Logic and Provability

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Citation for published version (APA):

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In this chapter, we consider Löb's axiom in extensions of $\vdash_V$. [Vis81] axiomatized the consequence relation $\vdash_F$ of formal propositional logic by adding Löb's inference rule $(T \supset A) \supset A$ to $\vdash_V$. $\vdash_F$ is also obtained by adding Löb's axiom $((T \supset p) \supset p) \supset (T \supset p)$ to $\vdash_V$ (cf. [AR99] and [SWZ98]). However, most of the extensions of $\vdash_V$ obtained by adding an inference rule to $\vdash_V$ cannot be obtained by adding the corresponding axiom to $\vdash_V$. For instance, the consequence relation $\vdash_I$ of intuitionistic propositional logic is obtained by adding the inference rule $T \supset A$ to $\vdash_V$, while it cannot be obtained by adding the axiom $(T \supset p) \supset p$ to $\vdash_V$. So, it is natural to ask what axiomatizations have such a property as the axiomatization by Löb's axiom (or inference rule), and what extensions have an axiomatization with this property. Here we consider this problem. We prove that if an extension has an axiomatization with the property, then so does every axiomatization of the extension, and that the maximum one among such extensions is $\vdash_F$. We end up with some other results about extensions with the property.

4.1 Extensions of $\vdash_V$

There are two possible axiomatic ways to extend a consequence relation $\vdash_L$; one is by adding an axiom, and the other by adding an inference rule. First, we define extensions of $\vdash_V$ in these two different ways.

4.1.1. Definition. By $\vdash_{L+A}$, we mean the consequence relation obtained by adding an axiom $A$ to $\vdash_L$. By $\vdash_{L+A/B}$, we mean the consequence relation obtained from $\vdash_L$ by adding an inference rule $\frac{A}{B}$, where $A$ and $B$ are schemas obtained from formulas $A$ and $B$ by replacing all the propositional variables $a_i$ occurring in $A$.
There are two important extensions of $\vdash_V$. One is the consequence relation $\vdash_F$ of formal propositional logic, and the other the consequence relation $\vdash_I$ of intuitionistic propositional logic. These extensions are obtained from $\vdash_V$ by adding Löb's inference rule

$$LR(A) = \frac{(T \supset A) \supset A}{T \supset A}$$

and the rule of modus ponens

$$\frac{T \supset A}{A},$$

respectively, and so, they are expressed as follows:

$$\vdash_F = \vdash_V + LR(p),$$

$$\vdash_I = \vdash_V + T \supset p / p.$$

[AR99] showed that $\vdash_F$ is also obtained by adding Löb's axiom

$$L(p) = ((T \supset p) \supset p) \supset (T \supset p)$$

to $\vdash_V$. In other words,

$$\vdash_F = \vdash_V + L(p).$$

Hence, we have

4.1.2. Lemma.

$$\vdash_F = \vdash_V + L(p) = \vdash_V + LR(p)$$

On the other hand, considering the intermediate propositional logics, we immediately have

$$\vdash_{I + A/B} = \vdash_{I + A \supset B}.$$

So, Lemma 4.1.2 seems to be obvious. However, considering the extensions of $\vdash_V$, it is not obvious. There is a pair of extensions of $\vdash_V$ such that

$$\vdash_{V + A/B} \neq \vdash_{V + A \supset B}.$$

For instance, we can show
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4.1.3. Lemma. ([SWZ98])

$$\vdash_I \vdash_V \neg (\top \vdash p) \vdash p$$

Proof. We note that every implication is true at $\alpha$ in $\langle \{\alpha\}, \emptyset, P \rangle$ for any $P$. Let it be that $P(p) = \emptyset$. Then we have $\langle \{\alpha\}, \emptyset, P, \alpha \rangle \models \langle (\top \vdash A) \supset A, \top \vdash p \rangle$ for any $A$, and $\langle \{\alpha\}, \emptyset, P, \alpha \rangle \not\models p$.

So, we may well say that Löb's axiom or rule has a nice property. Also it is natural to ask what consequence relations can be axiomatized by adding an axiom or a rule with such property as Löb's one has. In this chapter, we consider this problem. In other words, we investigate the set of consequence relations

$$\mathcal{R} = \{ \vdash \models \vdash_V \neg A/B = \vdash_V \neg A \supset B, \text{ for some } A, B \in \text{WFF} \}.$$  

First, we show some examples of consequence relations in $\mathcal{R}$ and not in $\mathcal{R}$. We immediately confirm

$$\vdash_V \in \mathcal{R} \text{ and } \vdash_F \in \mathcal{R}.$$  

Using the same proof as for $\vdash_F \in \mathcal{R},$

$$\vdash_V \neg L_R(\alpha) \in \mathcal{R}$$

is also true for any formula $A$. However,

$$\vdash_I \not\in \mathcal{R}$$

is not clear. It is true that

$$\vdash_I = \vdash_V \top \vdash p \not\models \vdash_V \neg (\top \vdash p) \vdash p,$$

but there might exist another axiomatization $\vdash_V \neg A/B$ for $\vdash_I$ such that

$$\vdash_I = \vdash_V \neg A/B = \vdash_V \neg A \supset B.$$  

From this, we note that it is not easy to give an example of a consequence relation not in $\mathcal{R}$. We prove the following theorem in order to give such examples.

4.1.4. Theorem. $\vdash_V \neg A/B \in \mathcal{R}$ iff $\vdash_V \neg A/B = \vdash_V \neg A \supset B$.

Since some previous papers gave useful results, there are several possibilities to prove the theorem. We can use the proof of Theorem 1.9 in [Vis81], Proposition 4.1.4 in [AR99] or sequent system GVPL$^+$ for $\vdash_V$ introduced in chapter 2. Here we use the system GVPL$^+$, because it is useful not only for the proof of Theorem 4.1.4 but also for other results, which will be described below.

We also introduce extensions of GVPL$^+$. 
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4.1.5. DEFINITION. By

$$\text{GVPL}^+ + A_1, \ldots, A_n \rightarrow A_0,$$

we mean the system obtained by adding the new axiom $A_1, \ldots, A_n \rightarrow A_0$ to $\text{GVPL}^+$, where each $A_i$ is a schema obtained from $A_i$ by substituting all the propositional variables $a_{i,j}$ occurring in $A_i$ by formulas $B_{i,j}$, respectively.

For brevity’s sake, we write $\text{GVPL}^+ + A$ instead of $\text{GVPL}^+ + A \rightarrow A$.

4.1.6. COROLLARY.

(1) $\Sigma \vdash_{L+A} f(\Lambda)$ iff $\Lambda \vdash \text{GVPL}^+ + A$,

(2) $\Sigma \vdash_{V+A/B} f(\Lambda)$ iff $\Lambda \vdash \text{GVPL}^+ + A \rightarrow B$.

4.1.7. LEMMA. $\Sigma \vdash_{V+A/B} f(\Lambda)$ iff $\Sigma \vdash \text{GVPL}^+ + (A \supset B)^+$.

Proof. By Corollary 4.1.6, it is sufficient to show

$$\Sigma \vdash \Lambda \in \text{GVPL}^+ + A \rightarrow B \text{ iff } \Sigma \vdash \Lambda \in \text{GVPL}^+ + (A \supset B)^+.$$

The following two proof figures in $\text{GVPL}^+ + A \rightarrow B$ and in $\text{GVPL}^+ + (A \supset B)^+$ convince us of the equivalence:

$$
\begin{array}{c}
A \rightarrow B \\
\hline
\rightarrow (A \supset B)^+ \\
\end{array}
\begin{array}{c}
A \rightarrow A \\
B \rightarrow B \\
\hline
(A \supset B)^+, A \rightarrow B \\
\end{array}
\frac{}
A \rightarrow B
$$

Let $X$ be a formula in $\text{WFF}^+$. By $\text{Subst}(X)$, we mean the set of formulas obtained from $X$ by substituting each propositional variable in $X$ by a formula in $\text{WFF}$.

Now, we prove Theorem 4.1.4.

Proof of Theorem 4.1.4. The “if” part is obvious. We show the “only if” part. Suppose that $\vdash_{V+A/B} \in \mathcal{R}$. Then there exist formulas $C$ and $D$ such that

$$\vdash_{V+A/B} = \vdash_{V+C/D} = \vdash_{V+C \supset D}.$$ 

So, we have $\{C\} \vdash_{V+A/B} D$. Using Lemma 4.1.7,

$$C \rightarrow D \in \text{GVPL}^+ + (A \supset B)^+.$$
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Hence, there exist \((A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+ \) such that

\[
(A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+, C \rightarrow D \in \text{GVPL}^+.
\]

Using \((\to \supset)\),

\[
A_1 \supset B_1, \cdots, A_n \supset B_n \rightarrow C \supset D \in \text{GVPL}^+.
\]

Since \(A_1 \supset B_1, \cdots, A_n \supset B_n \in \text{Subst}(A \supset B)\),

\[
C \supset D \in \text{GVPL}^+ + A \supset B.
\]

Using Lemma 4.1.7,

\[
\emptyset \vdash_{V+A\supset B} C \supset D.
\]

Hence,

\[
\vdash_{V+A/B} = \vdash_{V+C \supset D} \subseteq \vdash_{V+A\supset B}.
\]

On the other hand, it is easily seen that

\[
\vdash_{V+A/B} \supset \vdash_{V+A\supset B}.
\]

Hence, we obtain the theorem.

From the theorem above, we have

\[
\vdash \not \in \mathcal{R}.
\]

We also have the following lemma in a way similar to the proof of Theorem 4.1.4.

4.1.8. LEMMA. \( \vdash_{V+A\supset B} \in \mathcal{R} \) if and only if \( \vdash_{V+A/B} = \vdash_{V+A\supset B} \).

Proof. The outline of the proof is similar to the proof of Theorem 4.1.4. All we have to do is to show that

\[
A_1 \supset B_1, \cdots, A_n \supset B_n \rightarrow C \supset D \in \text{GVPL}^+
\]

implies

\[
(A_1 \supset B_1)^+, \cdots, (A_n \supset B_n)^+, C \rightarrow D \in \text{GVPL}^+.
\]

If \(C \supset D = A_i \supset B_i\), then this is obtained by \((\supset \rightarrow)\) and \((T \rightarrow)\), if not, it is derived from Lemma 4.1.6.

4.1.9. COROLLARY. \( \mathcal{R} = \{ \vdash | \vdash = \vdash_{V+A} = \vdash_{V+T/A}, \text{ for some } A \in \text{WFF} \} \).

Proof. It is sufficient to note that \( \vdash_{V+A} = \vdash_{V+T/A} \).
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4.2 The maximum in \( \mathcal{R} \)

Our main theorem in this section is

4.2.1. Theorem. \( \vdash_F \) is the maximal consequence relation in \( \mathcal{R} \).

In order to prove the theorem above, we provide some preparations.

4.2.2. Lemma. Let \( \alpha \) be a world in a Kripke model \( M = (W, R, P) \) and let it be that \( \{ A \} \vdash_{V+A \supset B} B \). If \( (M, \alpha) \not\models C \) for some \( C \in \text{Subst}(A \supset B) \), then there exists a world \( \beta \in \alpha^\uparrow \) such that \( (M, \beta) \not\models D \) for some \( D \in \text{Subst}(A \supset B) \).

Proof. Suppose that \( (M, \alpha) \not\models A^* \supset B^* \) for some \( A^* \supset B^* \in \text{Subst}(A \supset B) \). So, there exists a world \( \beta \in \alpha^\uparrow \) satisfying the condition

\( (1) \ (M, \beta) \models A^* \) and \( (M, \beta) \not\models B^* \).

By \( \{ A \} \vdash_{V+A \supset B} B \), we have \( \{ A^* \} \vdash_{V+A \supset B} B^* \). So, for any finite set \( \Sigma \subseteq \text{Subst}(A \supset B) \),

\( (M, \beta) \models \{ A^* \} \cup \Sigma \) implies \( (M, \beta) \models B^* \).

Using (1), we have

\( (M, \beta) \not\models D \) for some \( D \in \Sigma \subseteq \text{Subst}(A \supset B) \).

So, we obtain the lemma.

4.2.3. Lemma. \( \{ A \} \vdash_{V+A \supset B} B \) implies \( \emptyset \vdash_F A \supset B \).

Proof. Suppose that

\( \{ A \} \vdash_{V+A \supset B} B \) and \( \emptyset \not\vdash_F A \supset B \).

[Vis81] showed that for any finite set \( \Sigma \) and for any \( A \), the following two conditions are equivalent:

(i) \( \Sigma \vdash_F C \),

(ii) for any finite irreflexive Kripke model \( M = (W, R, P) \) and for any \( \alpha \in W \), \( (M, \alpha) \models \Sigma \) implies \( (M, \alpha) \models \Sigma \).

So, by \( \emptyset \not\vdash_F A \supset B \), there exists a finite irreflexive Kripke model \( M = (W, R, P) \) and \( \alpha \in W \) satisfying the condition

\( (M, \alpha) \not\models A \supset B \).

Since \( M \) is finite, we can take \( n \) as the number of worlds in \( W \).

Let it be that \( \gamma \in W \). By \( C(\gamma) \), we mean the condition

\( (M, \gamma) \not\models D \), for some \( D \in \text{Subst}(A \supset B) \).
4.2. The maximum in $\mathcal{R}$

We note that

(1) the condition $C(\alpha)$ holds.

Let $\beta$ be a world in $W$. Using Lemma 4.2.2,

(2) if $C(\beta)$ holds, then there exists a world $f(\beta) \in \beta$ satisfying $C(f(\beta))$.

Using (1) and (2) $n$ times, we obtain the sequence

$$\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n,$$

where $\alpha_0 = \alpha, \alpha_{k+1} = f(\alpha_k)$ and $C(\alpha_k)$. Since $W$ has only $n$ worlds, there exists a pair $(i, j)$ such that $0 \leq i < j \leq n$ and $\alpha_i = \alpha_j$. On the other hand, from $f(\beta) \in \beta$, we have $\alpha_k R \alpha_{k+1}$. So, using transitivity of $M$, we have $\alpha_i R \alpha_j$. Hence, $\alpha_i R \alpha_j$. This is in contradiction with the irreflexivity of $M$.  

4.2.4. LEMMA. $\vdash \in \mathcal{R}$ implies $\vdash \subseteq \vdash_F$.

Proof. Suppose that $\vdash \in \mathcal{R}$. So, there exist formulas $A$ and $B$ such that

$$\vdash = \vdash_{V+A/B} = \vdash_{V+A \supset B}.$$

Since $\{A\} \vdash_{V+A/B} B$, we have $\{A\} \vdash_{V+A \supset B} B$. Using Lemma 4.2.3, $\emptyset \vdash_F A \supset B$.

So, $\vdash_{V+A \supset B} \subseteq \vdash_F$. Hence, $\vdash \subseteq \vdash_F$.  

Now, Theorem 4.2.1 follows from Lemma 4.1.2 and Lemma 4.2.4.

4.2.5. COROLLARY.

(1) $\mathcal{R} \subseteq \{\vdash \vdash_{V+} \vdash \vdash_F\}$,

(2) min $\mathcal{R} = \vdash_V$,

(3) max $\mathcal{R} = \vdash_F$.

Although one might conjecture that the converse of Lemma 4.2.4 also holds, the following lemmas provide counterexamples.

4.2.6. LEMMA. Let it be that $B = ((\top \supset p) \supset p) \lor L(p)$. Then

$$\vdash_{V+B} \subseteq \vdash_F \text{ and } \vdash_{V+B} \not\in \mathcal{R}.$$  

Proof. Since $\emptyset \vdash_F L(p)$, we have $\vdash_{V+B} \subseteq \vdash_F$. Let it be that $W = \{\alpha, \beta, \gamma\}, R = \{\alpha, \beta\}, (\alpha, \gamma), (\beta, \beta\}, P(p) = \emptyset, M = (W, R, P)$.

We can easily check that

$$\vdash \text{ Subst}(\top \supset B), \vdash \top \text{ and } (M, \alpha) \not\in B.$$
So, we have

\[[\{T\} \cup \text{Subst}(T \supset B)] \not\vdash B.\]

Hence, \(\vdash_{V+T} \not\vdash_{V+T/B} B\). Using Theorem 4.1.4, we obtain \(\vdash_{V+T/B} \not\in \mathcal{R}\). Hence, \(\vdash_{V+T/B} \not\in \mathcal{R}\).

Similarly, we have the following example with \(A\) and \(B\) having only the connective \(\supset\).

**4.2.7. Lemma.** \(\vdash_{V+A/B} \supset F\) and \(\vdash_{V+A/B} \not\in \mathcal{R}\), where \(A = ((T \supset p) \supset p) \supset q, B = (L(p) \supset q) \supset (T \supset q)\).

### 4.3 Kripke semantics for extensions of \(\vdash_V\)

In section 4.2, we obtained that

\[\{\vdash_{V+L(A)} | A \in \text{WFF}\} \subseteq \mathcal{R}.\]

Also we note that every examples of consequence relations in \(\mathcal{R}\) in the previous sections can be axiomatized as \(\vdash_{V+L(A)}\) for some \(A\). In addition, the maximal consequence relation of \{\(\vdash_{V+L(A)} | A \in \text{WFF}\}\} is \(\vdash_F\) and the minimum one is \(\vdash_V\). So, it is natural to conjecture that

\[\{\vdash_{V+L(A)} | A \in \text{WFF}\} = \mathcal{R}.\]

Using Proposition 4.1.21 in [AR99], we obtain

\[\vdash_{V+L(A)} \vdash_{V+A}\]

if \(T \supset A \vdash_V A\). So, if we can prove

\[\vdash_{V+A} \in \mathcal{R} \text{ implies } T \supset A \vdash_V A \cdots (1),\]

then the conjecture is trivial. However, it is difficult to show (1). It is true that if \(\vdash_{V+A} \in \mathcal{R}\), then \(\vdash_{V+T \supset A} A\), and so,

\[\{T \supset A_1, \cdots, T \supset A_n\} \vdash_V A\]

for some substitution instances \(A_1, \cdots, A_n\) of \(A\), but it does not mean

\[\{T \supset A\} \vdash_V A.\]

In this section, we do not give the answer to the conjecture above. We consider relations between \(\mathcal{R}\) and finite Kripke models and show the difficulty to give a counterexample of the conjecture.

The main theorem in this section is
4.3. **Theorem.** Let it be that $\vdash_{V+A} \in \mathcal{R}$. Then for any finite Kripke model $M$,

$$M \models A \iff M \models L(A).$$

The theorem says that for no $\vdash_{V+A} \in \mathcal{R}$, there exists a finite Kripke model that distinguishes $A$ from $L(A)$ even if $\vdash_{V+A}$ does not equal $\vdash_{V+L(A)}$. So, it is difficult to give an example $\vdash \in \mathcal{R}$ such that $\vdash \neq \vdash_{V+L(A)}$ for any $A$.

In order to prove this theorem, we provide some preparations.

4.3.2. **Notation.** Let $M = \langle W, R, P \rangle$ be a Kripke model. For any $\alpha \in W$, we put

- $R_\alpha = R \cap (\alpha \times \alpha)$,
- $P_\alpha(\alpha) = P(\alpha) \cap \alpha$,
- $M_\alpha = \langle \alpha, R_\alpha, P_\alpha \rangle$.

4.3.3. **Lemma.** Let $\alpha$ be a world in $W$ and let $\beta$ be a world in $\alpha^\uparrow$. Then for any formula $A$,

$$\langle (W, R, P), \beta \rangle \models A \iff \langle (\alpha, R, P_\alpha), \beta \rangle \models A.$$  

4.3.4. **Lemma.** Let it be that $\emptyset \vdash_{V+TDA} A$ and let $M = \langle W, R, P \rangle$ be a finite Kripke model. If $M \not\models A$, then there exists a substitution instance $A_1 \in \text{Subst}(A)$ and worlds $\alpha \in W$ and $\beta \in \alpha^\uparrow$ such that

1. $\beta R \beta$,
2. $(M, \beta) \not\models A_1$,
3. for every $\gamma \in \alpha^\uparrow$, $\beta \not\models \gamma^\uparrow$ implies $(M, \gamma) \not\models A_1$.

**Proof.** We use an induction on the number $\#(W)$ of elements in $W$.

**Basis** ($\#(W) = 1$): We can put $W = \{\alpha\}$. By $\emptyset \vdash_{V+TDA} A$, there exist $A_1, \ldots, A_n \in \text{Subst}(A)$ such that $\{T \supset A_1, \ldots, T \supset A_n\} \vdash V A$. Using $(M, \alpha) \not\models A$, we have $(M, \alpha) \not\models T \supset A_i$ for some $i = 1, \ldots, n$. Without loss of generality, we assume that $i = 1$. Then there exists $\beta \in \alpha^\uparrow = \{\alpha\}$ such that $(M, \beta) \not\models A_1$. Since $\beta = \alpha$, we obtain (1), (2) and (3).

**Induction step** ($\#(W) > 0$): Suppose that the lemma holds for any $W^*$ such that $\#(W^*) < \#(W)$. Similarly as in the Basis, there exists $A_1 \in \text{Subst}(A)$ and $\beta \in \alpha^\uparrow$ such that $(M, \beta) \not\models A_1$.

If $\alpha \neq \beta$, then $\#(\beta^\uparrow) < \#(W)$. By Lemma 4.3.3, $(M_\beta, \beta) \not\models A_1$. Also by $A_1 \in \text{Subst}(A)$ and $\emptyset \vdash_{V+TDA} A$, we have $\emptyset \vdash_{V+TDA} A_1$. So, using the induction hypothesis, there exists a substitution instance $A_2 \in \text{Subst}(A_1) \subseteq \text{Subst}(A)$ and worlds $\beta_1 \in \beta^\uparrow \subseteq \alpha^\uparrow$ and $\beta_2 \in \beta_1^\uparrow$ such that

4. $\beta_2 R \beta_2$, 

If $\alpha = \beta$, then $\#(\beta^\uparrow) = \#(W)$.

In this case, we have $\emptyset \vdash_{V+TDA} A$, and by Lemma 4.3.3, $(M_\beta, \beta) \not\models A_1$. Also by $A_1 \in \text{Subst}(A)$ and $\emptyset \vdash_{V+TDA} A$, we have $\emptyset \vdash_{V+TDA} A_1$. So, using the induction hypothesis, there exists a substitution instance $A_2 \in \text{Subst}(A_1) \subseteq \text{Subst}(A)$ and worlds $\beta_1 \in \beta^\uparrow \subseteq \alpha^\uparrow$ and $\beta_2 \in \beta_1^\uparrow$ such that

4. $\beta_2 R \beta_2$, 

**Conclusion**: The theorem holds for any finite Kripke model $M$. 

**Remark**: The theorem is a special case of the more general Kripke completeness theorem for modal logics. The proof involves a careful construction of a model $M$ that satisfies the conditions required by the theorem. The theorem is a powerful tool for understanding the expressive power of modal logics and their models.
(5) \((M, \beta_2) \not\models A_2\),
(6) for every \(\gamma \in \beta_1\), \(\beta_2 \not\subseteq \gamma\) implies \((M, \gamma) \models A_2\).

By Lemma 4.3.3 and (5), we have \((M, \beta_2) \not\models A_2\). Hence, we obtain the lemma.

If \(\alpha = \beta\) and (3) holds, then we also obtain the lemma.

So, we assume that \(\alpha = \beta\) and that (3) does not hold. Then there exists \(\gamma \in \alpha\) such that \(\beta \not\subseteq \gamma\) and \((M, \gamma) \not\models A_1\). Since \(\beta \in \alpha\) and \(\beta \not\subseteq \gamma\), we have \(\alpha \neq \gamma\). So, we have \(#(\gamma) < #(W)\). Hence, we obtain the lemma as in the proof of the case that \(\alpha \neq \beta\).

4.3.5. Lemma. Let it be that \(\emptyset \vdash_{V+\Sigma A} A\) and let \(M = \langle W, R, P \rangle\) be a finite Kripke model. If \(M \not\models A\), then \(M \not\models L(A_1)\) for some \(A_1 \in \text{Subst}(A)\).

Proof. By Lemma 4.3.4, there exists a substitution instance \(A_1 \in \text{Subst}(A)\) and worlds \(\alpha \in W\) and \(\beta \in \alpha\) such that

(1) \(\beta R \beta\),
(2) \((M, \beta) \not\models A_1\),
(3) for every \(\gamma \in \alpha\), \(\beta \not\subseteq \gamma\) implies \((M, \gamma) \models A_1\).

Let \(\gamma\) be a world in \(\alpha\). By (3), if \(\beta \not\subseteq \gamma\), then \((M, \gamma) \models A_1\). By (2), if \(\beta \in \gamma\), then \((M, \gamma) \models T \supset A_1\).

Hence, we have either \((M, \gamma) \not\models T \supset A_1\) or \((M, \gamma) \models A_1\) for any \(\gamma \in \alpha\), which means \((M, \alpha) \models (T \supset A_1) \supset A_1\). On the other hand, by (2) and \(\alpha R \beta\), we have \((M, \alpha) \not\models T \supset A_1\). Hence, \((M, \alpha) \not\models L(A_1)\).

4.3.6. Corollary. Let it be that \(\emptyset \vdash_{V+\Sigma A} A\) and let \(M = \langle W, R, P \rangle\) be a finite Kripke model. Then

\[ M \models \text{Subst}(L(A)) \implies M \models A. \]

Now, Theorem 4.3.1 follows Corollary 4.3.6 and \(\{A\} \vdash V L(A)\).

4.4 Cut-elimination theorem

In section 2.4, sequent system GFPL\(^+\) for formal propositional logic was introduced. The cut-elimination theorem for the system was proved using the method in [Val83]. Here we give another proof of the theorem using a property of L"ob's axiom\(^1\).

\(^1\)Using the same method, [Sas91] gives a proof of the cut-elimination theorem of GL.
4.4. Cut-elimination theorem

4.4.1. Definition. The expression $T^n A$ is defined inductively as follows:

1. $T^0 A = A$,
2. $T^{k+1} A = T \supset T^k A$.

Also the expression $(T^{k+1} A)^+$ denotes $T \supset T^k A$.

By Corollary 4.1.9, it is true that $\vdash_{\forall + \supset \Omega(p)} L(p)$, but Löb's axiom has the following stronger property.

4.4.2. Lemma. $T^n L(A) \rightarrow L(A) \in GVPL^+$, for any $n \geq 0$.

Proof. If $n = 0$, then the lemma is obvious. Suppose that $n > 0$ and $T^{n-1} L(A) \rightarrow L(A) \in GVPL^+$. It is easily seen that $T^1 L(A) \rightarrow L(A) \in GVPL^+$. On the other hand, by the following figure, we have that for any $k \geq 0$,

$T^{k+1} L(A) \rightarrow T^k L(A) \in GVPL^+$ implies $T^{k+2} L(A) \rightarrow T^{k+1} L(A) \in GVPL^+$:

\[
\begin{align*}
T \rightarrow T, & \\
T, T^{k+1} L(A) \rightarrow T^k L(A) & \\
T, (T^{k+2} L(A))^+ \rightarrow T^k L(A) & \\
T^{k+2} L(A) \rightarrow T^{k+1} L(A)
\end{align*}
\]

Hence, we have $T^n L(A) \rightarrow T^{n-1} L(A) \in GVPL^+$. Using the induction hypothesis and cut, we obtain the lemma.

By Lemma 2.4.2 and Lemma 4.4.2, we have

4.4.3. Lemma. For any $n \geq 0$,

$\Gamma \rightarrow \Delta \in GFPL^+$ iff $\Gamma \rightarrow \Delta \in GVPL^+ + T^n L(p)$.

Our main purpose in this section is to give another proof to the following theorem using the lemma above (cf. Theorem 2.4.3). The method in this section is also useful in chapter 6.

4.4.4. Theorem. If $\Gamma \rightarrow \Delta \in GFPL^+$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$.

In order to prove the theorem above, we provide some preparations.

4.4.5. Definition. By $GFPL^*$, we mean the system obtained from $GFPL^+$ by adding the inference rule ($\rightarrow \supset$) in $GVPL^+$. 

4.4.6. Definition. Let $P$ be a cut-free proof figure in GFPL*. We define $\text{dep}_I(P)$ as follows:

1. $\text{dep}_I(D \rightarrow D) = \text{dep}_I(\bot \rightarrow) = 0,$
2. $\text{dep}_I\left(\frac{P_1}{\Gamma \rightarrow \Delta} \text{ and } \frac{P_2}{\Gamma \rightarrow \Delta} \right) = \max\{\text{dep}_I(P_1), \text{dep}_I(P_2)\},$
3. $\text{dep}_I\left(\frac{\Gamma_1 \rightarrow \Delta_1}{\Gamma \rightarrow \Delta}\right) = \begin{cases} \text{dep}_I(P_1) + 1 & \text{if } \frac{\Gamma_1 \rightarrow \Delta_1}{\Gamma \rightarrow \Delta} \text{ is either } (\rightarrow \top) \text{ or } (\rightarrow \bot) \\ \text{dep}_I(P_1) & \text{otherwise.} \end{cases}$

4.4.7. Notation. We put

$$\text{Sub}^+(A_1, \cdots, A_n \rightarrow \Delta) = \bigcup_{1 \leq i \leq n} \text{Sub}^+(A_i) \cup \text{Sub}^+(f(\Delta)).$$

4.4.8. Lemma. Let $\Sigma_1$ and $\Sigma_2$ be finite sets of formulas in WFF and let $P$ be a cut-free proof figure for

$$\{\top^n A \mid A \in \Sigma_1\}, \{\top^{n+1} B \mid B \in \Sigma_2\}, \Gamma \rightarrow \Delta$$

in GFPL*, where $n \geq 1$. If $\text{dep}_I(P) < n$ and $(\Sigma_1 \cup \Sigma_2) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in GFPL*.

Proof. We use an induction on $P$.

Basis ($P$ is an axiom): By $(\Sigma_1 \cup \Sigma_2) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset$, $\Gamma \rightarrow \Delta$ is an axiom, and so, we obtain the lemma.

Induction step ($P$ is not axiom): Suppose that the lemma holds for any proper subfigure of $P$. Since $P$ is not axiom, there exists an inference rule $I$ that introduces the end sequent of $P$. We show only the following two typical cases.

The case that $I$ is $(\rightarrow \top)$: We have $0 < \text{dep}_I(P) < n$ and $P$ is of the form

$$P_1 \left\{ C, C \supset D, \{\top^n A \mid A \in \Sigma_1\}, \{\top^{n+1} B \mid B \in \Sigma_2\}, \Gamma^+ \rightarrow D \right\}$$

Another expression of the upper sequent of $I$ is

$$C, C \supset D, \{\top^{(n-1)+1} B \mid B \in \Sigma_1 \cup \{\top B' \mid B' \in \Sigma_2\}\}, \Gamma^+ \rightarrow D$$

Since $0 < \text{dep}_I(P) < n$, we have $n \geq 2$, and so, $n-1 \geq 1$. By $(\Sigma_1 \cup \Sigma_2) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset$, we have

$$(\Sigma_1 \cup \{\top B \mid B \in \Sigma_2\}) \cap \text{Sub}^+(C, C \supset D, \Gamma^+ \rightarrow D) = \emptyset.$$
Also we have
\[ \text{dep}_1(P_1) = \text{dep}_f(P) - 1 < n - 1. \]
So, by the induction hypothesis, there exists a cut-free proof figure for
\[ C, C \supset D, \Gamma^+ \rightarrow D. \]

Using \((-\supset f)\), we obtain the lemma.

The case that the principal formula of \( I \) is \((T^{n+1}B_1)^+\) for some \( B_1 \in \Sigma_2: P \)
is of the form
\[
\begin{array}{c}
P_1 \{ \Sigma_1, \Sigma_2, \Gamma \rightarrow \top \}
\end{array}
\begin{array}{c}
P_2 \{ \Sigma_1^*, \Sigma_2^*, \Gamma \rightarrow \Delta \}
\end{array}
\]

where \( \Sigma_1^* = \{ T^n A \mid A \in \Sigma_1 \} \) and \( \Sigma_2^* = \{ (T^{n+1}B)^+ \mid B \in \Sigma_2 - \{ B_1 \} \} \). Another
expression of the right upper sequent of \( I \) is
\[
\{ T^n A \mid A \in \Sigma_1 \cup \{ B_1 \} \}, \{ (T^{n+1}B)^+ \mid B \in \Sigma_2 - \{ B_1 \} \}, \Gamma \rightarrow \Delta.
\]

By \((\Sigma_1 \cup \Sigma_2) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset\), we have
\[
(\Sigma_1 \cup \{ B_1 \} \cup (\Sigma_2 - \{ B_1 \})) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset.
\]

Also we have
\[ \text{dep}_1(P_2) \leq \text{dep}_f(P) < n. \]
So, by the induction hypothesis, there exists a cut-free proof figure for
\[ \Gamma \rightarrow \Delta. \]

4.4.9. Notation. By \( \mathcal{P}(A \supset B) \), we mean the set of each cut-free proof figure
\( P \) such that the inference rule introducing the end sequent of \( P \) is either \((-\supset)\)
or \((-\supset f)\) and its principal formula in the succedent is \( A \supset B \).

4.4.10. Definition. We define a mapping \( h_{C+D} \) on the set of cut-free proof
figures in \( \text{GFPL}^* \) as follows:

1. \( h_{C+D}(A \rightarrow A) = A \rightarrow A \)
2. \( h_{C+D}(\bot \rightarrow) = \bot \rightarrow \)
3. \( h_{C+D}(\Gamma \rightarrow \Delta) = \frac{P_1}{\Gamma \rightarrow \Delta} \)
\[ \begin{align*}
C & \rightarrow C \\
D & \rightarrow D \\
C & \supset D, C \rightarrow D \\
C & \supset D, C \supset D \\
C & \supset D, C \supset D \\
C & \supset D, C \supset D \\
\text{using } (T \rightarrow \), possibly several times \\
C & \supset D, \Gamma \rightarrow C \supset D \\
\end{align*} \]

\[ \begin{align*}
\frac{h_{C \supset D}(P_1)}{C \supset D, \Gamma \rightarrow \Delta} & \quad \text{if } \frac{P_1}{\Gamma \rightarrow \Delta} \in \mathcal{P}(C \supset D) \\
\frac{h_{C \supset D}(P_1) \cdot h_{C \supset D}(P_2)}{C \supset D, \Gamma \rightarrow \Delta} & \quad \text{otherwise}
\end{align*} \]

(4) \( h_{C \supset D}(\Gamma \rightarrow \Delta) = \frac{h_{C \supset D}(P_1) \cdot h_{C \supset D}(P_2)}{C \supset D, \Gamma \rightarrow \Delta} \).

4.4.11. **COROLLARY.** Let \( P \) be a cut-free proof figure for \( \Gamma \rightarrow \Delta \). Then \( h_{C \supset D}(P) \) is a cut-free proof figure for \( C \supset D, \Gamma \rightarrow \Delta \) such that \( \text{dep}_I(P) \geq \text{dep}_I(h_{C \supset D}(P)) \).

**Proof.** Using an induction on \( P \).

4.4.12. **NOTATION.** By \( \#_I(P) \), we mean the sum of the number of inference rule \( (\rightarrow \supset) \) in \( P \) and the number of inference rule \( (\rightarrow \land) \) in \( P \).

4.4.13. **LEMMA.** Let \( P \) be a cut-free proof figure. If there exists a subfigure \( Q \in \mathcal{P}(A \supset B) \) of \( P \) satisfying \( \text{dep}_I(Q) \geq 2 \), then \( \#_I(P) > \#_I(h_{A \supset B}(P)) \).

**Proof.** We use an induction on \( P \).

If \( P \in \mathcal{P}(A \supset B) \), then \( \#_I(h_{A \supset B}(P)) = 1 \). Since there exists a subfigure \( Q \) of \( P \) such that \( \text{dep}_I(Q) \geq 2 \), \( \#_I(P) \geq 2 \). Hence \( \#_I(P) \geq 2 > 1 = \#_I(h_{A \supset B}(P)) \).

Suppose that \( P \not\in \mathcal{P}(A \supset B) \) and the lemma holds for any proper subfigure of \( P \). We only show the case that \( P \) is of the form

\[ \begin{align*}
P_1 \left\{ \begin{array}{c}
:: C, \Gamma \rightarrow \Delta \\
C \land D, \Gamma \rightarrow \Delta 
\end{array} \right. \]

By the induction hypothesis, \( \#_I(P_1) > \#_I(h_{A \supset B}(P_1)) \). Since \( h_{A \supset B}(P) \) is \( \frac{h_{A \supset B}(P_1)}{A \supset B, C \land D, \Gamma \rightarrow \Delta} \), we obtain

\[ \#_I(P_1) > \#_I(h_{A \supset B}(P_1)) = \#_I(h_{A \supset B}(P)). \]
4.4. Cut-elimination theorem

The other cases can be shown similarly.

4.4.14. DEFINITION. We define a mapping $h_{C \supset D}$ on the set of cut-free proof figures in GFPL$^*$ as follows:

1. $h_{C \supset D}(A \rightarrow A) = \begin{cases} 
A \rightarrow A & \text{if } A \neq C \supset D \\
A \rightarrow A & \text{otherwise}
\end{cases}$

2. $h_{C \supset D}(\bot \rightarrow) = \frac{\bot \rightarrow}{C \supset D}$

3. $h_{C \supset D}(\frac{P_1}{\Gamma \rightarrow \Delta}) = \begin{cases} 
\frac{C \supset D \rightarrow C \supset D}{\text{using } (T \rightarrow), \text{possibly several times}} & \text{if } \frac{P_1}{\Gamma \rightarrow \Delta} \in \mathcal{P}(C \supset D) \\
\frac{h_{C \supset D}(P_1)}{C \supset D, \Gamma \rightarrow \Delta} & \text{if } \frac{P_1}{\Gamma \rightarrow \Delta} \in \mathcal{P}(E \supset F) \\
\frac{h_{C \supset D}(P_1)}{C \supset D, \Gamma \rightarrow \Delta} & \text{for some } E \supset F \neq C \supset D \\
\frac{h_{C \supset D}(P_1) \cdot h_{C \supset D}(P_2)}{C \supset D, \Gamma \rightarrow \Delta} & \text{otherwise}
\end{cases}$

4.4.15. COROLLARY. Let $P$ be a cut-free proof figure for $\Gamma \rightarrow \Delta$. Then $h_{C \supset D}(P)$ is a cut-free proof figure for $C \supset D, \Gamma, \rightarrow \Delta$ such that $\text{dep}_1(P) \geq \text{dep}_1(h_{C \supset D}(P))$.

Proof. Using an induction on $P$ and Corollary 4.4.11.

4.4.16. LEMMA. Let $P$ be a cut-free proof figure. If there exists a subfigure $Q \in \mathcal{P}(A \supset B)$ of $P$ satisfying $\text{dep}_1(Q) \geq 2$, then $\#_1(P) > \#_1(h_{A \supset B}(P))$.

Proof. We use an induction on $P$.

Most of the cases can be shown as in Lemma 4.4.13. The only other case we should consider is $P \in \mathcal{P}(C \supset D)$ for $C \supset D \neq A \supset B$, but using Lemma 4.4.13 instead of the induction hypothesis, we also obtain the lemma.
4.4.17. **Lemma.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be finite sets of formulas in \( \text{WFF} \) and let \( P \) be a cut-free proof figure for

\[
\{T^{2n+3} A \mid A \in \Sigma_1\}, \{(T^{2n+4} B)^+ \mid B \in \Sigma_2\}, \Gamma \rightarrow \Delta
\]

in \( \text{GFPL}^* \), where \( n \) is the number of elements in \( \{A \supset B \mid A \supset B \in \text{Sub}^+(\Gamma \rightarrow \Delta)\} \). Then there exists a cut-free proof figure for \( \Gamma \rightarrow \Delta \) in \( \text{GFPL}^* \).

**Proof.** We use an induction on \( \#_1(P) + \omega(\text{dep}_1(P)) \). We note that \( \text{dep}_1(P) \leq \#_1(P) \). Also the end sequent of \( P \) is

\[
\{T^{n+2} A \mid A \in \{T^{n+1} A' \mid A' \in \Sigma_1\}\}, \{(T^{n+3} B)^+ \mid B \in \{T^{n+1} B' \mid B' \in \Sigma_2\}\}, \Gamma \rightarrow \Delta
\]

and

\[
\left(\{T^{n+1} A' \mid A' \in \Sigma_1\}\cup\{T^{n+1} B' \mid B' \in \Sigma_2\}\right) \cap \text{Sub}^+(\Gamma \rightarrow \Delta) = \emptyset.
\]

If \( \text{dep}_1(P) < n + 2 \), we obtain the lemma by Lemma 4.4.8. Suppose that \( \text{dep}_1(P) \geq n + 2 \) and the lemma holds for any \( P^* \) such that \( \#_1(P^*) + \omega(\text{dep}_1(P^*)) < \#_1(P) + \omega(\text{dep}_1(P)) \). Since \( \text{dep}_1(P) \geq n + 2 \), there exists a sequence of subfigures of \( P \)

\[P_1, P_2, \ldots, P_{n+1}, P_{n+2}, \ldots, P_{\text{dep}_1(P)}\]

such that

1. \( P_{i+1} \) is a proper subfigure of \( P_i \),
2. \( P_i \in \mathcal{P}(C_i \supset D_i) \) for some \( C_i \) and \( D_i \).

We note that if \( i \leq n + 1 \), then the sum of the number of inference rules \( (\rightarrow \supset) \) and \( (\rightarrow \supset) \) on the path from the end sequent to the lower sequent of \( P_i \) is \( i - 1 \). On the other hand, logical inference rules whose principal formula is of the form \( A \supset B \) are only \( (\rightarrow \supset) \) and \( (\rightarrow \supset) \). So, using an induction, we can easily show that the succedent of each sequent on the path contains only elements of \( \{T\} \cup \text{Sub}^+(\Gamma \rightarrow \Delta) \). Hence we have \( P_i \in \mathcal{P}(C_i \supset D_i) \) for some \( C_i \supset D_i \in \text{Sub}^+(\Gamma \rightarrow \Delta) \). Since \( n \) is the number of elements in \( \{A \supset B \mid A \supset B \in \text{Sub}^+(\Gamma \rightarrow \Delta)\} \), there exist \( i, j \) and \( C \supset D \in \text{Sub}^+(\Gamma \rightarrow \Delta) \) such that \( P_i, P_j \in \mathcal{P}(C \supset D) \) and \( 1 \leq i < j \leq n + 1 \). Using \( \text{dep}_1(P) \geq n + 2 \), we have \( \text{dep}_1(P_j) \geq 2 \). Let \( P_j' \) be the subfigure of \( P_i \) whose end sequent is the upper sequent of the inference rule introducing the end sequent of \( P_i \). Then by Lemma 4.4.15, \( \text{dep}_1(P_j') \geq \text{dep}_1(h_{C \supset D}(P_j')) \) and by Lemma 4.4.16, \( \#_1(P_j') > \#_1(h_{C \supset D}(P_j')) \). Let \( Q_i \) be the figure

\[
\frac{h_{C \supset D}(P_j')}{\Pi \rightarrow C \supset D'}
\]

We note that \( Q_i \) is a cut-free proof figure satisfying \( \#_1(P_i) > \#_1(Q_i) \) and \( \text{dep}_1(P_i) \geq \text{dep}_1(Q_i) \). Let \( Q \) be a figure obtained from \( P \) by replacing \( P_i \) by \( Q_i \). Then \( Q \) is a cut-free proof figure for the end sequent of \( P \) satisfying \( \#_1(P) > \#_1(Q) \) and \( \text{dep}_1(P) \geq \text{dep}_1(Q) \). By the induction hypothesis, we obtain the lemma. \( \blacksquare \)
4.5. Other results

4.4.18. LEMMA. Let $P$ be a cut-free proof figure for $\Gamma \rightarrow \Delta$ in GFPL*. Then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in GFPL+

Proof. By replacing each inference rule

\[
\frac{A, \Gamma^+ \rightarrow B}{\Gamma \rightarrow A \supset B}
\]

by

\[
\frac{A, \Gamma^+ \rightarrow B}{A, A \supset B, \Gamma^+ \rightarrow B}
\]

\[
\frac{A, \Gamma^+ \rightarrow B}{\Gamma \rightarrow A \supset B}
\]

we obtain a cut-free proof figure in GFPL+

Proof of Theorem 4.4.4. Suppose that $\Gamma \rightarrow \Delta \in GFPL^+$. Using Lemma 4.4.3, we have

\[
\Gamma \rightarrow \Delta \in GVPL^+ + \sum_{2n+3} L(p),
\]

where $n$ is the number of elements in $\{A \supset B \mid A \supset B \in \text{Sub}^+(\Gamma \rightarrow \Delta)\}$. So, there exist formulas $A_1, \ldots, A_m$ such that

\[
\sum_{2n+3} L(A_1), \ldots, \sum_{2n+3} L(A_m), \Gamma \rightarrow \Delta \in GVPL^+.
\]

Using Theorem 2.2.6, there exists a cut-free proof figure $P$ for the sequent above in GVPL+. It is easily seen that $P$ is also a proof figure in GFPL*. Using Lemma 4.4.17, there exists a cut-free proof figure $Q$ for

\[
\Gamma \rightarrow \Delta
\]

in GFPL*. Using Lemma 4.4.18, we obtain the theorem.

4.5 Other results

In this section, we show some other results concerning $R$.

We say that a consequence relation $\vdash$ has the disjunction property if

\[
\emptyset \vdash A \lor B \text{ implies either } \emptyset \vdash A \text{ or } \emptyset \vdash B
\]

(cf. [CZ97]).
4.5.1. **Proposition.** Every consequence relation in \( \mathcal{R} \) has the disjunction property.

Proof. By Corollary 2.2.12, it was proved that for any formula \( C \supset D, \vdash V+C \supset D \) has disjunction property. So, using Corollary 4.1.9, we obtain the proposition.

A superintuitionistic logic, we mean a set of formulas containing intuitionistic propositional logic closed under modus ponens and substitution. We also consider the cardinality of \( \mathcal{R} \) by comparing it with the set \( SI \) of all the finite axiomatizable superintuitionistic logics.

4.5.2. **Lemma.** \( \mathcal{R} \) is homomorphic to \( SI \).

Proof. It suffices to provide an example of a homomorphism from \( \mathcal{R} \) to \( SI \). We define a mapping \( f \) from \( \mathcal{R} \) to \( SI \) as follows:

\[
f(\vdash V+A) = \vdash I+A.
\]

In other words,

\[
f(\vdash V+A) = \vdash V+A + \because p/p.
\]

It is easily seen that \( \vdash_1 \subseteq \vdash_2 \) implies \( f(\vdash_1) \subseteq f(\vdash_2) \) and so, we confirm that \( f \) is a mapping from \( \mathcal{R} \) to \( SI \). Hence, all we have to do is to show that \( f \) is a surjection. Since

\[
((T \supset A) \supset A)^+, L(A) \rightarrow A \in GVPL^+,
\]
we have

\[
\vdash I+A = \vdash I+L(A).
\]

So,

\[
f(\vdash V+L(A)) = \vdash I+L(A) = \vdash I+A.
\]

Since \( \vdash V+L(A) \in \mathcal{R} \), \( f \) is a surjection.

4.5.3. **Proposition.** There are infinitely many consequence relations in \( \mathcal{R} \).

Proof. It is known that there are infinitely many finitely axiomatizable superintuitionistic logics (cf. [CZ97]). So, by Lemma 4.5.2, we obtain the proposition.

Also we have the following result.
4.6. Corresponding results in modal logics

4.5.4. Proposition. Let it be that

\[ IF = \{ \models \models \vdash \forall x A \} \]

Then

\[ \mathcal{R} \cap IF = \{ \models \}. \]

Proof. It is easily seen that \( \{ \mathcal{R} \} \subseteq \mathcal{R} \cap IF \). So, we only have to show \( \{ \models \} \supseteq \mathcal{R} \cap IF \). Suppose that \( \forall x A \in \mathcal{R} \cap IF \). By Theorem 4.1.4 and Theorem 4.2.1, we have

\[ \forall x A = \vdash \forall x \forall y A \subseteq \vdash \forall y. \]

So, \( \{ A \} \vdash \forall y A \). Using Corollary 2.4.4, \( \{ A \} \vdash \forall A \), and so, \( \forall x A \vdash \forall x \forall y A \subseteq \vdash \forall y. \)

4.6 Corresponding results in modal logics

In this section, we extend the results in section 4.3 to normal modal logics. Results in the other previous sections in this chapter can also be extended in a similar way. The modal operator is denoted by \( \Box \) (necessity). Modal formulas are defined, as usual. If there is no confusion, we simply call them formulas. A normal modal logic is a set of formulas containing all the tautologies of classical logic and

\[ \Box (p \supset q) \supset (\Box p \supset \Box q), \]

which is closed under modus ponens, substitution and necessitation,

\[ \begin{array}{c}
A \\
\Box A
\end{array} \]

By K, we mean the smallest normal modal logic. Let L be a normal modal logic. The expression \( L + A \) denotes the closure under modus ponens, substitution and necessitation of \( L \cup \{ A \} \). The normal modal logics K4 and GL are defined as follows:

\[ K4 = K + \Box p \supset \Box \Box p \quad \text{and} \quad GL = K4 + L^2(p). \]

where \( L^2(p) = \Box (\Box p \supset p) \supset \Box p \).

A Kripke frame for the modal language is a pair \( \langle W, R \rangle \), in which \( R \) is a binary relation on a set \( W \neq \emptyset \). A Kripke model for the modal language is a triple \( M = \langle W, R, P \rangle \), where \( \langle W, R \rangle \) is a Kripke frame and \( P \) is a mapping from the set of all propositional variables to the set \( 2^W \). The truth valuation \( \models \) differs from that for the non-modal propositional language in the following respects: (K5) in section 3.1 is replaced by
Chapter 4. Löb’s axiom in propositional logics

(K5)' \((M, \alpha) \models A \supset B\) iff \((M, \alpha) \models A\) implies \((M, \alpha) \models B\), and we add the condition

(K6) \((M, \alpha) \models \Box A\) iff for any \(\beta \in \alpha^+, (M, \beta) \models A\).

Similarly to the non-modal case, we use the expression \(M \models A\).

4.6.1. Lemma. (cf. [CZ97])

\(A \in K\) iff for any Kripke model \(M, M \models A\),

\(A \in K4\) iff for any transitive Kripke model \(M, M \models A\),

\(A \in GL\) iff for any finite irreflexive transitive Kripke model \(M, M \models A\).

Now, we consider the set:

\[ \mathcal{ML}/L_0 = \{ L \mid L = L_0 + A = L_0 + \Box A, \text{ for some formula } A\}, \]

which corresponds to \(\mathcal{R}\) if \(L_0 = K4\). Also the following lemma can be proved similarly to the proof of Lemma 4.2.4.

4.6.2. Lemma. Let \(L_0\) be a normal modal logic contained in \(GL\). Then

\[ L_0 + A \in \mathcal{ML}/L_0 \text{ implies } L_0 + A \subseteq GL. \]

4.6.3. Theorem. \(GL\) is the maximal modal logic in \(\mathcal{ML}/K4\).

Proof. By Lemma 4.6.2, it is sufficient to prove \(GL \in \mathcal{ML}/K4\). It is easily seen that for any transitive Kripke frame \(M, M \models \Box L^\alpha(p) \supset L^\alpha(p)\).

So, by Lemma 4.6.1, we have \(\Box L^\alpha(p) \supset L^\alpha(p) \in K4\). Using modus ponens, \(L^\alpha(p) \in K4 + \Box L^\alpha(p)\). By necessitation, we also have \(\Box L^\alpha(p) \in K4 + L^\alpha(p)\).

Hence, we obtain the theorem.

4.6.4. Corollary.

(1) \(\mathcal{ML}/K4 \subseteq \{ L \mid K4 \subseteq L \subseteq GL\}\),

(2) \(\min \mathcal{ML}/K4 = K4\),

(3) \(\max \mathcal{ML}/K4 = GL\).

However \(GL\) is not the maximal modal logic in \(\mathcal{ML}/K\), since \(GL \notin \mathcal{ML}/K\).