Logic for social software
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Citation for published version (APA):
Appendix A

Fixpoint Facts

Chapters 4 and 6 make use of fixpoint constructions to define long-term ability and iteration, respectively. This appendix recalls some standard results about fixpoints, namely, the Knaster-Tarski fixpoint theorem and the upward and downward hierarchies for fixpoint approximation. The material is standard, with the possible exception of theorem A.2, a generalization of the Knaster-Tarski fixpoint theorem.

Consider any monotonic operation on the nonempty set of states \( S \), i.e., any function \( F : \mathcal{P}(S) \rightarrow \mathcal{P}(S) \) such that \( X \subseteq Y \) implies \( F(X) \subseteq F(Y) \). We say that a set \( Z \subseteq S \) is a fixpoint of \( F \) iff \( F(Z) = Z \). \( Z \) is a least (greatest) fixpoint of \( F \) iff (1) \( Z \) is a fixpoint and (2) \( Z \) is a subset (superset) of every fixpoint of \( F \). Note that least and greatest fixpoints are unique. We denote the least fixpoint of \( F \) as \( \mu X. F(X) \) (the smallest set \( X \) such that \( F(X) = X \)) and the greatest fixpoint of \( F \) as \( \nu X. F(X) \) (the greatest set \( X \) such that \( F(X) = X \)).

For repeated application of the operation \( F \), we define the following downward and upward hierarchies by ordinal induction:

\[
F^\varnothing(X) = X \\
F^{\kappa+1}(X) = F(F^{\kappa}(X)) \\
F^{\lambda}(X) = \bigcup_{\kappa < \lambda} F^{\kappa}(X) \\
F^\varnothing(X) = X \\
F^{\kappa+1}(X) = F(F^{\kappa}(X)) \\
F^{\lambda}(X) = \bigcap_{\kappa < \lambda} F^{\kappa}(X)
\]

where \( \kappa \) and \( \lambda \) are ordinals and \( \lambda \) is a limit ordinal. In most cases, the upward hierarchy will be used for \( X = \emptyset \) and the downward hierarchy for \( X = S \), and for ease of notation, we use \( F^{\kappa} \) for \( F^{\kappa}(\emptyset) \) and \( F^{\kappa} \) for \( F^{\kappa}(S) \). A central result on fixpoints is the well-known Knaster-Tarski fixpoint theorem:

**Theorem A.1 (Tarski [118])**. If \( F : \mathcal{P}(S) \rightarrow \mathcal{P}(S) \) is any monotonic operation, then

1. \( \mu X. F(X) = \bigcap \{ Y \subseteq S | F(Y) = Y \} = \bigcap \{ Y \subseteq S | F(Y) \subseteq Y \} = \bigcup \{ F^{\kappa} \}
\]

where \( \kappa \) ranges over all ordinals of cardinality at most \( |S| \) and \( F^{\varnothing} \subseteq F^{\varnothing} \subseteq F^{\varnothing} \subset \ldots \).
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2. \( \nu X.F(X) = \bigcup \{ Y \subseteq S \mid F(Y) = Y \} = \bigcup \{ Y \subseteq S \mid F(Y) \supseteq Y \} = \bigcap_{\kappa} F^{i\kappa} \), where \( \kappa \) ranges over all ordinals of cardinality at most \( |S| \) and \( F^{i0} \supseteq F^{i1} \supseteq F^{i2} \) ....

In section 6.5, a less well-known generalization of this theorem will allow us to reduce the complexity of a model-checking algorithm substantially. It is an easy consequence of the previous result.

**Corollary A.2 (Emerson & Lei [46]).** If \( F : \mathcal{P}(S) \to \mathcal{P}(S) \) is any monotonic operation, then

1. \( \mu X.F(X) = \bigcup_{\kappa} F^{i\kappa}(X_0) \) for any \( X_0 \subseteq F(X_0) \cap \mu X.F(X) \), where \( \kappa \) ranges over all ordinals of cardinality at most \( |S| \) and \( F^{i0}(X_0) \subseteq F^{i1}(X_0) \subseteq F^{i2}(X_0) \) ....

2. \( \nu X.F(X) = \bigcap_{\kappa} F^{i\kappa}(X_0) \) for any \( X_0 \supseteq F(X_0) \cup \nu X.F(X) \), where \( \kappa \) ranges over all ordinals of cardinality at most \( |S| \) and \( F^{i0}(X_0) \supseteq F^{i1}(X_0) \supseteq F^{i2}(X_0) \) ....

Finally, there are cases in which the fixpoint approximation provided by the upward and downward hierarchies is guaranteed to reach the fixpoint after at most \( \omega \) stages. A well-known sufficient condition for such a closure at \( \omega \) is disjunctivity. As defined in section 2.4.3, an operation \( F : \mathcal{P}(S) \to \mathcal{P}(S) \) is disjunctive iff for all \( V \subseteq \mathcal{P}(S) \) we have \( F(\bigcup_{X \in V} X) = \bigcup_{X \in V} F(X) \). Recall that disjunctivity implies monotonicity and that \( F(\emptyset) = \emptyset \). As an analogue to disjunctivity, call \( F \) conjunctive iff for all \( V \subseteq \mathcal{P}(S) \) we have \( F(\bigcap_{X \in V} X) = \bigcap_{X \in V} F(X) \). Also conjunctivity implies monotonicity and furthermore that \( F(S) = S \).

The following result shows that indeed disjunctivity (conjunctivity) is a sufficient condition for approximating the fixpoint after at most \( \omega \) steps. Note that there are weaker conditions such as continuity which are also sufficient (see, e.g., [37, 91]), but for our purposes the following result is exactly what we need.

**Theorem A.3.** If \( F \) is disjunctive then \( \mu X.F(X) = F^{i\omega} \), and if \( F \) is conjunctive then \( \nu X.F(X) = F^{i\omega} \).

**Proof.** Disjunctivity immediately implies that \( \bigcup_{i<\omega} F^{ii} \) is a fixpoint of \( F \), and given any fixpoint \( Z \) of \( F \) one can show by induction on \( i \) that \( F^{ii} \subseteq Z \) and consequently \( \bigcup_{i<\omega} F^{ii} \subseteq Z \).