also pose some interesting meta-theoretic questions, some of which we will not be able to answer in the present section. The most notable opportunities for future work concern axiomatization and the complexity of the satisfiability problem, even though chapter 6 will shed some light on the difficulties involved here since eventual goal achievement is in fact very similar to iteration in Game Logic.

We start by considering the relationship between local and global properties. Given that certain conditions such as playability are imposed on the local effectivity function $E$, will these conditions be preserved on the global level by $E^t$? For playability, we shall answer this question affirmatively, also pointing out the link to the game-theoretic concept of strategic normal form.

As might be expected, the additional modalities of Extended Coalition Logic will increase the complexity of the model checking problem, and below we will give a precise upper bound which should be compared to the upper bound obtained for basic Coalition Logic. Furthermore, we take up the issue of coalitional expressiveness: We saw in section 3.7.1 that in extensive games without simultaneous moves, coalitions did not add any expressive power beyond individuals. We will see in section 4.3.3 that when we also consider effectivity in the long run, coalitions do add expressive power even for these games.

4.3.1 Local vs. Global Properties of Ability

The different kinds of coalition frames associated, e.g., with extensive games have been defined in terms of local requirements, i.e., properties such as weak playability which the local effectivity functions have to satisfy. Some of these properties will be maintained globally or terminally. An important case is weak playability: One can show that for games without infinite plays, the total (= partial) terminal effectivity function is playable.

\textbf{Theorem 4.10.} If $\mathcal{F} = (S, \{E_C|C \subseteq N\})$ is a weakly playable coalition frame such that $sE^t_\emptyset S$, then $E^t(s)$ is strongly playable.

\textbf{Proof.} The strong playability conditions can be checked one by one: For the first condition, one can check that $E^t_C(\emptyset) = \emptyset$. For the second condition, $E^t_\emptyset(S) \subseteq E^t_C(S)$ since $E(s)$ is coalition monotonic in case $s \not\in E_\emptyset(\emptyset)$. For $N$-maximality, if $s \not\in E^t_\emptyset(X)$, since $s \in E^t_\emptyset(S)$, we have $s \not\in E^t_\emptyset(X)$ by theorem 4.4 and hence $sE^t_N X$ by theorem 4.3.

For superadditivity, we show generally that for all $C_1 \cap C_2 = \emptyset$ we have $E^t_{C_1}(X_1) \cap E^t_{C_2}(X_2) \subseteq E^t_{C_1 \cup C_2}(X_1 \cap X_2)$; we proceed again by transfinite induction,
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as in the proof of theorem 4.4. Let

\[ F_1(Y) = (E_0(\emptyset) \cap X_1) \cup (E_\emptyset(\emptyset) \cap E_{C_1}(Y)) \]
\[ F_2(Y) = (E_0(\emptyset) \cap X_2) \cup (E_\emptyset(\emptyset) \cap E_{C_2}(Y)) \]
\[ F_3(Y) = (E_\emptyset(\emptyset) \cap X_1 \cap X_2) \cup (E_\emptyset(\emptyset) \cap E_{C_1 \cup C_2}(Y)) \]

We show that \( F_1^{\kappa} \cap F_2^{\kappa} \subseteq F_3^{\kappa} \). The heart of the proof is the inductive step for \( \kappa + 1 \), where one can check that

\[(E_\emptyset(\emptyset) \cap X_1) \cup (E_\emptyset(\emptyset) \cap E_{C_1}(F_1^{\kappa})) \cap ((E_\emptyset(\emptyset) \cap X_2) \cup (E_\emptyset(\emptyset) \cap E_{C_2}(F_2^{\kappa})))\]

is a subset of

\[(E_\emptyset(\emptyset) \cap X_1 \cap X_2) \cup (E_\emptyset(\emptyset) \cap E_{C_1 \cup C_2}(F_3^{\kappa}))\].

The only two cases possible are \( s \in E_\emptyset(\emptyset) \cap X_1 \cap X_2 \) and \( s \in E_\emptyset(\emptyset) \cap E_{C_1}(F_1^{\kappa}) \cap E_{C_2}(F_2^{\kappa}) \). The latter case makes use of the superadditivity of \( E \).

Consequently, for every extensive game \( G \) without infinite plays, there is a nonempty strategic game \( G' \) such that the total terminal effectivity function of \( G \) is the \( \alpha \)-effectivity function of \( G' \). In fact, one such strategic game \( G' \) is simply the strategic normal form of \( G \). As for the logical analogue of theorem 4.10, the preservation of local properties on the global level, observe that the four strong playability conditions for \( E^\kappa \) can be translated into the logical language:

1. \( \neg [C^I] \bot \)
2. \( [C^I] \top \)
3. \( \neg [\emptyset^I] \varphi \rightarrow [N^I] \varphi \)
4. \( ([C_1^I] \varphi_1 \land [C_2^I] \varphi_2) \rightarrow [(C_1 \cup C_2)^I] (\varphi_1 \land \varphi_2) \) where \( C_1 \cap C_2 = \emptyset \)

Theorem 4.10 thus shows that all four axiom schemata are valid for extensive games without infinite plays.

As another example, consider majority voting which we will discuss in more detail in the next chapter. If the game linked to a particular state is a voting game where all of the players can choose between a number of alternatives, we might want to demand that every majority of players can completely determine the outcome. This property can be captured in terms of effectivity: Call an effectivity function \( E : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}(S)) \) majorative iff for every coalition \( C \) with \( |C| > \frac{1}{2}|N| \) we have \( E(N) \subseteq E(C) \).

The notion of majorativity is a local notion which guarantees that at every stage of a voting process, voting is democratic. As one would hope, this property holds for the voting procedure as a whole as well: If \( E \) is majorative at every state of an coalition frame, then so is \( E^I \), so a democratic procedure will maintain democracy overall. That the converse is not true can be gathered from the extensive game in figure 4.2, where at the initial state, \( E^I \) is majorative while \( E \) is not.
4.3. Some Meta-Theory

4.3.2 Model checking

As may be expected, the presence of modalities for eventual goal achievement and goal maintenance leads to a more complex model checking problem. Since these modalities are defined by fixpoint constructions, a model checking algorithm and its complexity will resemble algorithms used for model checking of the modal \( \mu \)-calculus. For the \( \mu \)-calculus, the best known upper bound is \( \text{NP} \cap \text{co-NP} \), while for bounded alternation depth, the problem can be solved in deterministic polynomial time. Since Extended Coalition Logic does not allow for nestings of fixpoint operators which could yield formulas of the form \( \mu X.\nu Y[C]X \land [D]Y \), one can expect model checking to be solvable in polynomial time where the polynomial is of very low degree. This should be compared to the analysis of the model checking problem for Game Logic in section 6.5.

\begin{theorem}
Given a formula \( \varphi \) of Extended Coalition Logic and a coalition model \( \mathcal{M} \), there is an algorithm for calculating \( \varphi^\mathcal{M} \) which runs in time \( O(|\mathcal{M}|^2 \times |\varphi|) \).
\end{theorem}

\begin{proof}
The proof extends the argument used for basic Coalition Logic by two additional cases, eventual goal achievement and goal maintenance: In case \( \varphi_{k+1} = [C^*]\varphi_k \), after determining all the \( \varphi_i^\mathcal{M} \) for \( i \leq k \) in time \( O(k \times |\mathcal{M}|^2) \), we now need to calculate

\[ E^\mathcal{M}_C(\varphi_k^\mathcal{M}) = \mu X.\varphi_k^\mathcal{M} \cup (E_\emptyset(\emptyset) \cap E_C(X)) = \bigcup_{0 \leq i \leq |S|} F^{\uparrow i} \]

where \( F(X) = \varphi_k^\mathcal{M} \cup (E_\emptyset(\emptyset) \cap E_C(X)) \). We can assume that at the very beginning we have marked all states \( s \) according to whether or not \( sE_\emptyset \emptyset \) holds. Initially, we then label all states satisfying \( \varphi_k \) with \([C^*]\varphi_k \). Next, for every state \( t \notin E_\emptyset(\emptyset) \) we check whether there is a set \( X \) with \( tE_C^c X \) such that all states in \( X \) are labeled with \([C^*]\varphi_k \), and if so, we label \( t \) with \([C^*]\varphi_k \) as well. We repeat this step at most \( |S| \) times, and since each step can be done in \( O(|\mathcal{M}|) \) time, \( E^\mathcal{M}_C(\varphi_k^\mathcal{M}) \) can be calculated in \( O(|\mathcal{M}|^2) \), yielding a \( O((k+1) \times |\mathcal{M}|^2) \) bound for calculating \( \varphi_{k+1} \).

In case \( \varphi_{k+1} = [C^\times]\varphi_k \), after determining all the \( \varphi_i^\mathcal{M} \) for \( i \leq k \) in time \( O(k \times \)
we need to calculate

\[ E_C^k(\varphi^M) = \nu X.\varphi^M_k \cap (E_\emptyset(\emptyset) \cup E_C(X)) = \bigcap_{0 \leq i \leq |S|} F^{|i|} \]

where \( F(X) = \varphi^M_k \cap (E_\emptyset(\emptyset) \cup E_C(X)) \).

Initially, we label all states satisfying \( \varphi_k \) with \([C \times] \varphi_k \). Next, for every state \( t \notin E_\emptyset(\emptyset) \) labeled with \([C \times] \varphi_k \), we check whether there is a set \( X \) with \( tE_C^0X \) such that all states in \( X \) are labeled with \([C \times] \varphi_k \), and if not, we remove the \([C \times] \varphi_k \) label from \( t \). Again, this yields an \( O((k+1) \times |M|^2) \) bound for calculating \( \varphi_{k+1} \).

Roughly speaking, while model checking for basic Coalition Logic is linear time, model checking for Extended Coalition Logic is quadratic. Note again that as with basic Coalition Logic, this is a rather rough estimate: Inspecting the proof carefully, the real calculation time is actually \( O(|S| \times |M| \times |\varphi|) \) where \( S \) is the universe of \( M \). Since \( |S| \) is usually much smaller than \( |M| \), we consider the result as an argument for the feasibility of doing model checking in practice, also for Extended Coalition Logic.

4.3.3 Expressiveness

In theorem 3.35 of the previous chapter, we have compared the expressiveness of full Coalition Logic to the expressiveness of its individual fragment. The result stated that while over Mon, Play and MaxPlay full Coalition Logic is more expressive than its individual fragment, this is not the case for Ind. In other words, for extensive games without simultaneous moves, group modalities do not add anything to individual modalities in terms of expressive power. For Extended Coalition Logic, it turns out that even for extensive games without simultaneous moves, coalitional modalities do add expressive power to individual modalities.

\textbf{Theorem 4.12.} Over Mon, Play, MaxPlay and Ind, Extended Coalition Logic is more expressive than its individual fragment, provided that \(|N| > 1\).

\textbf{Proof.} For Mon, Play and MaxPlay, the proof of theorem 3.35 for basic Coalition Logic easily extends to Extended Coalition Logic. For Ind, consider the weakly individualistic coalition model \( \mathcal{M} = ((S, \{E_C|C \subseteq N\}), V) \) with \( S = \{s_i, t_i|i \geq 0\}, \Phi_0 = \emptyset \) and \( N = \{1, 2\} \), which \( \alpha \)-corresponds to the extensive game below:
Every state $s_i$ has only a single successor state so that it is irrelevant whose turn it is at $s_i$. At $t_i$ for $i > 0$ on the other hand, player 1 moves if $i$ is odd, player 2 moves if $i$ is even. As the game indicates, player 1 wins at state $t_0$ whereas player 2 wins at state $s_0$, so the empty games associated with these states are $\{1\}$ and $\{2\}$, respectively. Given these empty games, note that $\mathcal{M}$ is $C$-maximal for all coalitions $C$ at every state.

We will use this model to show that the formula $[N^*][1] \bot$ cannot be expressed in the individual fragment of Extended Coalition Logic. The formula states that the players together have a strategy to achieve a win for player 1. Note that $[N^*][1] \bot$ is true at all $t_i$ and false at all $s_i$. This means that the denotation of $[N^*][1] \bot$ in $\mathcal{M}$ is neither finite nor co-finite since it is true at an infinite number of states and also false at an infinite number of states. To establish our result, we shall show by induction that the denotation of all formulas of the individual fragment of ECL are either finite or co-finite.

The base case and the boolean cases are easy to check, so we only consider the modalities for player 1. For $[1] \varphi$, if $\varphi^M$ is finite, the denotation of $[1] \varphi$ must also be finite, simply because $E_C(X)$ is finite provided that $X$ is, with the proviso that there is only a finite number of terminal states. If $(\neg \varphi)^M$ is finite, then $([1] \varphi)^M = ([2] \neg \varphi)^M$ using maximality and the latter set must again be finite.

Consider now $[1^*] \varphi$ and assume that $\varphi^M$ is finite. We can also assume that $\varphi^M \neq \emptyset$, for otherwise $[1^*] \varphi$ will be false everywhere and hence of finite denotation. We consider two different cases: (1) Suppose first that there is some state $s_c$ such that $\mathcal{M}, s_c \models \varphi$. As a result, $[1^*] \varphi$ will hold at all states to the right of $s_{c+1}$, i.e., for all $d > c + 1$ we have $\mathcal{M}, s_d \models [1^*] \varphi$ and $\mathcal{M}, t_d \models [1^*] \varphi$, the strategy for player 1 being to choose a state in the top row as soon as possible. Consequently, $[1^*] \varphi$ can fail to hold only to the left of $s_{c+1}$ (top or bottom row), hence $(\neg [1^*] \varphi)^M$ is finite. In the second case (2), there is no state $s_c$ such that $\mathcal{M}, s_c \models \varphi$. Let $t_c$ be the rightmost state at which $\mathcal{M}, t_c \models \varphi$ (since $\varphi^M$ is finite such a largest $c$ exists). Then for all $d > c + 1$ we have $\mathcal{M}, s_d \not\models [1^*] \varphi$ and $\mathcal{M}, t_d \not\models [1^*] \varphi$ since player 2 can simply choose a state in the top row as soon as possible. Consequently, $[1^*] \varphi$ can hold only to the left of $t_{c+1}$ (top or bottom row), hence $(1^* \varphi)^M$ is finite.

The final situation to consider is when $\varphi^M$ is co-finite. Then by theorem 4.3, $(\neg [1^*] \varphi)^M = ([2^*] \neg \varphi)^M$, and since $E_C^X(X)$ must be finite in case $X$ is, $([1^*] \varphi)^M$ is co-finite.

The proof of this expressiveness result actually establishes a difference in expressive power in a very strong form: It would have been sufficient to show that there is an ECL-formula which is not equivalent to any formula of the individual fragment of ECL over all coalition models. The proof of theorem 4.12 on the other hand establishes something stronger, namely that there is an ECL-formula and a model such that no formula of the individual fragment is equivalent to that formula in that model. So in a very strong sense, long-term coalitional effectivity
cannot be reduced to individual effectivity.

Note also that the model used in the proof had to be infinite, i.e., for every finite model \( \mathcal{M} \), every coalitional ECL-formula \( \varphi \) is equivalent (on \( \mathcal{M} \)) to a formula of the individual fragment of ECL. The reason is that since \( \mathcal{M} \) is finite, \( \varphi \) can be rewritten into a CL formula \( \varphi' \) which is true at exactly the same states, simply by expanding the fixpoint definition at most \( n \) times, where \( n \) is the number of states in the model. Formula \( \varphi' \) can in turn be rewritten into an equivalent formula \( \varphi'' \) of the individual fragment of CL. Note, however, that this argument does not show that over finite individualistic models, the individual fragment of ECL is equally expressive as full ECL, for \( \varphi'' \) will only be equivalent to \( \varphi \) in \( \mathcal{M} \), not in all models. In fact, we conjecture that also over finite individualistic models, full ECL will be more expressive than its individual fragment.

### 4.4 Alternating Temporal Logic

As shown in [56] on which the following discussion is based, Extended Coalition Logic is closely related to Alternating Temporal Logic (ATL) [3], a generalization of temporal logic for reasoning about open systems. Traditional linear-time and branching-time temporal logics such as LTL and CTL describe closed systems, i.e., systems whose behavior is completely determined by the present state of the system. In open systems on the other hand, the system interacts with the environment and hence the behavior of the system depends on the present state and the behavior of the environment. The picture can be generalized to multi-agent systems where the different agents may represent different components of the system and the environment. An open system is modeled as an alternating transition system:

**Definition 4.13 (Alternating Transition System).** For agents \( N \) and a set of atomic propositions \( \Phi_0 \), an alternating transition system (ATS) is a triple \( \mathcal{M} = (S,V,\delta) \) where \( S \) is a nonempty set of states, \( V : \Phi_0 \to \mathcal{P}(S) \) is the valuation function, and \( \delta : S \times N \to \mathcal{P}(\mathcal{P}(S)) \) models the effectivity of a player at a state. The function \( \delta \) must satisfy the following intersection property: Assume that \( \delta(s,i) = X \) for all \( i \in N \), then for every state \( s \in S \) and all sets \( X_1, \ldots, X_n \) such that \( X_i \in \delta(s,i) \) for all \( i \in N \), we have \( |\bigcap_{i \in N} X_i| = 1 \).

Thus, \( \delta \) essentially associates an individual effectivity function with each state. The only requirement on \( \delta \) is that every set of choices which the agents make determines precisely one resulting state, the new state of the system.

There is a tight connection between alternating transition systems and strongly playable coalition models. Recall that strongly playable coalition models associate a strategic game form with every state. Given an alternating transition system, one can easily define a corresponding strongly playable coalition model: For every state \( s \in S \), associate a strategic game form with \( s \) such that the strategies \( \Sigma_i \) of
player $i$ are all the sets for which she is effective, i.e., $G(s) = (N, \{\Sigma_i | i \in N\}, o, S)$ where $\Sigma_i = \{X_i \subseteq S | X_i \in \delta(s, i)\}$ and $o(X_1, \ldots, X_n) = \bigcap_{i \in N} X_i$. The two models correspond in the sense that at every state, the coalitional effectivity is the same in both models: Extending the individual effectivity $\delta$ of ATSs to coalitions by defining $X \in \delta(s, C)$ iff $X \supseteq \bigcap_{i \in C} X_i$ and for all $i \in C$, $X_i \in \delta(s, i)$, one can show that $X \in \delta(s, C)$ holds in an ATS iff $sE_{C^c}X$ holds in its corresponding coalition model.

Conversely, it is not the case that every strongly playable coalition model corresponds to an alternating transition system. Consider the following 2-player strategic game form:

$$
\begin{array}{c|cc}
  & m & e \\
\hline
m & \text{win} & \text{loss} \\
e & \text{loss} & \text{win}
\end{array}
$$

The game form models, e.g., the coordinated attack problem where two generals have to decide independently when to attack a common enemy. If both attack in the morning ($m$) they will win, and similarly if both attack in the evening ($e$). If they attack at different times however (i.e., they fail to coordinate), they will lose. While this strategic game can be associated to a state in a strongly playable coalition model (where $\text{win}$ and $\text{loss}$ are states), there is no individual effectivity map $S$ which correctly captures the individual effectivity in this game and which satisfies the intersection property. For note that in this game player 1 is $\alpha$-effective for $\{\text{win}, \text{loss}\}$ and so is player 2, and there are no smaller sets for which they are $\alpha$-effective. Since the intersection of $\{\text{win}, \text{loss}\}$ with itself is not a singleton, there can be no map $S$ which correctly captures this game. Consequently, there are strongly playable coalition models which do not correspond to an ATS. (The class of game forms which can be correctly captured by an individual effectivity map $S$ satisfying the intersection property is the class of rectangular game forms.)

Various types of ATSs are discussed in [3]: In a turn-based synchronous ATS, there is a single agent at every state which determines the next state of the system. Since the agent may be different at every state, turn-based synchronous ATSs thus correspond to strongly individualistic coalition models where each state knows a local dictator. In a lock-step synchronous ATS, each state of the system is divided into local states for each agent, and at every state, each agent can determine its next local state, possibly dependent on the current local states of the other agents, but independent of the actions of the other agents. In a turn-based asynchronous ATS finally, there is a designated agent called the scheduler who chooses at every state an agent who gets to determine the next state. Usually the scheduling policy will be subject to various fairness constraints, prohibiting, e.g., that the scheduler assigns the same agent to every state.

The central feature of the language of ATL are its three modalities $\langle C \rangle X \varphi$ (next), $\langle C \rangle G \varphi$ (always) and $\langle C \rangle \varphi U \psi$ (until). The formula $\langle C \rangle X \varphi$ is true at a state in case coalition $C$ is locally effective for $\varphi$, i.e., it corresponds to the Coali-
tion Logic formula $\langle C \rangle \varphi$. The formula $\langle C \rangle G \varphi$ is true at a state in case coalition $C$ has a joint long-term strategy which maintains $\varphi$ in the future. Consequently, $\langle C \rangle G \varphi$ is the temporal analogue of the Extended Coalition Logic formula $[C^x] \varphi$. Finally, $\langle C \rangle \varphi U \psi$ expresses that coalition $C$ has a joint long-term strategy which will guarantee $\psi$ at some point in the future and maintain $\varphi$ until that point.

Coalition Logic and Extended Coalition Logic over strongly playable coalition models form a fragment of ATL. We have seen that the language of (E)CL forms a sublanguage of ATL. Furthermore, while not every strongly playable coalition model corresponds to an ATS, for every strongly playable coalition model there is an ATS satisfying the same formulas of (E)CL. The reason can easily be demonstrated using the earlier coordinated attack game: While that game cannot be modeled by an ATS, the following game can easily be modeled by an ATS:

$$
\begin{array}{cc}
  & m & e \\
 m & \text{win}_1 & \text{loss}_1 \\
e & \text{loss}_2 & \text{win}_2
\end{array}
$$

If we see to it that the states $\text{win}_1$ and $\text{win}_2$ on the one hand and $\text{loss}_1$ and $\text{loss}_2$ on the other hand are identical in terms of observable properties, the game cannot be distinguished from the original coordinated attack game by any formula of the language of CL or ATL. Hence, the axiomatization we gave in section 3.4 can be mapped into an axiomatization for a fragment of ATL. In [56], this axiomatization has been extended to full ATL as well. Note, however, that the coordinated attack example shows that a satisfying ATS may be much larger than a satisfying coalition model, so complexity results regarding the satisfiability problem do not immediately transfer from one logic to the other.

Coalition Logic has a number of advantages over ATL: First, since coalition models allow for terminal states, we can model endpoints of a process explicitly. In an ATS, such terminal states would have to be represented via terminal loops, where all transitions lead to the same state again. The problem with this solution is that it does not allow us to distinguish partial from terminal effectivity anymore, since all runs will be infinite. Second, we saw that coalition models can associate arbitrary strategic games to states while ATSs can only capture rectangular games. In fact, coalition models can even capture interactions which cannot be modeled by strategic games at all, for example situations where coalition monotonicity is violated. Third, while the expressive power of basic and Extended Coalition Logic is weaker than that of ATL, Coalition Logic will be computationally less complex. This means that satisfiability problems which may be intractable for ATL may still be feasible for Coalition Logic. This will certainly be the case for basic Coalition Logic, and as chapter 5 will show, many applications will not require more.
4.5 Distributed Artificial Intelligence

The research area of distributed artificial intelligence has developed various logics for reasoning about multi-agent systems, formalizing not only the ability of an agent but also its beliefs, desires, intentions, and so on. Hence, it is useful to point out some similarities and differences between Coalition Logic and the approach taken in the artificial intelligence literature.

The semantics used here to formalize multi-agent ability is based on minimal models with a neighborhood relation for each agent. For the single agent case, such models have been used in [28] to study the logic of ability. This logic of ability is a very weak modal logic since properties such as $\Diamond(A \lor B) \rightarrow (\Diamond A \lor \Diamond B)$ fail. The example given to illustrate the failure of this principle refers to a deck of cards turned face down. Since the colors (red or black) are concealed, the agent is not able to draw a red card nor a black card, while he is able to draw a card which is either black or red. From our perspective, we interpret the situation as a game against nature, i.e., as a 2-player game where Nature chooses which card to give to the agent. The advantage of this approach is that it makes the roles of the players explicit, and hence one can point out that if the situation were in fact a 1-player game, the modal distribution principle would hold after all.

The approach taken here to formalize the ability of groups of agents differs somewhat from the existing literature on multi-agent ability. Among the earliest works, Tennenholtz and Moses in [119] conceive of an agent as a set of finite state machines whose transitions model the agent’s actions. As in an extensive game with simultaneous moves, the joint actions of the agents determine the new configuration of the system. Of central concern is the cooperative goal achievement (CGA) problem: Is there a run of the system in which all agents achieve their goal? It is argued that this problem is PSPACE-complete, which seems to correspond nicely to the complexity of the satisfiability problem of basic Coalition Logic, but the decision problems are quite different. Note also that our logical approach is more general in that it is not specifically tailored to the CGA problem alone, but also allows to ask, for example, whether an infinite run can be forced by some group of agents.

While the work of Werner in [125] is more directly related in its logical approach, his framework includes much more than just ability, covering also time, intentions, actions and knowledge. Given the more complex aims, his semantics is much more complicated than what is proposed here, and some of the fundamental issues which arise purely on the level of abilities are not investigated, e.g., the relationship between local and global ability, basic cooperative axioms such as superadditivity, and so on.

Related to Werner’s work, Wooldridge and Fisher in [126] also take a logical approach to multi-agent interaction which includes communication between agents. Their notion of goal achievement essentially corresponds to $\alpha$-effectivity. Their logic is very expressive (including “at least” first-order logic) and hence also
much more complex than the rather simple system presented here. Some of their axioms, however, have direct analogues in Coalition Logic. As an example, one of their axioms states that bigger groups cannot achieve less (coalition-monotonicity) which they write as $\forall x \forall y ((\text{Can}(x, \varphi) \land x \subseteq y) \rightarrow \text{Can}(y, \varphi))$ where $x$ and $y$ refer to groups of agents.

Note that our general approach is different from the works cited and from the approach taken in multi-agent systems more generally: Our aim is to provide a formal logical theory of ability in a multi-agent setting, without adding any other notions such as beliefs, intentions, etc. which would complicate the picture. What is more important, we want a general model of ability, and this is what effectivity functions allow us to do. Effectivity in game-like situations (which is taken as basic in the other approaches) is only a special case which can be characterized by certain axioms, precisely the properties of group ability which characterize strategic games. This approach still allows us to model situations which would be beyond the scope, e.g., of [126] because they violate the coalition-monotonicity mentioned. Even our relatively simple model, however, is sufficient to ask many of the questions raised in the literature such as the CGA problem.

### 4.6 Summary

Extended Coalition Logic is a sufficiently expressive yet computationally simple extension of basic Coalition Logic. In this sense, ECL is similar to temporal logics like CTL which are simple and expressive fragments of the highly complex modal $\mu$-calculus. Theorem 4.11 has shown that the complexity of model checking is still relatively low, in particular when compared to Game Logic whose complexity we shall discuss in section 6.5. On the other hand, ECL can express the existence of a winning strategy in an extensive game, one of the central properties we will want to express in applications. Further evidence of the expressive power of ECL is given by theorem 4.12 which shows that even over extensive games of perfect information, the existence of a coalitional winning strategy cannot be reduced to individual strategies. Consequently, there can be no logic for reasoning about individual ability only which is as expressive as ECL.

Of the various long-term ability notions we have considered, we saw that at least for determined extensive games, all can be reduced to eventual goal achievement (theorem 4.3). For reasoning about undetermined games, we also need the other primitive modality $[C^x]$ defined in terms of goal maintenance $E^x$.

Finally, the generality of coalition models has allowed us to obtain a generalization of Dijkstra's relations between partial and total program correctness. These relations state that for programs, total correctness implies partial correctness and that for programs without infinite runs, the converse implication holds as well. Formulated in our framework, these relations amount to the following...
two claims:
\[ E_0^T(X) \subseteq E_0^T(X) \text{ and } E_0^T(S) \cap E_0^T(X) \subseteq E_0^T(X) \]
holds for the class of models 1-Play. Theorem 4.4 has generalized this result from programs to games with/without simultaneous moves, formally, from 1-Play to Play and from the empty coalition \( \emptyset \) to general coalitions \( C \).

4.7 Bibliographic Notes

Parts of the material in this chapter has been published in [103, 102].

For books on Dijkstra's theory of partial and total program correctness, see [38, 39, 73, 7]. For an overview of temporal logic, see [43].

All the results concerning the relation between Coalition Logic and Alternating Temporal Logic are from [56]. ATL has been extended and modified in various ways (also in [3]), yielding ATL* (which mirrors the step from CTL to CTL*), the alternating-time \( \mu \)-calculus, and ATL over ATSSs with incomplete information, where agents have only partial knowledge of the current state of the system. Alternating refinement relations for ATSSs and their computational complexity have been studied in [4] which also contains a notion of alternating bisimulation similar to what was proposed in section 2.5.
1.6 Summary

The logical language ECL is a sufficiently expressive yet computationally simple language. In the sense, ECL is similar to temporal logic. It admits simple and expressive fragments of the highly complex temporal logic. The result shows that the complexity of model checking is sub-exponential in the number of vertices compared to temporal logic whose complexity is even exponential in the number of vertices. The other usual LTL can express the various desirable properties in an imperative form. Hence, one of the essential properties we will concentrate on is to further evidence of the expressive power of ECL. A conclusion of the chapter can be written to reasoning about ECL.

It is also noted that in some cases, the constraints can be knowledge for resolving partial correctness and global correctness in the system. It proves that ECL is very powerful in the formal verification of concurrent systems.