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Inequivalent classes of interference experiments with non-Abelian anyons

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We present a theoretical analysis of inequivalent classes of interference experiments with non-Abelian anyons using an idealized Mach-Zender-type interferometer. Because of the non-Abelian nature of the braid group action one has to distinguish the different possibilities in which the experiment can be repeated, which lead to different interference patterns. We show that each setup will, after repeated measurement, lead to a situation where the two-particle (or multiparticle) state gets locked into an eigenstate of some well-defined operator. Also, the probability to end up in such an eigenstate is calculated. Some representative examples are worked out in detail.

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I. INTRODUCTION

Physics in lower dimensions is not always simpler. A non-trivial instance in two dimensions is the possibility of *anyons*, i.e., excitations that exhibit statistical properties that transcend the conventional possibilities of bosons and fermions [1]. It is striking that such exotic theoretical possibilities appear to be realized in nature. With the experimental advances in condensed matter physics related to the fractional quantum-Hall effect, quasiparticles have been discovered that appear to exhibit anyonic behavior.

There is also a more mundane interest in the two-dimensional physics of anyons in the emerging field of quantum computation. In quantum computation, as the number of qubits grows, the problem of beating decoherence becomes a major threat: the larger the quantum system to be manipulated gets, the more it interacts with the (noisy) environment, causing the quantum state to collapse. In order to compensate the inaccuracies caused by decoherence, the idea of fault-tolerance and error-correction has become an important subject, and quantum codes were devised where single qubit-information is encoded into multiple qubits [2]. If errors in the physical qubits can be corrected in time, the stored quantum information remains intact, or at least the decoherence time of the encoded qubit is increased. The progress achieved in the theory of quantum error codes, however, is mainly mathematical in nature. Current experimental realizations of this idea of qubit-encoding do not yet share the optimism of the theorists and are still far from being decoherence free.

It may, therefore, be profitable to go back to some basic physics and search for intrinsically decoherence-free quantum states. Promising candidates are “global quantum states” carrying quantum information that is not “stored” locally, in single spins, for instance, but in a robust collective property of a multiparticle system—a topological feature, for example. Such a global characteristic of the state cannot be destroyed by local interactions, especially not by those with the noisy environment. As anyons exhibit such a global, topological robustness as a basic property, it seems anyons are

especially suited for this kind of the quantum computer game, which goes by such names as topological quantum computation [3] and geometrical quantum computation [4].

Anyons for quantum computation can be divided into two groups: the (conventional) Abelian anyons that are manipulated by mere geometric (Abelian) *phases*, as in Refs. [4,5], and, most promising, the so-called *non-Abelian anyons* (which are strictly speaking a generalization of the Abelian anyons). Non-Abelian anyons may carry nontrivial internal degrees of freedom that can have *non-Abelian topological interactions* that allow (in principle) for a rather clean manipulation of entangled states. *These non-Abelian anyons may serve as intrinsically fault-tolerant decoherence free qubits* [3,6–8]. One might think of non-Abelian anyons in particular media with, for example, broken symmetries that support quasi-particle excitations with anyonic properties as in model systems such as the so-called discrete gauge theories [9–11].

A sometimes underestimated issue but necessary requirement for quantum computation is the ability to perform a measurement of the quantum state. In Refs. [6–8], the so-called fusion¹ properties of the non-Abelian anyons are used to perform a quantum measurement. But a perhaps more generally applicable way to approach measurements, is to look at *interference experiments* with the quasiparticles, both for non-Abelian anyons, as suggested by Ref. [3], and Abelian anyons, [12]. Interference experiments with non-Abelian anyons are generalizations of the Aharonov-Bohm effect, and have been studied in the past [13,14]; however, these papers focused primarily on determining the cross section via the initial probability distribution, and neglected the *change* in the quantum state of the target particle. For topological quantum computation, the quantum state of the target particle, i.e., the qubit one likes to measure, is obviously important.

Motivated by these considerations, we have analyzed certain types of idealized interference experiments that one

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¹In the discrete gauge theories [10,11] the quasiparticles are allowed to fuse and their fusion product is a gauge invariant, and, therefore, likely to be a measurable observable. However, a detailed description of the mechanism of such a measurement is yet to appear.

could perform with non-Abelian anyons, thereby taking into account the changes in the quantum states of the quasiparticles. This leads to a critical study of the non-Abelian generalization of the Aharonov-Bohm effect and how it could be measured by a Mach-Zender-type device. It turns out that there are essentially different possibilities to perform the repeated experiment leading to very different interference patterns.

The paper is organized as follows. After a brief introduction to non-Abelian anyons and the braid group, we discuss in Sec. II the general setup of experiments with the Mach-Zender interferometer. In Sec. III, we use this setup to analyze the different classes of anyonic experiments that could be performed and show to what kinds of interference patterns they may lead. We distinguish the so-called *one-to-one*, *many-to-one*, and *many-to-many* experiments. The observable patterns may be very distinct, because after a sufficient number of repetitions the system will get locked into a particular eigenstate or subspace of some well-determined operator that depends on the type of experiment. The proof of this locking is relegated to the appendix. The probabilities of finding certain patterns is also calculated. At the end of the paper, we have summarized the results for the various classes in a comprehensive way in Table I.

A. Non-Abelian anyons and the braid group

If we physically move two particles around each other and return both to their original positions, it corresponds to a closed path in the two-particle configuration space. In three or more dimensions, this closed loop will be contractible to a point, which means that the monodromy operator describing the net effect on the two-particle Hilbert space will just be the identity operator. If we take two identical (indistinguishable) particles, one may also consider the exchange operator defined as the square root of the monodromy operator, which consequently can have only two distinct eigenvalues, namely, $+1$ or -1 , one corresponding to bosons and fermions, respectively. It is well known that in two dimensions the situation is fundamentally different; the loop is no longer contractible so that the (unitary) monodromy operator can generate an arbitrary phase. Similarly, the exchange operator R may yield *any* phase on a state with two identical particles that are therefore denoted as *anyons*. For such particles, the counterclockwise exchange operator $\mathcal{R}$ and the clockwise exchange $\mathcal{R}^{-1}$ have to be distinguished.

On an algebraic level, this leads to a considerable generalization because it allows also the possibility of nontrivial braidings between different types of particles. In short, the simple representation theory of the *symmetric group* S_n (of permutations) has to be replaced by the representation theory of the *braid group* B_n that is in principle an infinite group. The one-dimensional unitary representations of B_n are labeled by an arbitrary phase, but its higher-dimensional unitary irreducible representations are far more complicated. In general, the n -particle Hilbert space can be decomposed into subspaces that will carry irreducible representations of this braid group B_n , and if such representations turn out to be higher dimensional we call the particles involved, *non-Abelian anyons*.

FIG. 1. A pictorial representation of the Yang-Baxter equations $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ and $\tau_i \tau_j = \tau_j \tau_i$ with $|i-j| \geq 2$ for $i=1$ and $j=3$; the similarity with braids of strings of ropes is crystal-clear.

The braid group B_n is generated by $n-1$ elements τ_i and their inverses. The τ_i obey the Yang-Baxter equations, also known as the braid relations, which are

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad i=1, \dots, n-2, \quad (1.1)$$

$$\tau_i \tau_j = \tau_j \tau_i, \quad |i-j| \geq 2. \quad (1.2)$$

The correspondence between the Yang-Baxter equations and braids of strings of rope is best described pictorially, as in Fig. 1.

The way in which non-Abelian anyons are usually described is by endowing the particles with some internal degrees of freedom, the nontrivial braid statistics of these particles can then be implemented by coupling these new internal degrees of freedom to a non-Abelian Chern-Simons (or statistical) gauge field that mediates the appropriate type of topological interactions.

As different braidings need not commute, it becomes important to be able to distinguish between them. A convenient way to accomplish this is to order the system in the following way: map all particles in the two-dimensional plane on some fixed line and number the particles (we will take the line to be horizontal and we number r from right to left, starting with one). If particles i and $i+1$ pass each other on the virtual line, we have to apply the appropriate exchange operator: $\mathcal{R}_i$ if the exchange is counterclockwise, $\mathcal{R}_i^{-1}$ if clockwise (not that after the exchange the original particle i now is particle $i+1$ and vice versa). Every braid can be decomposed as a product of these braids of adjacent particles, $\mathcal{R}_i$ and $\mathcal{R}_i^{-1}$; also known as the braid operators, $\mathcal{R}_i$ and $\mathcal{R}_i^{-1}$ clearly satisfy the braid relations (1.1), (1.2). An example is shown in Fig. 2.

Associated with each particle is an internal space, and identical particles carry identical internal spaces. Typically, and this probably is the main feature of non-Abelian anyons, the internal states of different particles may become *entangled* through braiding: this interaction of the non-abelian anyons is of a topological nature, independent of the distance between the anyons, and it is a global property of the system, unperturbable by local interactions. The total system's multiparticle, possibly entangled, internal state denoted by $|\psi\rangle$ lies in the tensor product of the internal spaces of the individual particles. In such tensor products we use the same ordering of particles as introduced above, in particles that are left (right) on the virtual line should appear left (right) in the

FIG. 2. An example on ordering a system with anyons and classifying braids. We assign a number to all six particles and map them on a (horizontal) line. The shown braid, which moves particle 3 around some of the other particles and back to its original position, can now be unambiguously decomposed in adjacent particle exchanges; read from right to left, this particular braid is given by $\mathcal{R}_2^{-2}\mathcal{R}_3\mathcal{R}_4\mathcal{R}_5^2\mathcal{R}_4^{-1}\mathcal{R}_3$.

tensor product. For instance, imagine a system with one particle of type *A* (with internal space V^A) and two particles of type *B* (with internal spaces V^B) that lie on the left of particle *A*. The three-particle internal state $|\psi\rangle$ is then an element of $V^B \otimes V^B \otimes V^A$.

The exchange caused by braid operators occurs explicitly in the tensor product as well, i.e., $\mathcal{R}, \mathcal{R}^{-1}: V^B \otimes V^A \rightarrow V^A \otimes V^B$. We can think of $\mathcal{R}$ and $\mathcal{R}^{-1}$ as combination of some matrix and an operator σ , where σ only swaps the two vector spaces. The matrix components of $\mathcal{R}$ and $\mathcal{R}^{-1}$ depend only on the types of the two particles that is operated upon. The square of $\mathcal{R}$, the monodromy operator $\mathcal{R}^2$, obviously does no net swapping and can be regarded as a unitary matrix, of which the explicit matrix-components depend on the particles on which it operates.

For the work we are about to present, the knowledge of the braid matrices is assumed *a priori*. It is clearly of great importance to understand the way one gets to this knowledge from an underlying algebraic structure. For two-dimensional physics, there turns out to be an important generalization of ordinary group theory—the theory of quantum groups and Hopf algebras—which provides the natural descriptions of non-Abelian anyons, in that it generates natural representations of the braid operator R on the tensor product of its representations. For the fractional quantum Hall effect, for example, this program has been carried out quite explicitly [15,16]. This has evolved into a rapidly expanding field of research on its own, we will, however, not make explicit use of this machinery in the following.

II. THE SETUP OF EXPERIMENTS WITH THE MACH-ZENDER INTERFEROMETER

We introduce the setup of our thought experiments by using the Mach-Zender interferometer in this section. We will first treat an ordinary interference experiment, then move to the Aharonov-Bohm experiment, and generalize it further to the non-Abelian anyon experiment. We will see that for non-Abelian anyons, there is a nontrivial difference between the interference pattern and the probability distribution.

FIG. 3. The setup of the Mach-Zender interferometer for ordinary interference experiments. At point A, a particle may be injected into the device, where it will encounter beam splitter BS_1 . It may traverse two different paths (thereby passing either mirror M_1 or M_2): the counterclockwise path I and pick up the phase factor $e^{i\theta_1}$, or the clockwise path II and acquire $e^{i\theta_2}$. The particle is not restricted to one path exclusively: it is allowed to travel both paths at the same time. Beam splitter BS_2 unites both paths, which now interfere with each other, and the particle emerges at one of two possible exits, B or C, which will be observed by a detector, D_1 or D_2 .

A. Ordinary interference

We are interested in describing interference experiments with non-Abelian anyons, which basically means that we investigate non-Abelian generalizations of the Aharonov-Bohm effect. We should realize that it is not the topological interactions leading to the well-known (phase) factors that cause the interference, for these interactions will only be able to modify existing interference patterns. So, the starting point is a basic device that produces ordinary interference effects, as such a device, we may, for example, choose the Mach-Zender interferometer. We will introduce some terminology that we would like to use in the following sections without further reference.

The Mach-Zender interferometer, as depicted in Fig. 3, is a two-dimensional device, built of two lossless beam splitters² BS_1 and BS_2 , two 100% reflecting mirrors M_1 and M_2 , and two detectors D_1 and D_2 . A particle³ enters the apparatus at point A on the left, where it encounters a beam splitter BS_1 . The beam splitter will direct the particle along

²We make use of the classical terminology of “splitting a beam,” but this should not distract us from the quantum nature of the experiment: one particle at a time may propagate through the apparatus and it is the two components of a single-particle wave packet that get split by the beam splitter. When probed, by a detector like D_1 or D_2 , one would find only one particle and that it followed one path, not both.

³We use “particle” in a rather abstract sense; it may be any kind of particle as long as it exhibits both wave and particle behavior. One may, for instance, think of electrons, or photons, but in the present context also of quasiparticles with non-Abelian braiding properties.

FIG. 4. The coefficients t_i , r_i , t'_i , r'_i determine the lossless beam splitter BS_i (where $i=1,2$); t_i indicates transmission for incoming beams from the left, r_i stands for reflection when incident from the left, t'_i and r'_i denote transmission and reflection when incident from below.

the counterclockwise path I and/or the clockwise path II. Both paths include a 100% reflecting mirror, M_1 or M_2 , and both paths end up at beam splitter BS_2 . The particle may now emerge from the apparatus at either point B and be registered by detector D_1 or at point C and be observed by detector D_2 .

Ordinary interference is caused by the difference in relative phase of particles traveling along path I or path II, which is due to a difference in path length. We assume that we can adjust the apparatus in such a way that we can tune the difference in path length between paths I and II with high precision. We also have to include the transmission and reflection coefficients t_i, r_i, t'_i, r'_i of the beam splitters BS_i , see Fig. 4, because these complex factors also affect the relative phase. Since we assume the beam splitters to be perfect, its coefficients form a unitary matrix and we have that [17,18]

$$r_i^* = r'_i, \quad t_i^* = -t'_i, \quad |r_i| + |t_i|^2 = 1. \quad (2.1)$$

We will not explicitly need the lengths of paths I and II, we will need only the relative phases that the particle will acquire when traversing path I or II. We will indicate the associated phase factors by $e^{i\theta_I}$ and $e^{i\theta_{II}}$. The amplitude $A[D_1]$ of the particle's wave-packet component at detector D_1 is the sum of the contributions of both paths with the appropriate factors

$$A[D_1] = t_1 r'_2 e^{i\theta_I} + r_1 t_2 e^{i\theta_{II}}, \quad (2.2)$$

and for $A[D_2]$,

$$A[D_2] = t_1 t'_2 e^{i\theta_I} + r_1 r_2 e^{i\theta_{II}}. \quad (2.3)$$

The probabilities $P[D_1]$ and $P[D_2]$, that indicate the probability for the particle to be detected at D_1 or D_2 , are the absolute squares of the amplitudes $A[D_1]$ and $A[D_2]$:

$$P[D_1] = |t_1 r'_2|^2 + |r_1 t_2|^2 + 2 \operatorname{Re}(t_1 r'_2 r_1^* t_2^* e^{i(\theta_I - \theta_{II})}), \quad (2.4)$$

$$P[D_2] = |t_1 t'_2|^2 + |r_1 r_2|^2 + 2 \operatorname{Re}(t_1 t'_2 r_1^* r_2^* e^{i(\theta_I - \theta_{II})}) \quad (2.5)$$

$$= |t_1 t'_2|^2 + |r_1 r_2|^2 - 2 \operatorname{Re}(t_1 t'_2 r_1^* r_2^* e^{i(\theta_I - \theta_{II})}), \quad (2.6)$$

where we used Eq. (2.1) to go from Eq. (2.5) to Eq. (2.6). Adding Eqs. (2.4) and (2.6) confirms that the total probability is of course preserved:

$$P[D_1] + P[D_2] = 1. \quad (2.7)$$

In a real experimental setup, the particles that are directed at the apparatus will not all acquire precisely the same phase factors $e^{i\theta_I}$ or $e^{i\theta_{II}}$, for instance, because of a slight difference in the momenta of the incoming particles. Therefore, we introduce a variable q and a density function $\rho(q)$ to incorporate all such factors that influence the relative phase factors (which are due to the experimental setup). When calculating probabilities, we have to integrate over the, now q dependent, phase factors $e^{i\theta_I(q)}$, $e^{i\theta_{II}(q)}$:

$$\int \exp[i(\theta_I(q) - \theta_{II}(q))] \rho(q) dq \equiv Q e^{i\theta}, \quad (2.8)$$

where Q is real and between 0 and 1,⁴ and θ is some sort of average phase difference. After integration over q , the probabilities of Eqs. (2.4) and (2.6) then become

$$P[D_1] = |t_1 r'_2|^2 + |r_1 t_2|^2 + 2Q \operatorname{Re}(t_1 r'_2 r_1^* t_2^* e^{i\theta}), \quad (2.9)$$

$$P[D_2] = |t_1 t'_2|^2 + |r_1 r_2|^2 - 2Q \operatorname{Re}(t_1 t'_2 r_1^* r_2^* e^{i\theta}). \quad (2.10)$$

We will use $e^{i\theta_I}$ and $e^{i\theta_{II}}$ in our notation for amplitudes, and Q and $e^{i\theta}$ for probabilities, without further mentioning the implied q integration.

So far, we have only talked about probabilities for single particles to be injected and observed. Obviously, we want to repeat many of such single-particle “runs” and call the result the interference pattern. In other words, the interference pattern $I[D_1]$ is the number of times $\mathcal{N}[D_1]$ that a particle is injected into the apparatus and observed by detector D_1 divided by the total number n of injected particles, and likewise for $I[D_2]$,

$$I[D_1] = \frac{\mathcal{N}[D_1]}{n}, \quad I[D_2] = \frac{\mathcal{N}[D_2]}{n}, \quad (2.11)$$

$$\mathcal{N}[D_1] + \mathcal{N}[D_2] = n. \quad (2.12)$$

If we repeat the experiment for a large number of particles, i.e., perform many runs under the same conditions, we ex-

⁴If Q is zero, no interference is observed, because the phase factors average out to zero; this is a common aspect in every interference experiment: the density distribution $\rho(q)$ needs to be narrow enough to keep observable interference.

FIG. 5. The Aharonov-Bohm effect in the setup of the Mach-Zender interferometer: the presence of Φ contributes to additional topological phase factors. Following path I gives rise to the factor $e^{i\lambda_I}$, and traversing path II yields $e^{i\lambda_{II}}$. The incoming particle still emerges at either detector D_1 or D_2 , but the probabilities for either exit have been altered. The interference pattern now depends explicitly on the difference of the relative phases $e^{i\lambda} = e^{i(\lambda_I - \lambda_{II})}$.

pect *by definition* that the observed interference patterns become equal to the probability distributions:

$$\lim_{n \rightarrow \infty} I[D_1] = P[D_1], \quad \lim_{n \rightarrow \infty} I[D_2] = P[D_2]. \quad (2.13)$$

On several occasions we will give examples, where for definiteness we have chosen the following values for the coefficients and factors that completely determine the Mach-Zender apparatus:

$$r_j = r'_j = \frac{1}{\sqrt{2}}, \quad t_j = -t'_j = i \frac{1}{\sqrt{2}} \quad (j=1,2), \quad (2.14)$$

$$Q = 1, \quad \theta = \arccos\left(\frac{4}{5}\right).$$

These values were chosen such that, when substituted in Eqs. (2.9) and (2.10), the probabilities $P[D_1]$ and $P[D_2]$ become simple, but still representative, numbers:

$$P[D_1] = \frac{9}{10} \quad P[D_2] = \frac{1}{10}. \quad (2.15)$$

So, in this particular example, it is by far more likely to detect the emerging particle at detector D_1 ; only 10% of the incident particles is observed at detector D_2 .

B. The Aharonov-Bohm effect: the topological phase factor

We will now discuss the Aharonov-Bohm version of the Mach-Zender interferometer. The Aharonov-Bohm effect is the most logical step in going from ordinary interference experiments to interference experiments with non-Abelian anyons, as it is both a well-understood extension of ordinary interference and at the same time features a topological aspect.

The Mach-Zender apparatus is extended with a physical object Φ that we put at its center, as depicted in Fig. 5, and we assume that this object has a topological interaction with

the injected particles.⁵ The incoming particle now acquires an additional phase factor that *only* depends on the topology of the path (where, from a topological point of view, Φ is a puncture of the two-dimensional plane). Paths I and II are topologically distinct and contribute the additional phase factors $e^{i\lambda_I}$ and $e^{i\lambda_{II}}$. The amplitude $A_{AB}[D_1]$ of the particle's wave-packet component at detector D_1 for the Aharonov-Bohm effect obviously includes phase factors

$$A_{AB}[D_1] = t_1 r'_2 e^{i\theta_I} e^{i\lambda_I} + r_1 t_2 e^{i\theta_{II}} e^{i\lambda_{II}}, \quad (2.16)$$

and likewise for the amplitude $A_{AB}[D_2]$ of the component at detector D_2 .

The probabilities $P_{AB}[D_i]$ to observe a particle at either detector depend on the (gauge invariant) phase difference $e^{i(\lambda_I - \lambda_{II})} \equiv e^{i\lambda}$, which is the topological phase factor that a particle picks up when it circumvents Φ in a counterclockwise way. The probabilities $P_{AB}[D_i]$ are

$$P_{AB}[D_1] = |t_1 r'_2|^2 + |r_1 t_2|^2 + 2Q \operatorname{Re}(t_1 r'_2 r_1^* t_2^* e^{i\theta} e^{i\lambda}), \quad (2.17)$$

$$P_{AB}[D_2] = |t_1 t'_2|^2 + |r_1 r_2|^2 - 2Q \operatorname{Re}(t_1 r'_2 r_1^* t_2^* e^{i\theta} e^{i\lambda}). \quad (2.18)$$

As expected, the particle still emerges from the apparatus with unit probability,

$$P_{AB}[D_1] + P_{AB}[D_2] = 1. \quad (2.19)$$

We will use the above expressions for $P_{AB}[D_i]$ later on, where we also want to explicitly include the dependence on $e^{i\lambda}$. For this purpose we introduce $P_{e^{i\lambda}}[D_i]$, which is defined as

$$P_{e^{i\lambda}}[D_i] \equiv P_{AB}[D_i]. \quad (2.20)$$

The single-particle run can be repeated for many particles, which gives rise to the interference patterns $I_{AB}[D_i]$ similar to those of Eq. (2.11). And again, when the number n of particles grows, the interference patterns approach the probability distributions

$$\lim_{n \rightarrow \infty} I_{AB}[D_1] = P_{AB}[D_1] = P_{e^{i\lambda}}[D_1], \quad (2.21)$$

$$\lim_{n \rightarrow \infty} I_{AB}[D_2] = P_{AB}[D_2] = P_{e^{i\lambda}}[D_2]. \quad (2.22)$$

This is about all there is to say about the Aharonov-Bohm interference experiment apart from an example that demonstrates how $e^{i\lambda}$ changes the interference pattern. Let us substitute $e^{i\lambda} = -1$ and the values from Eq. (2.14) into Eqs. (2.17) and (2.18); this yields

$$P_{AB}[D_1] = \frac{1}{10}, \quad P_{AB}[D_2] = \frac{9}{10}, \quad (2.23)$$

⁵In the case of the famous Aharonov-Bohm-effect experiment, where electrons are the incoming particles, Φ represents a magnetic flux, and the electron picks up the phase factor $e^{iae\Phi}$ (a is some constant and e the charge of the electron) encircling the flux once.

FIG. 6. The setup for interference experiments with non-Abelian anyons using the Mach-Zender interferometer. At the center of the apparatus resides a non-Abelian anyon, particle A. The non-Abelian anyon incident from the left is particle B. The non-Abelian anyons have a topological interaction; if particle B traverses the counterclockwise path this corresponds to the braid operator $\mathcal{R}$ acting on the two-particle internal state, the clockwise path yields the inverse operation $\mathcal{R}^{-1}$. The probabilities to emerge at detector D_1 and D_2 will explicitly depend on the expectation value of the monodromy operator $\mathcal{R}^2$.

where the particle is now with 90% chance detected by detector D_2 in contrast with the values of the probabilities in the example of the ordinary interference experiment given by Eq. (2.15).

C. The setup with non-Abelian anyons

The setup for what up to now should be considered thought experiments with non-Abelian anyons using the Mach-Zender interferometer, is very similar to that for the Aharonov-Bohm effect. The incoming particle will be a non-Abelian anyon and the object Φ at the center of the apparatus will be replaced by another particle which is *also* a non-Abelian anyon. So, it is no longer a single-particle problem (one-particle scattering of an invariant external object Φ), but a two-particle problem (one-particle scattering of another one, both being non-Abelian anyons). We will designate the incident non-Abelian anyon as particle B and the non-Abelian anyon fixed at the apparatus' center as particle A. The idea of the experiment remains the same as for the conventional interference experiments: we inject the particle B, in the Mach-Zender apparatus and detect it again with one of the two detectors D_1 and D_2 , also see Fig. 6.

In calculating the amplitudes and probabilities for the non-Abelian anyon experiment, we have to include the two-particle internal state $|\psi\rangle$, which is an element of the tensor product of the internal spaces V^B and V^A of particles B and A:

$$|\psi\rangle \in V^B \otimes V^A. \quad (2.24)$$

The topological phase factors in the Aharonov-Bohm effect will be replaced by braid operators acting on the state $|\psi\rangle$. If particle B follows path I within the apparatus, it picks up a counterclockwise braid on the two-particle internal state:

$\mathcal{R}|\psi\rangle$; if particle B would traverse path II, it would correspond to a clockwise braid, $\mathcal{R}^{-1}|\psi\rangle$.

Let us write down the amplitudes $A_{\text{NA}}[D_i]$ for the interference experiment with non-Abelian anyons:

$$A_{\text{NA}}[D_1] = t_1 r_2' e^{i\theta_I} \mathcal{R}|\psi\rangle + r_1 t_2 e^{i\theta_{II}} \mathcal{R}^{-1}|\psi\rangle, \quad (2.25)$$

$$A_{\text{NA}}[D_2] = t_1 t_2' e^{i\theta_I} \mathcal{R}|\psi\rangle + r_1 r_2 e^{i\theta_{II}} \mathcal{R}^{-1}|\psi\rangle. \quad (2.26)$$

In calculating the probability distributions $P_{\text{NA}}[D_i]$, we assume that the braid operators are unitary and $|\psi\rangle$ is properly normalized:

$$P_{\text{NA}}[D_1] = |t_1 r_2'|^2 + |r_1 t_2|^2 + 2Q \operatorname{Re}(t_1 r_2' r_1^* t_2^* e^{i\theta} \langle \psi | \mathcal{R}^2 | \psi \rangle), \quad (2.27)$$

$$P_{\text{NA}}[D_2] = |t_1 t_2'|^2 + |r_1 r_2|^2 - 2Q \operatorname{Re}(t_1 r_2' r_1^* t_2^* e^{i\theta} \langle \psi | \mathcal{R}^2 | \psi \rangle). \quad (2.28)$$

Compared with the probabilities $P_{\text{AB}}[D_i]$ for the Aharonov-Bohm effect, Eqs. (2.17) and (2.18), the topological phase factor $e^{i\lambda}$ has been replaced by the expectation value $\langle \psi | \mathcal{R}^2 | \psi \rangle$ of the monodromy operator $\mathcal{R}^2$ in Eqs. (2.27) and (2.28).

We can rewrite the probabilities $P_{\text{NA}}[D_i]$ when we assume that the monodromy operator is unitary and thus has eigenvalues of the form $e^{i\lambda}$. If we introduce the projection operators E_λ that project onto the eigenspace of $e^{i\lambda}$, we have that

$$\mathcal{R}^2 = \sum_\lambda e^{i\lambda} E_\lambda, \quad p_\lambda \equiv \langle \psi | E_\lambda | \psi \rangle, \quad (2.29)$$

$$\langle \psi | \mathcal{R}^2 | \psi \rangle = \sum_\lambda p_\lambda e^{i\lambda}, \quad \sum_\lambda E_\lambda = 1, \quad \sum_\lambda p_\lambda = 1, \quad (2.30)$$

where we indicated the real-valued expectation value of E_λ by p_λ . With Eq. (2.20) in mind, the probability distribution of Eq. (2.27) can now be cast in the following form:

$$\begin{aligned} P_{\text{NA}}[D_1] &= |t_1 r_2'|^2 + |r_1 t_2|^2 + 2Q \operatorname{Re} \left(t_1 r_2' r_1^* t_2^* e^{i\theta} \sum_\lambda p_\lambda e^{i\lambda} \right) \\ &= \sum_\lambda p_\lambda [|t_1 r_2'|^2 + |r_1 t_2|^2 \\ &\quad + 2Q \operatorname{Re}(t_1 r_2' r_1^* t_2^* e^{i\theta} e^{i\lambda})] \\ &= \sum_\lambda p_\lambda P_{e^{i\lambda}}[D_1]. \end{aligned} \quad (2.31)$$

This equation is simpler because it gets rid of the uninteresting coefficients r_i, Q , etc; it states that the probability to observe particle B at detector D_1 is a sum over probabilities

of the Aharonov-Bohm version of the experiment, where the eigenvalues $e^{i\lambda}$ of the monodromy operator $\mathcal{R}^2$ play the role of the topological phase factor. The weight p_λ in the sum is precisely the same weight as in the decomposition of Eq. (2.30) of $\langle\psi|\mathcal{R}^2|\psi\rangle$ into eigenvalues $e^{i\lambda}$. Of course, there is a similar decomposition for $P_{\text{NA}}[D_2]$, so we can write

$$P_{\text{NA}}[D_i] = \sum_{\lambda} p_{\lambda} P_{e^{i\lambda}}[D_i], \quad (2.32)$$

and the total probability still equals unity,

$$P_{\text{NA}}[D_1] + P_{\text{NA}}[D_2] = \sum_{j,\lambda} p_{\lambda} P_{e^{i\lambda}}[D_j] = 1. \quad (2.33)$$

Clearly, the values of the probabilities $P_{\text{NA}}[D_i]$ depend on the two-particle internal state $|\psi\rangle$, via the weights p_{λ} . Let us give an example to illustrate this. We plug the coefficients of the Mach-Zender apparatus given by Eq. (2.14) into Eq. (2.32), and assume that the monodromy operator $\mathcal{R}^2$ has two eigenvalues: $+1, -1$ [Eq. (3.46) shows a specific example of such a matrix $\mathcal{R}^2$]. In the following table, we have given $P_{\text{NA}}[D_i]$ for three internal states, each of which is characterized by the p_{λ} ; we have also included the expectation value $\langle\psi|\mathcal{R}^2|\psi\rangle$.

p_1	p_{-1}	$\langle\psi \mathcal{R}^2 \psi\rangle$	Probability distributions	
			$P_{\text{NA}}[D_1]$	$P_{\text{NA}}[D_2]$
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	$\frac{9}{10}$	$\frac{1}{10}$
0	1	-1	$\frac{1}{10}$	$\frac{9}{10}$

It is worth mentioning that one is not restricted to the Mach-Zender interferometer to make the decomposition of the probabilities, like in expression (2.32). This is not so strange, since any dependence on the Mach-Zender apparatus, i.e., the coefficients r_i, Q etc., is absent in Eq. (2.32). For instance, for double slit or plain scattering interference experiments with non-Abelian anyons, one can write down a similar decomposition. However, it is not conventional to do so; one usually keeps the expectation value of $\mathcal{R}^2$ explicit in one's expressions.

D. From probabilities to interference patterns

The interference experiment with non-Abelian anyons is obviously a generalization of the Aharonov-Bohm experiment, by replacing the phase factor $e^{i\lambda}$ by a unitary operator $\mathcal{R}^2$, which has eigenvalues of the form $e^{i\lambda}$. But the picture of the interference experiment with non-Abelian anyons is not complete yet, and requires further analysis. We have to consider the question of interference patterns, which, for the case of non-Abelian anyons, we avoided so far.

What one wants is to predict what one would observe in a realistic experiment. The interference pattern is the number of particles the detectors count if the experiment is repeated with many particles. For the ordinary interference and the Aharonov-Bohm experiments, it was perfectly legitimate to identify the interference patterns with the probability distributions, since by definition these are equal when the same

FIG. 7. The *many-to-many* experiment. To be able to measure the initial probability distribution as the interference patterns requires that exactly the same run be repeated often. This means that after each observation of particle B at either detector both particles A and B have to be replaced by new ones having the same initial internal state. The old particles cannot be reused because the detection of particle B at D_i causes a change in the internal state. This way to repeat the runs is called the *many-to-many* experiment as it involves many particles B and many particles A .

experiment is repeated many times. Is this also true for the experiment with non-Abelian anyons? The answer to this question turns out to be: not necessarily. The problem lies in the definition: in order to be the same one needs to repeat exactly the same experiment. For the non-Abelian anyons experiment, this would imply that for each consecutive run, *both* particle A and B would have to be discarded, to be replaced by new particles A and B , as is also shown in Fig. 7.

Why do we need a new particle A and the same particle B each time, why not use the old particles? The answer is, the two-particle internal state $|\psi\rangle$ not only affects the probabilities to observe particle B at either detector, but also alters itself by the detection, becoming a new, different state $|\psi'\rangle$. It is obvious that an experiment with the internal state $|\psi\rangle$ gives another result than an experiment with internal state $|\psi'\rangle$; so indeed, if we want to find the calculated probabilities as the observed interference pattern, we need a large supply of particles A and B , all with the same internal states $|\psi\rangle$. Then we have

$$\lim_{n \rightarrow \infty} I_{\text{NA}}^{m-m}[D_i] = \lim_{n \rightarrow \infty} \frac{\mathcal{N}[D_i]}{n} = P_{\text{NA}}[D_i], \quad (2.34)$$

where $I_{\text{NA}}^{m-m}[D_i]$ indicate the interference patterns for the experiment with non-Abelian anyons that is repeated with many particles B directed to many particles A . We will refer to this type of experiment as a *many-to-many* ($m-m$) experiment.

However, we may decide to repeat the experiment in a different way, i.e., not using a new A and B each time, but then we will have to include the change in the internal state. The change in the two-particle internal state from $|\psi\rangle$ to $|\psi'\rangle$ itself is also worth examining, and observable as well; if we want to observe the effect of this change of $|\psi\rangle$, it is *necessary* that we repeat the experiment while keeping at least one

of the two old particles A and B . We will discuss two such ways to repeat the experiment. First we will describe an experiment in which we keep using the same particle A and particle B for all consecutive runs, we call this the *one-to-one* experiment. The second way to repeat the experiment is to keep only particle A and replace each particle B by a new one for each run; we will call this type of experiment *many-to-one*.⁶

The reason that the two-particle internal state $|\psi\rangle$ gets changed, is the fact that the detection of particle B is a quantum mechanical measurement; there are two components of the two-particle wave packet, one associated with detector D_1 , the other with D_2 , and the measurement process projects out one of these components. The amplitudes $A_{\text{NA}}[D_i]$, Eqs. (2.25) and (2.26), describe these components: the D_1 component is $t_1 r'_2 e^{i\theta_1} \mathcal{R}|\psi\rangle + r_1 t_2 e^{i\theta_2} \mathcal{R}^{-1}|\psi\rangle$ and the D_2 component is $t_1 t'_2 e^{i\theta_1} \mathcal{R}|\psi\rangle + r_1 r_2 e^{i\theta_2} \mathcal{R}^{-1}|\psi\rangle$. If particle B is detected by D_1 , the two-particle internal state $|\psi\rangle$ changes to $|\psi'\rangle$, which is the normalized D_1 component:

$$|\psi'\rangle = \frac{1}{\sqrt{K}} (t_1 r'_2 e^{i\theta_1} \mathcal{R}|\psi\rangle + r_1 t_2 e^{i\theta_2} \mathcal{R}^{-1}|\psi\rangle), \quad (2.35)$$

where the real factor K is the normalization constant. There is a similar expression for the case particle B is detected by detector D_2 .

A priori, we may not assume that we know the precise values of t_i , r_i , r'_i , $e^{i\theta_1}$, $e^{i\theta_2}$ in the above expression (2.35); it is also doubtful whether we know $|\psi\rangle$ beforehand, but even if we do, we know little of the new state $|\psi'\rangle$. The best we can say is that $|\psi'\rangle$ will be an entangled state, i.e., it cannot be factorized as a simple tensor product $|x\rangle \otimes |y\rangle$, $|x\rangle \in V^A$, $|y\rangle \in V^B$; this is because $|\psi'\rangle$ is, see Eq. (2.35), some linear combination of $\mathcal{R}$ and $\mathcal{R}^{-1}$ acting on $|\psi\rangle$, and for nontrivial $\mathcal{R}$ and $\mathcal{R}^{-1}$ this action usually results in entanglement, independent of whether the initial state $|\psi\rangle$ was entangled. However, as we will see in the one-to-one and the many-to-one experiment, it turns out that we can work with this expression for $|\psi'\rangle$ and eventually, after observing many incident particles at the detectors, we can even determine $|\psi'\rangle$ with precision.

III. THE INEQUIVALENT CLASSES OF INTERFERENCE EXPERIMENTS

In this section we want to consider the various possibilities for the interference experiments in detail. We start with the one-to-one experiment followed by the many-to-one experiment (we already discussed the many-to-many experi-

⁶One may wonder if there are more ways to repeat an interference experiment with non-Abelian anyons. There are endless ways to do so, but these are all hybrids of the many-to-many, one-to-one, and many-to-one experiments, and introduce no new concepts. One might be tempted to say that a one-to-many experiment is missing; however, the setup is symmetrical under the exchange of particle A and B , and a one-to-many experiment would yield the same results as the many-to-one experiment does.

FIG. 8. The setup for the *one-to-one* experiment, which uses only one particle B and one particle A . Additional (intelligent) mirrors M have been placed around the Mach-Zender device to route particle B back to its original position after it has been detected by D_1 or D_2 , where it will be injected again in the apparatus. As particle B is returned, it passes particle A in a counterclockwise fashion, which means the braid operator $\mathcal{R}$ should act on the two-particle internal state. Particle A remains all the time unperturbed at the center of the apparatus. With each detection of particle B , the two-particle internal state may change.

ment in the previous section). We conclude by summarizing the different results in a Table I.

A. The one-to-one experiment

The one-to-one experiment requires only one particle A and one particle B . What we will show in this section is that after repeating the one-to-one experiment a sufficient number of times, the system will get locked into an eigenstate of the monodromy operator $\mathcal{R}^2$ with a probability determined by the initial state. We inject particle B at the left of the Mach-Zender apparatus, with particle A at its center, and detect particle B at detector D_1 or D_2 , and we repeat this procedure using the same particles B and A . This of course requires that we route particle B back to its original position—the entrance point at the left of the apparatus. We assume that we can extend the Mach-Zender device with some additional mirrors that take care of this, as depicted in Fig. 8.^{7,8} As we move non-Abelian anyons around, we should be aware that this may include some action of the braid operators; as we move particle B back, we exchange it with particle A in a counterclockwise way, which means that an additional braid operation $\mathcal{R}$ acts on the two-particle internal state $|\psi'\rangle$.

Let us take a closer look at the one-to-one experiment, paying special attention to the internal state. We start with an

⁷These mirrors should be somewhat smart, in the sense that they may adjust their position based on whether particle B emerged at D_1 or D_2 , to make sure that particle B is returned at the same position independent of the detector at which it was observed at the previous run.

⁸Particle B might need some preparation as to become a wave packet suitable to “split” at the beam splitter, for instance, it may need to be accelerated. Devices, needed for such purposes, can be placed to the left of the Mach-Zender apparatus, but we will not bother ourselves with such devices as they do not contribute to the essence of the problem.

initial two-particle internal state $|\psi\rangle \equiv |\psi_0\rangle$, and insert particle B in the apparatus. We then observe particle B at one of the detectors D_i . The probability to emerge at detector D_i is of course given by Eq. (2.27): $P_{\text{NA}}[D_i] = P_{\text{NA}}[D_i]_{|\psi_0\rangle}$, to show explicitly the dependence on the internal state $|\psi_0\rangle$. Since the probability for each detector will in general be nonzero, there are two possible outcomes for the first run: particle B is detected at D_1 or at D_2 . If it is at detector D_1 and we return particle B to its initial position, the new two-particle internal state $|\psi_1\rangle$ (we will use $|\psi_j\rangle$ to indicate the internal state after the j th run) will be

$$\begin{aligned} |\psi_1\rangle &= \mathcal{R} \frac{1}{\sqrt{K}} (t_1 r'_2 e^{i\theta_1} \mathcal{R} |\psi\rangle + r_1 t_2 e^{i\theta_2} \mathcal{R}^{-1} |\psi\rangle) \\ &= \frac{1}{\sqrt{K}} (t_1 r'_2 e^{i\theta_1} \mathcal{R}^2 + r_1 t_2 e^{i\theta_2}) |\psi_0\rangle. \end{aligned} \quad (3.1)$$

If we would find particle B at detector D_2 , the new state $|\psi_1\rangle$ would be

$$|\psi_1\rangle = \frac{1}{\sqrt{K'}} (t_1 t'_2 e^{i\theta_1} \mathcal{R}^2 + r_1 r_2 e^{i\theta_2}) |\psi_0\rangle, \quad (3.2)$$

where K' is a normalization constant that will in general be different from K .

We can easily continue this for a second run. The probability to observe particle B at detector D_i at the second run is given by $P_{\text{NA}}[D_i]_{|\psi_1\rangle}$, where we note that $|\psi_1\rangle$, albeit implicitly, depends on the outcome of the first run. Of course, there are two possible outcomes for the second run, and the resulting internal state $|\psi_2\rangle$ will depend on the outcomes of the first and second run. We could follow this procedure for a large number of runs, say n . However, this means we would have to work with an internal state $|\psi_n\rangle$ depending on the outcomes of all n runs, meaning there would be 2^n possible outcomes to consider. Although this would probably lead to the correct answer, the number 2^n grows so incredibly fast, that this way is not advisable, and indeed we will follow another path to the answer.⁹

We are basically interested in two things after we repeated the experiment for n runs: first the interference pattern, i.e., how many times $\mathcal{N}[D_i]$ out of the total n we observed the incident particle at detector D_i , and second, what we can say about the internal state $|\psi_n\rangle$ given the interference pattern.

The interference pattern has something to do with the probabilities $P_{\text{NA}}[D_i]_{|\psi_j\rangle}$, $j=1,2,\dots,n$. Since $|\psi_1\rangle$ is different from $|\psi_0\rangle$, which is in the general case true and inde-

pendent of the outcome of the first run, we can write down the following inequality concerning their probability distributions:

$$P_{\text{NA}}[D_i]_{|\psi_1\rangle} \neq P_{\text{NA}}[D_i]_{|\psi_0\rangle}. \quad (3.3)$$

One may wonder if such an inequality holds also for the j th and $(j+1)$ th run. This would mean that the probabilities keep fluctuating from run to run and the resulting interference pattern would become random, and thereby also independent of the initial internal state $|\psi_0\rangle$. However, it turns out that this is not so, and after a large number of runs, the inequality of Eq. (3.3) converges to an equality,

$$P_{\text{NA}}[D_i]_{|\psi_{n+1}\rangle} = P_{\text{NA}}[D_i]_{|\psi_n\rangle} \quad \text{for } n \rightarrow \infty. \quad (3.4)$$

The equality in Eq. (3.4) is automatically satisfied if $|\psi_{n+1}\rangle$ and $|\psi_n\rangle$ would differ only by a phase factor, i.e.,

$$|\psi_{n+1}\rangle = e^{i\alpha} |\psi_n\rangle \quad \text{for arbitrary } e^{i\alpha}. \quad (3.5)$$

There is a class of internal states that obey Eq. (3.5): the eigenstates of the monodromy operator $\mathcal{R}^2$. If $|\psi_n\rangle$ would be an eigenstate of $\mathcal{R}^2$ with eigenvalue $e^{i\lambda}$ then, after the $(n+1)$ th run, the internal state $|\psi_{n+1}\rangle$ would be, according to Eq. (3.1) if the outcome of the n th run had been observation at detector D_1 ,

$$|\psi_{n+1}\rangle = e^{i\alpha} |\psi_n\rangle, \quad e^{i\alpha} = \frac{1}{\sqrt{K}} (t_1 r'_2 e^{i\theta_1} e^{i\lambda} + r_1 t_2 e^{i\theta_2}). \quad (3.6)$$

If at the $(n+1)$ th run, particle B was found at detector D_2 , the new internal state would have been, using Eq. (3.2),

$$|\psi_{n+1}\rangle = e^{i\alpha'} |\psi_n\rangle, \quad e^{i\alpha'} = \frac{1}{\sqrt{K'}} (t_1 t'_2 e^{i\theta_1} e^{i\lambda} + r_1 r_2 e^{i\theta_2}). \quad (3.7)$$

The probability distributions $P_{\text{NA}}[D_i]_{|\psi_n\rangle}$ indeed turn out to converge to a fixed value that is associated with an eigenvalue $e^{i\lambda}$ of the monodromy operator, when the number of runs n becomes large, i.e.,

$$\begin{aligned} \lim_{n \rightarrow \infty} P_{\text{NA}}[D_1]_{|\psi_n\rangle} &= |t_1 r'_2|^2 + |r_1 t_2|^2 \\ &\quad + 2Q \operatorname{Re}(t_1 r'_2 r_1^* t_2^* e^{i\theta} e^{i\lambda}). \end{aligned} \quad (3.8)$$

In Eq. (3.8) we still used the notation that includes the Mach-Zender apparatus' coefficients; let us write Eq. (3.8) in the preferred device-independent notation,

$$\lim_{n \rightarrow \infty} P_{\text{NA}}[D_i]_{|\psi_n\rangle} = P_{e^{i\lambda}}[D_i]. \quad (3.9)$$

It is not only the probability distributions that converge, for the internal state $|\psi_n\rangle$ also becomes fixed: as the eigenstate of $\mathcal{R}^2$ that carries the eigenvalue $e^{i\lambda}$, in other words,

⁹Actually, the first thing we did was a numerical simulation. We did not consider all 2^n outcomes, but tried to simulate the “real” thought experiment: guided by probabilities we followed branches in the “tree” of all 2^n outcomes. These numerical simulations lead us straight to the results we are presenting here.

$$\lim_{n \rightarrow \infty} \langle \psi_n | \mathcal{R}^2 | \psi_n \rangle = e^{i\lambda}. \quad (3.10)$$

So, we conclude that *the internal state is locked into an eigenstate of $\mathcal{R}^2$* ; the only change that subsequent runs of the experiment can accomplish, is an uninteresting overall phase factor $e^{i\alpha}$, as follows from Eqs. (3.6) and (3.7).

But $\mathcal{R}^2$ has in general more than one eigenvalue. What can we say about which eigenvalue $e^{i\lambda}$ will come out, i.e., to which eigenvalue and eigenstate do $P_{NA}[D_i]|\psi_n\rangle$ and $|\psi_n\rangle$ will converge? This is where the initial internal state $|\psi\rangle = |\psi_0\rangle$ comes in; if the expectation value of $\mathcal{R}^2$ and $|\psi\rangle$ is given by

$$\langle \psi | \mathcal{R}^2 | \psi \rangle = \sum_{\lambda} p_{\lambda} e^{i\lambda}, \quad (3.11)$$

then the probability that the two-particle system gets locked in the eigenstate of $e^{i\lambda}$ is given by p_{λ} .

We will now consider the actual interference pattern $I_{NA}^{o-o}[D_i]$ of the one-to-one (*o-o*) experiment with non-Abelian anyons and argue that it equals the fixed probability distribution of Eq. (3.9). Let us assume that projection, i.e., locking, is not achieved until the m th run. Then the outcomes of the first m runs were governed by probabilities other than those of Eq. (3.9) and the interference pattern, i.e., the collection of outcomes at the time of the m th run, will not look like Eq. (3.9). However, after the m th run, we can repeat the experiment for an additional n runs, where n can be much greater than m ; all these subsequent outcomes will obey Eq. (3.9) and, since $n \gg m$, completely dominate the interference pattern, rendering the first m contributions negligible,

$$\lim_{n \rightarrow \infty} I_{NA}^{o-o}[D_i] = P_{e^{i\lambda}}[D_i]. \quad (3.12)$$

Not only does the monodromy operator $\mathcal{R}^2$, in general, have multiple distinct eigenvalues, these eigenvalues can be degenerate as well. This degeneracy has no effect on the resulting interference pattern, which remains of the form of Eq. (3.12). However, the internal state does not become projected onto a particular eigenstate, but rather onto the eigenspace of eigenvalue $e^{i\lambda}$:

$$\lim_{n \rightarrow \infty} |\psi_n\rangle = \frac{E_{\lambda} |\psi\rangle}{\sqrt{\langle \psi | E_{\lambda} | \psi \rangle}}, \quad (3.13)$$

where the square root in the denominator is there to normalize the state; E_{λ} is the projection operator projecting onto the eigenspace of $e^{i\lambda}$. The probability p_{λ} can also be given in terms of the projection operator E_{λ} ,

$$p_{\lambda} = \langle \psi | E_{\lambda} | \psi \rangle. \quad (3.14)$$

We have showed that eigenstates of the monodromy operator are fixed states, and said that all initial internal states eventually become locked. The complete (nontrivial) proof of this convergence of arbitrary initial states is rather lengthy, and we have relegated it to the Appendix.

Projection is achieved after a large but finite number of runs. How do we know when we have reached such a projected, fixed, state? This, we determine from the interference patterns: if we can identify the interference pattern $I_{NA}^{o-o}[D_i]$ with extreme certainty with one and only one $P_{e^{i\lambda}}[D_i]$, we know the state is projected onto the eigenspace of $e^{i\lambda}$. The harder it is to distinguish the $P_{e^{i\lambda}}[D_i]$ for different $e^{i\lambda}$, the longer (more runs) it takes to realize the projection.

Let us conclude the one-to-one experiment with an example, with the same values (2.14) for the coefficients of the Mach-Zender interferometer, and a monodromy matrix $\mathcal{R}^2$ with eigenvalues ± 1 , as we used before. In the following table we give the interference patterns $I_{NA}^{o-o}[D_i]$ and their probability to come out of the experiment for four different initial internal states (which are characterized by p_1 and p_{-1}). So, in this example, there are two possible observable interference patterns: either we observe nine out of ten times particle B at detector D_1 and one out of ten at detector D_2 or the other way around (see the Table). Of the two situations we may end up with, we can only calculate the probabilities, as it should in this quantum mechanical setting.

p_1	p_{-1}	$I_{NA}^{o-o}[D_1]$	$I_{NA}^{o-o}[D_2]$	Probability
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{9}{10}$	$\frac{1}{10}$	$\frac{1}{2}$
		$\frac{1}{10}$	$\frac{9}{10}$	$\frac{1}{2}$
1	0	$\frac{9}{10}$	$\frac{1}{10}$	1
0	1	$\frac{1}{10}$	$\frac{9}{10}$	1
$\frac{3}{8}$	$\frac{5}{8}$	$\frac{9}{10}$	$\frac{1}{10}$	$\frac{3}{8}$
		$\frac{1}{10}$	$\frac{9}{10}$	$\frac{5}{8}$

B. The many-to-one experiment: probe one particle with many others

In the many-to-one experiment, the experiment is repeated with the same particle A but with a new particle B each time. This will lead to a different result, for now the system will get locked into an eigenstate of a different operator called U , which will be defined below.

The setup of the Mach-Zender apparatus in this case no longer requires the extension with mirrors as needed for the one-to-one experiment, but it does require a large supply, i.e., a “bag” with many identical particles B in it, on the left of the apparatus, see Fig. 9. We will be using the notation/convention for braids of these multiple particles as we introduced in Sec. I (i.e., writing particles left in the experiment left in the tensor product, $\mathcal{R}_i$ interchanging particles i and $(i+1)$ counterclockwise, numbering from right to left beginning with one). The total system’s initial internal state $|\psi\rangle$ is an element of the tensor product of the internal spaces V^B of the B particles and V^A of the A particle:

$$|\psi\rangle \in V^B \otimes \dots \otimes V^B \otimes V^B \otimes V^A. \quad (3.15)$$

Once a B particle has been used, it becomes uninteresting but we will not discard it completely, merely collect such B particles somewhere on the right of the apparatus.

For reasons of simplicity and to give a full solution, we will subject the internal state $|\psi\rangle$ to the following conditions: the initial internal state $|\psi\rangle$ should factorize completely, i.e.,

FIG. 9. The *many-to-one* experiment: the same particle A remains continuously at the center of the apparatus during the experiment, but particle B gets replaced by a new one after detection. All B particles initially carry the same internal state. The multiparticle internal state changes with each observation of the current particle B at detector D_1 or D_2 and may become highly entangled.

the initial internal state should be *unentangled*, and furthermore, the B particles should have trivial braiding,

$$\mathcal{R}_i|\psi\rangle = \mathcal{R}_i^{-1}|\psi\rangle = \sigma_i|\psi\rangle = |\psi\rangle, \quad \forall i \geq 2, \quad (3.16)$$

where σ_i is the operator that only swaps particle i and $(i+1)$ without affecting their internal state. Applying both conditions to the internal state $|\psi\rangle$, implies that $|\psi\rangle$ can be written in the following form:

$$|\psi\rangle = y^{i_n} \dots y^{i_2} y^{i_1} x^\alpha |e_{i_n}\rangle \dots |e_{i_2}\rangle |e_{i_1}\rangle |f_\alpha\rangle, \quad (3.17)$$

where $\{|e_i\rangle\}$ is a basis for the internal space V^B of the j th particle B , and $\{|f_\alpha\rangle\}$ a basis for V^A .

Let us start the first run of the thought experiment: we insert the first particle B in the Mach-Zender interferometer and observe it again at either detector D_1 or D_2 . The probabilities $P_{NA}[D_i]$ for detector D_i are of course given by Eqs. (2.31), (2.32), and depend on the expectation value $\langle\psi_0|\mathcal{R}_1^2|\psi_0\rangle$ of the monodromy operator $\mathcal{R}_1^2$ that acts on particle A and the first particle B , with respect to the initial internal state $|\psi_0\rangle \equiv |\psi\rangle$. Let us assume we observe the first incident particle at detector D_1 , then the new, changed, internal state, which we indicate by $|\psi_1\rangle$, is given by

$$|\psi_1\rangle = \frac{1}{\sqrt{K}} (t_1 r_2' e^{i\theta_1} \mathcal{R}_1 + r_1 t_2 e^{i\theta_2} \mathcal{R}_1^{-1}) |\psi_0\rangle. \quad (3.18)$$

Because the first particle B is now to the right of particle A , the new internal state $|\psi_1\rangle$ is, strictly speaking, an element of a different vector space than that of $|\psi_0\rangle$,

$$|\psi_1\rangle \in V^B \otimes \dots \otimes V^B \otimes V^B \otimes V^A \otimes V^B. \quad (3.19)$$

Furthermore, the first particle B and particle A are now entangled, so the internal state $|\psi_1\rangle$ cannot be completely factorized anymore:

$$|\psi_1\rangle = y^{i_n} \dots y^{i_2} z^{\alpha i_1} |e_{i_n}\rangle \dots |e_{i_2}\rangle |f_\alpha\rangle |e_{i_1}\rangle, \quad (3.20)$$

where the components $z^{\alpha i_1}$ are determined by the coefficients of Eq. (3.18) and the components x^α and y^i . The second up

to the n th B particle, still have trivial braiding, which we can generalize to the j th run of the experiment by the statement

$$\mathcal{R}_i|\psi_j\rangle = |\psi_j\rangle, \quad i > j+1. \quad (3.21)$$

So far for the first run.

For the second run, the next particle B is injected in the Mach-Zender apparatus, and the probabilities to observe it at detector D_i are now governed by the expectation value $\langle\psi_1|\mathcal{R}_2^2|\psi_1\rangle$ of the current internal state $|\psi_1\rangle$ and the monodromy operator $\mathcal{R}_2^2$ that operates on particle A and the second particle B . This expectation value will in general be different from the expectation value of the first run: $\langle\psi_1|\mathcal{R}_2^2|\psi_1\rangle \neq \langle\psi_0|\mathcal{R}_1^2|\psi_0\rangle$, and thus the probabilities for the first and second run will be different. But, just as in the case of the one-to-one experiment, the probabilities and thus the expectation values will converge to a final, fixed value,

$$\langle\psi_{n+1}|\mathcal{R}_{n+2}^2|\psi_{n+1}\rangle = \langle\psi_n|\mathcal{R}_{n+1}^2|\psi_n\rangle, \quad n \rightarrow \infty. \quad (3.22)$$

We would like to know which states obey this equality (3.22) and what the expectation value will be. These will not be eigenstates and eigenvalues of a monodromy operator, as for the one-to-one experiment, but something related.

Let us now start solving Eq. (3.22) if we assume that the $(n+1)$ th particle B has been observed at detector D_1 , and use that we know $|\psi_{n+1}\rangle$ in terms of $|\psi_n\rangle$, as in Eq. (3.18),

$$\begin{aligned} \langle\psi_{n+1}|\mathcal{R}_{n+2}^2|\psi_{n+1}\rangle &= \frac{1}{K} (|t_1 r_2'|^2 \langle\psi_n|\mathcal{R}_{n+1}^{-1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}|\psi_n\rangle \\ &\quad + t_1 r_2' r_1^* t_2^* e^{i(\theta_1 - \theta_2)} \langle\psi_n|\mathcal{R}_{n+1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}|\psi_n\rangle \\ &\quad + |r_1 t_2|^2 \langle\psi_n|\mathcal{R}_{n+1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}^{-1}|\psi_n\rangle \\ &\quad + t_1^* r_2'^* r_1 t_2 e^{-i(\theta_1 - \theta_2)} \langle\psi_n|\mathcal{R}_{n+1}^{-1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}^{-1}|\psi_n\rangle) \\ &= \langle\psi_n|\mathcal{R}_{n+1}^2|\psi_n\rangle, \end{aligned} \quad (3.23)$$

where the normalization constant K is given by

$$\begin{aligned} K &= |t_1 r_2'|^2 + |r_1 t_2|^2 \\ &\quad + 2 \operatorname{Re}(t_1 r_2' r_1^* t_2^* e^{i(\theta_1 - \theta_2)} \langle\psi_n|\mathcal{R}_{n+1}^2|\psi_n\rangle). \end{aligned} \quad (3.24)$$

These two expressions, (3.23), (3.24), may look horrid [and we can write down another pair for the case that outcome of the $(n+1)$ th run is observation at detector D_2], but they are not that bad if we apply the braid relations to them, i.e., use the fact that the $\mathcal{R}_n$ satisfy the Yang-Baxter relations. Using Eq. (3.21) as well, two of the four expectation values in Eq. (3.23) simplify to

$$\begin{aligned} \langle\psi_n|\mathcal{R}_{n+1}^{-1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}|\psi_n\rangle &= \langle\psi_n|\mathcal{R}_{n+2} \mathcal{R}_{n+1}^2 \mathcal{R}_{n+2}^{-1}|\psi_n\rangle \\ &= \langle\psi_n|\mathcal{R}_{n+1}^2|\psi_n\rangle \\ &\equiv \kappa, \end{aligned} \quad (3.25)$$

$$\begin{aligned}
\langle \psi_n | \mathcal{R}_{n+1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}^{-1} | \psi_n \rangle &= \langle \psi_n | \mathcal{R}_{n+2}^{-1} \mathcal{R}_{n+1}^2 \mathcal{R}_{n+2} | \psi_n \rangle \\
&= \langle \psi_n | \mathcal{R}_{n+1}^2 | \psi_n \rangle \\
&= \kappa,
\end{aligned} \tag{3.26}$$

where we introduced κ to indicate the expectation value $\langle \psi_n | \mathcal{R}_{n+1}^2 | \psi_n \rangle$. Equation (3.23) will be solved if $|\psi_n\rangle$ is such that

$$\langle \psi_n | \mathcal{R}_{n+1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1} | \psi_n \rangle = \kappa^2, \tag{3.27}$$

$$\langle \psi_n | \mathcal{R}_{n+1}^{-1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}^{-1} | \psi_n \rangle = \kappa \kappa^*. \tag{3.28}$$

Let us rewrite these expectation values too, by applying then Yang-Baxter equation and Eq. (3.21) once more:

$$\langle \psi_n | \mathcal{R}_{n+1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1} | \psi_n \rangle = \langle \psi_n | \mathcal{R}_{n+1}^2 \sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2} | \psi_n \rangle, \tag{3.29}$$

$$\begin{aligned}
\langle \psi_n | \mathcal{R}_{n+1}^{-1} \mathcal{R}_{n+2}^2 \mathcal{R}_{n+1}^{-1} | \psi_n \rangle &= \langle \psi_n | \mathcal{R}_{n+1}^{-2} \sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2} | \psi_n \rangle.
\end{aligned} \tag{3.30}$$

Although $\mathcal{R}_{n+1}^2$ and $\sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2}$ are distinct operators, their matrix elements $(R^2)_{j\beta}^{i\alpha}$ are equal:

$$(R^2)_{j\beta}^{i\alpha} = \langle e^i | \langle f^\alpha | \mathcal{R}^2 | e_j \rangle | f_\beta \rangle. \tag{3.31}$$

We will decompose the right-hand side of Eq. (3.29) in a similar way in components relative to the bases $\{|e_i\rangle\}$, $\{|f_\alpha\rangle\}$ of the vector spaces V^B and V^A . The internal state $|\psi_n\rangle$ decomposes as

$$|\psi_n\rangle = y^i y^j z^{\alpha i_n \dots i_1} |e_i\rangle |e_j\rangle |f_\alpha\rangle |e_{i_n}\rangle \dots |e_{i_1}\rangle. \tag{3.32}$$

Next, we let the combined operators $\mathcal{R}_{n+1}^2$ and $\sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2}$ act on a basis state (ignoring the $|e_{i_n}\rangle \dots |e_{i_1}\rangle$ part) to give

$$\begin{aligned}
&\mathcal{R}_{n+1}^2 \sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2} |e_i\rangle |e_j\rangle |f_\alpha\rangle \\
&= (R^2)_{i\alpha}^{k\beta} (R^2)_{j\beta}^{l\gamma} |e_k\rangle |e_l\rangle |f_\gamma\rangle.
\end{aligned} \tag{3.33}$$

If we also introduce the density matrices ρ_A and ρ_B of particles A and B ,

$$(\rho_A)_\beta^\alpha = z^{\alpha i_n \dots i_1} z_{\beta i_n \dots i_1}, \quad (\rho_B)_j^i = y^i y_j, \tag{3.34}$$

then the expectation value of Eq. (3.29) can be written as follows:

$$\langle \psi_n | \mathcal{R}_{n+1}^2 \sigma_{n+2} \mathcal{R}_{n+1}^2 \sigma_{n+2} | \psi_n \rangle \tag{3.35}$$

$$\begin{aligned}
&= (R^2)_{i\alpha}^{k\beta} (R^2)_{j\beta}^{l\gamma} (\rho_B)_k^i (\rho_B)_l^j (\rho_A)_\gamma^\alpha \\
&= [(\rho_B)_l^j (R^2)_{j\beta}^{l\gamma}] [(\rho_B)_k^i (R^2)_{i\alpha}^{k\beta}] (\rho_A)_\gamma^\alpha \\
&= U_\beta^\gamma U_\alpha^\beta (\rho_A)_\gamma^\alpha \\
&= \text{Tr}(U U \rho_A).
\end{aligned} \tag{3.36}$$

Here, in the last two steps of Eq. (3.36), we introduced—as announced—an operator U that operates on V^A and is defined as the partial trace over the B space of the monodromy operator $\mathcal{R}^2$ and the density matrix of particle B :

$$U_\beta^\alpha = (\rho_B)_i^j (R^2)_{j\beta}^{i\alpha}, \quad U_A = \text{Tr}_B(\rho_B \mathcal{R}_{BA}^2). \tag{3.37}$$

Let us now reconsider our original problem in terms of the U and κ . We can express κ as the trace over U and the density matrix of particle A ,

$$\kappa = \text{Tr}(U \rho_A). \tag{3.38}$$

The condition to satisfy equality (3.23), as given by Eq. (3.27), becomes in terms of U ,

$$\text{Tr}(U^2 \rho_A) = \kappa^2. \tag{3.39}$$

Since the trace of an operator and a density matrix can be regarded as the expectation value of that operator, we have that

$$\langle U^2 \rangle = \langle U \rangle^2, \tag{3.40}$$

which can only be true if κ is an eigenvalue of U and ρ_A lies in the eigenspace of the eigenvalue κ .

Just as in the one-to-one experiment, the probability distributions for the incident particles converge to a fixed distribution associated with the eigenvalue of an operator, and the final distribution determines the interference pattern. Also, the system becomes projected onto the eigenspace of the observed eigenvalue. However, the eigenvalues and eigenstates are no longer given by the monodromy operator $\mathcal{R}^2$ as in the one-to-one experiment; in the many-to-one (m - o) experiment, we observe an eigenvalue κ of the operator U and have projection onto the eigenspace of that eigenvalue. If we denote the probability distributions by $P_{\text{NA}}^{m-o}[D_i]$, we see that for large n they become equal to the interference patterns $I_{\text{NA}}^{m-o}[D_i]$:

$$\begin{aligned}
\lim_{n \rightarrow \infty} P_{\text{NA}}^{m-o}[D_1]_{|\psi_n\rangle} &= \lim_{n \rightarrow \infty} I_{\text{NA}}^{m-o}[D_1] \\
&= |t_1 r_2'|^2 + |r_1 t_2|^2 + 2Q \text{Re}(t_1 r_2' r_1^* t_2^* e^{i\theta} \kappa) \\
&\equiv P_\kappa[D_1],
\end{aligned} \tag{3.41}$$

where we used $P_\kappa[D_1]$ to indicate the locked probability distribution that is associated with the eigenvalue κ ; of course there are similar expressions for D_2 . As far as the projection of the internal state is concerned, there is, strictly speaking, only a projection in a subsystem of the total system's internal state, namely, the subsystem describing the internal state of particle A , because U acts on V^A only. It is the density matrix of particle A that becomes projected onto the associated eigenspace of κ ,

$$\rho_A \rightarrow \frac{E_\kappa \rho_A E_\kappa}{\text{Tr}(E_\kappa \rho_A)} \quad \text{for } n \rightarrow \infty, \tag{3.42}$$

where E_κ is the projection operator of the eigenspace of κ . Just as the monodromy operator $\mathcal{R}^2$ can have multiple (de-

generate) eigenvalues, so can U . Each eigenvalue κ has a probability p_κ to be observed in the end; p_κ is determined by the initial density matrix ρ_A :

$$p_\kappa = \text{Tr}(E_\kappa \rho_A), \quad (\rho_A)^\alpha_\beta = x^\alpha x_\beta. \quad (3.43)$$

We have showed above that there exist states for which the probability distributions are locked, and that there is a chance p_κ to end up in each locked state. The proof of it is almost the same as that of the convergence in the one-to-one experiment and can be found in the Appendix.

To solve the many-to-one experiment, we introduced two conditions: the initial internal state should factorize completely and the B particles should have trivial braiding. We believe these conditions are too restrictive: the many-to-one experiment should be possible for arbitrary entangled initial states, subjected only to the condition that the B particles have Abelian braiding,

$$\mathcal{R}_j |\psi_k\rangle = e^{is_B} |\psi_k\rangle, \quad j > k, \quad (3.44)$$

where the s_B in the phase factor e^{is_B} is also known as the spin of the B particles. In this generalized many-to-one experiment, the interference patterns also depend explicitly on κ :

$$\lim_{n \rightarrow \infty} I_{\text{NA}}^{m-o}[D_i] = P_\kappa[D_i], \quad (3.45)$$

but κ can no longer be regarded as the eigenvalue of the operator U ; the definition of U becomes ambiguous, because U is the trace of $\mathcal{R}^2$ and the density matrix ρ_B of the B particles, and in this case ρ_B is no longer constant. We have not yet succeeded in proving this for the generalized many-to-one experiment, so the claim of Eq. (3.45) with the condition of Eq. (3.44) remains for the moment a *conjecture*.¹⁰

Let us now give an example to illustrate the many-to-one experiment, which is borrowed from an example in Ref. [19]. We will take the internal space V^A of the A particle to be three-dimensional and to have a basis $\{|1\rangle, |2\rangle, |3\rangle\}$. The internal space V^B will be two-dimensional and it will have a basis $|+\rangle, |-\rangle$. The monodromy operator $\mathcal{R}^2$ is thus six-dimensional. Relative to the chosen bases, the monodromy operator we will use is, in a basis $\{|+\rangle|1\rangle, |-\rangle|1\rangle, |+\rangle|2\rangle, |-\rangle|2\rangle, |+\rangle|3\rangle, |-\rangle|3\rangle\}$, given by

$$\mathcal{R}^2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & & & & \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & & & & \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ & & & & -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ & & & & -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}. \quad (3.46)$$

This specific $\mathcal{R}^2$ has two eigenvalues: $+1, -1$, which are both threefold degenerate. Of course, it is a unitary matrix. We choose the incident B particles to be in the state $|+\rangle$, which means that their density matrix in a basis $\{|+\rangle, |-\rangle\}$ is

$$\rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.47)$$

The matrix of the operator U can be simply obtained by performing the partial trace of $\mathcal{R}^2$ and ρ_B , in a basis $\{|1\rangle, |2\rangle, |3\rangle\}$:

$$U = \text{Tr}_B(\mathcal{R}^2 \rho_B) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \quad (3.48)$$

So, there are two different eigenvalues κ : $+1$ and $-\frac{1}{2}$. If we now plug in the values for the Mach-Zender apparatus' coefficients, Eq. (2.14), in the interference patterns $I_{\text{NA}}^{m-o}[D_i]$ associated with each eigenvalue, these interference patterns take the form given in the Table below. One outcome of this example of the many-to-one experiment is equal to the outcome of the ordinary interference experiment and the one-to-one experiment: nine out of ten times the incident particle is found at detector D_1 . The other possible outcome is rather different: three out of ten times it is detected by detector D_1 and with a 70% chance the B particle is observed at detector D_2 .

Eigenvalue κ	Interference patterns	
	$I_{\text{NA}}^{m-o}[D_1]$	$I_{\text{NA}}^{m-o}[D_2]$
$+1$	$\frac{9}{10}$	$\frac{1}{10}$
$-\frac{1}{2}$	$\frac{3}{10}$	$\frac{7}{10}$

¹⁰Proving Eq. (3.45) with Eq. (3.44) certainly requires some additional properties of non-Abelian anyons and physical restrictions on the allowed internal states, especially concerning the creation and fusion of non-Abelian anyons, for which the description of non-Abelian anyons through a quantum group may be instrumental. This is beyond the scope of the present paper.

The above example also demonstrates that the eigenvalues of the operator U need not have an absolute value of one. This would have been the case if U were a unitary operator, but U is merely a *normal* operator, i.e., it commutes with its adjoint

TABLE I. Summary of the results for the three different thought interference experiments with non-Abelian anyons we discussed. Notice that the results are independent of the exact setup of the Mach-Zender interferometer and may be applied to other setups as well. The one-to-one and many-to-one experiment lock the system into an eigenpattern or eigenvalue associated with an operator (either the monodromy operator $\mathcal{R}^2$, or U : the partial trace over $\mathcal{R}^2$ and the density matrix of particle B) acting on the internal state, and project the internal state onto the eigenspace. The many-to-many experiment merely determines the expectation value of the monodromy operator $\mathcal{R}^2$, and ignores any change in the internal state due to the measurements. Although interference patterns cannot be identified with probability distributions, there is still some conservation of probability: the average of the outcomes of all three different schemes is the same. For ordinary interference experiments or Aharonov-Bohm experiments there is no need to distinguish between the many-to-many, the one-to-one and the many-to-one schemes, as all schemes will yield the same results in these cases, the difference becomes important for non-Abelian anyons only.

	Type of interference experiment		
	Many-to-many	One-to-one	Many-to-one ^a
The measured expectation/eigen value, $\langle \psi_n \mathcal{R}_{n+1}^2 \psi_n \rangle$ for $n \rightarrow \infty$	$\langle \psi \mathcal{R}^2 \psi \rangle$	$e^{i\lambda}$	κ
The diagonalized operator	Not relevant	$\mathcal{R}^2$	$U = \text{Tr}_B(\mathcal{R}^2 \rho_B)$
The observed interference pattern, $I_{\text{NA}}[D_i] = \lim_{n \rightarrow \infty} P_{\text{NA}}[D_i]_{\langle \psi_n \mathcal{R}_{n+1}^2 \psi_n \rangle}$	$\sum_{\lambda} p_{\lambda} P_{e^{i\lambda}}[D_i]$	$P_{e^{i\lambda}}[D_i]$	$P_{\kappa}[D_i]$
Change of internal state	Not relevant	$ \psi\rangle \rightarrow \frac{E_{\lambda} \psi\rangle}{\sqrt{\langle \psi E_{\lambda} \psi \rangle}}$	$\rho_A \rightarrow \frac{E_{\kappa} \rho_A E_{\kappa}}{\text{Tr}_A(E_{\kappa} \rho_A)}$
Probability of eigenvalue/pattern	1	$p_{\lambda} = \langle \psi E_{\lambda} \psi \rangle$	$p_{\kappa} = \text{Tr}_A(E_{\kappa} \rho_A)$
Average of all outcomes	$\langle \psi \mathcal{R}^2 \psi \rangle$	$\sum_{\lambda} p_{\lambda} e^{i\lambda} = \langle \psi \mathcal{R}^2 \psi \rangle$	$\sum_{\kappa} p_{\kappa} \kappa = \langle \psi \mathcal{R}^2 \psi \rangle$
Aharonov-Bohm pattern $I_{\text{AB}}[D_i]$ with $\mathcal{R}^2 = e^{i\lambda} 1$	$P_{e^{i\lambda}}[D_i]$	$P_{e^{i\lambda}}[D_i]$	$P_{e^{i\lambda}}[D_i]$
The plain interference pattern $I[D_i]$ with $\mathcal{R}^2 = 1$	$P_1[D_i]$	$P_1[D_i]$	$P_1[D_i]$

^aRecall that to solve the many-to-one experiment we posed some restrictions on the internal state of the B particles; the operator U can only be defined if the initial internal state is completely unentangled.

$$UU^{\dagger} = U^{\dagger}U, \quad (3.49)$$

which follows from the unitarity of $\mathcal{R}^2$ and the hermicity of density matrices. Because of the definition of U in terms of the monodromy operator $\mathcal{R}^2$, one can easily deduce that κ can be written as a linear combination of eigenvalues $e^{i\lambda}$ of $\mathcal{R}^2$ and that its absolute value cannot exceed one,

$$\kappa = \sum_{\lambda} p_{\lambda} e^{i\lambda}, \quad \sum_{\lambda} p_{\lambda} = 1, \quad \Rightarrow |\kappa| \leq 1. \quad (3.50)$$

C. Summary

Let us summarize the results of the three different thought experiments we have described in this paper: the many-to-many, one-to-one, and many-to-one experiments. Table I gives a compact survey of the results.

The topological interactions of non-Abelian anyons are generally agreed upon to be observable in interference ex-

periments. We have demonstrated that such interference experiments are generalizations of interference experiments of the Aharonov-Bohm type. However, whereas in the Aharonov-Bohm versions, the probability distributions are identified with the interference patterns, this cannot be readily carried over to experiments involving non-Abelian anyons. There it is crucial to explicitly describe the way in which each run of an experiment is repeated.

In the many-to-many experiment, each new run is conducted with fresh particles A and B carrying the initial internal state $|\psi\rangle$. In this case, by definition, the interference pattern reproduces the probability distribution, and is characterized by the expectation value $\langle \psi | \mathcal{R}^2 | \psi \rangle$ of the monodromy operator $\mathcal{R}^2$.

The scheme of the one-to-one experiment uses the same two particles A and B over and over again. The resulting interference pattern now does not depend on $\langle \psi | \mathcal{R}^2 | \psi \rangle$, but rather on an eigenvalue $e^{i\lambda}$ of the unitary monodromy operator $\mathcal{R}^2$. Because there is a clear analogy between the eigen-

values $e^{i\lambda}$ and the associated interference patterns $P_{e^{i\lambda}}[D_i]$ we could use the term *eigenpattern* to indicate the eigenvalue-pattern $P_{e^{i\lambda}}[D_i]$. So, in the one-to-one experiment we observe an eigenpattern, or in other words, we measure an eigenvalue. The measurement of the eigenvalue projects the two-particle internal state onto the eigenspace of $e^{i\lambda}$. The probability to observe a specific eigenpattern is determined by p_λ : the fraction of the internal state $|\psi\rangle$ that carries eigenvalue $e^{i\lambda}$.

The many-to-one experiment describes the way one intuitively (and conventionally) envisages the repetition of an interference experiment with non-Abelian anyons: fix particle A inside the interferometer and direct many identically prepared particles B at it. However, the many-to-one turns out mathematically to be the hardest of all three schemes to analyze, as we need to deal with the total system's internal state for each run. If we restrict ourselves to initially unentangled internal states, we can identify the resulting eigenpattern $P_\kappa[D_i]$ with the eigenvalue κ of an operator U that acts on the internal space of particle A ; the operator U explicitly depends on the internal state of particle B , as U is the partial trace over $\mathcal{R}^2$ and ρ_B . The possible eigenvalues κ of U thus also depend on ρ_B . The projection of the system onto the eigenspace of κ effectively only affects the A particle: it is its density matrix ρ_A that gets projected.

Interference experiments, including thought experiments, obviously need a device that can create interference; the topological interaction of the non-Abelian anyons does not cause interference, it only alters existing interference. We chose to use the Mach-Zender interferometer and explicitly used its determining coefficients t_1 , r'_2 , Q , etc., in many equations and our examples. In the end, however, we switched to a notation that is independent of the exact setup, and only depends on eigenvalues and eigenpatterns. The results in Table I are thus not restricted to experiments with the Mach-Zender interferometer, but apply to interference experiments in general, for instance, double-slit experiments.

Although the three schemes yield different results, the probability distribution for the first incident particle is the same for each scheme, and we expect to see some conservation of this initial probability. This is achieved by taking the *average* of all outcomes. When averaged, the many-to-many, one-to-one, and many-to-one experiments all return the same result, which reflects the equal probability for the first run of either experiment.

Being a generalization, the non-Abelian anyon experiments should of course cover the Aharonov-Bohm effect and the ordinary interference experiments as well. If we let the monodromy operator act on a trivial internal space, i.e., $\mathcal{R}^2 = e^{i\lambda}\mathbb{1}$ or just $\mathcal{R}^2 = e^{i\lambda}$, we retrieve the Aharonov-Bohm effect: the interference pattern for all three schemes will be the same $P_{e^{i\lambda}}[D_i]$, and the question of projection becomes irrelevant on the trivial internal space. If we let $\mathcal{R}^2$ act as unity, we recover the ordinary interference experiment, and the three different ways to repeat the experiment obviously yield the same interference patterns. The difference between interference patterns and probability distributions becomes nontrivial *only* in the case of non-Abelian anyons.

APPENDIX: PROOF OF LOCKING

Here in the Appendix, we will give the proof that in the one-to-one and many-to-one experiments, all initial internal states converge to a locked state. We will begin from the point of view of the one-to-one experiment, and we will see later on that the proof for the many-to-one experiment is essentially the same. We will repeat some of the more important equations from Secs. II and III, and we will state what needs to be proven from the physical point of view. From there on, the proof is purely mathematical, and we will change notation as to clearly discriminate physics from mathematics. The pivot of proof is to construct a new probability distribution function, which has a clear behavior as a function on n , and shows that $|\psi_n\rangle$ becomes projected for large n with the correct probability.

So, focusing on the one-to-one experiment now, we will first determine an expression for $\langle\psi_n|E_\mu|\psi_n\rangle$ as a function of the outcomes D_{i_j} ($j \in \{1, \dots, n\}$, $i_j \in \{1, 2\}$) of the performed runs of the experiment. If the two-particle internal state after the j th run is $|\psi_j\rangle$, then based on the outcome of the $(j+1)$ th run, $D_{i_{j+1}}$, the new internal state $|\psi_{j+1}\rangle$ becomes [Eqs. (3.1), (3.2)]

$$|\psi_{j+1}\rangle = \frac{1}{\sqrt{K'}}(t_1 r'_2 e^{i\theta_1} \mathcal{R}^2 + r_1 t_2 e^{i\theta_2} \mathbb{1}) |\psi_j\rangle, \quad (\text{A1})$$

when the outcome is observation at detector D_1 ; if the incident particle emerges at detector D_2 , the new state is given by

$$|\psi_{j+1}\rangle = \frac{1}{\sqrt{K''}}(t_1 t'_2 e^{i\theta_1} \mathcal{R}^2 + r_1 r_2 e^{i\theta_2} \mathbb{1}) |\psi_j\rangle. \quad (\text{A2})$$

The dependence on the outcome $D_{i_{j+1}}$ is not explicit yet in the above expressions. However, we can rewrite the parts $(\dots \mathcal{R}^2 + \dots \mathbb{1})$ in terms of E_λ and $e^{i\lambda}$ if we recall Eqs. (2.29), (2.30):

$$\mathcal{R}^2 = \sum_\lambda e^{i\lambda} E_\lambda, \quad \mathbb{1} = \sum_\lambda E_\lambda,$$

for then

$$\begin{aligned} t_1 r'_2 e^{i\theta_1} \mathcal{R}^2 + r_1 t_2 e^{i\theta_2} \mathbb{1} &= \sum_\lambda (t_1 r'_2 e^{i\theta_1} e^{i\lambda} + r_1 t_2 e^{i\theta_2}) E_\lambda \\ &\equiv \sum_\lambda f_\lambda[D_1] E_\lambda. \end{aligned}$$

Here, we introduced $f_\lambda[D_i]$ for the sake of abbreviation, but also the dependence on both the eigenvalues $e^{i\lambda}$ and the outcomes D_i becomes more clear. If we take the absolute square of $f_\lambda[D_i]$ (and implicitly integrate over q) we recognize the Aharonov-Bohm probability distribution $P_{e^{i\lambda}}[D_i]$,

$$|t_1 r'_2 e^{i\theta_1} e^{i\lambda} + r_1 t_2 e^{i\theta_2}|^2 = |f_\lambda[D_1]|^2 = P_{e^{i\lambda}}[D_1].$$

If we write $|\psi_n\rangle$ in terms of $|\psi_0\rangle$, according to Eqs. (A1), (A2) and by substituting $f_\lambda[D_i]$, we obtain a product of n similar factors:

$$|\psi_n\rangle = \frac{1}{\sqrt{K}} \left(\sum_{\lambda_n} f_{\lambda_n}[D_{i_n}] E_{\lambda_n} \right) \cdots \left(\sum_{\lambda_1} f_{\lambda_1}[D_{i_1}] E_{\lambda_1} \right) |\psi_0\rangle.$$

Most of the projectors E_{λ_j} cancel each other, if we apply to them the projector algebra

$$E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda.$$

Now, we are ready to express the expectation value $\langle \psi_n | E_\mu | \psi_n \rangle$ with explicit dependence on the outcomes $\{D_{i_j}\}$ of the n runs, in the following way:

$$\langle \psi_n | E_\mu | \psi_n \rangle = \frac{1}{K} P_{e^{i\mu}}[D_{i_1}] \cdots P_{e^{i\mu}}[D_{i_n}] \langle \psi_0 | E_\mu | \psi_0 \rangle. \quad (\text{A3})$$

The normalization constant K is simply given by

$$K = \sum_{\lambda} P_{e^{i\lambda}}[D_{i_1}] \cdots P_{e^{i\lambda}}[D_{i_n}] \langle \psi_0 | E_\lambda | \psi_0 \rangle \quad (\text{A4})$$

$$\equiv p_n(D_{i_1}, D_{i_2}, \dots, D_{i_n}), \quad (\text{A5})$$

but this K turns out to have a physical meaning too: it is equal to the probability $p_n(D_{i_1}, D_{i_2}, \dots, D_{i_n})$ to observe the particular sequence of n outcomes at the detectors $D_{i_1}, D_{i_2}, \dots, D_{i_n}$. With this probability in mind, proving that the state locks means we need to show that for large values of n the following is true for a certain eigenvalue $e^{i\mu}$:

$$\begin{aligned} \langle \psi_n | E_\mu | \psi_n \rangle &\rightarrow 1, \\ \langle \psi_n | E_\lambda | \psi_n \rangle &\rightarrow 0, \quad \forall \lambda \neq \mu, \end{aligned} \quad (\text{A6})$$

or equivalently:

$$\langle \psi_n | E_\mu | \psi_n \rangle \gg \langle \psi_n | E_\lambda | \psi_n \rangle, \quad \forall \lambda \neq \mu. \quad (\text{A7})$$

The probability p_μ for this particular eigenvalue $e^{i\mu}$ to come out is determined by the initial state $|\psi_0\rangle$,

$$p_\mu = \langle \psi_0 | E_\mu | \psi_0 \rangle. \quad (\text{A8})$$

We will now assume that there are only two distinct eigenvalues $e^{i\lambda_1}$ and $e^{i\lambda_2}$; later on, we will give a generalization for cases with more than two eigenvalues. As the following part of the proof is purely mathematical, it is convenient to use an abstract notation that does not directly relate to the thought-experimental setup of the interference experiments. Let us introduce the functions $A(i)$, $B(i)$, $p(i_1, \dots, i_n)$ and the constants α^2 , β^2 , by making the following identifications:

$$A(i) \equiv P_{e^{i\lambda_1}}[D_i], \quad (\text{A9})$$

$$B(i) \equiv P_{e^{i\lambda_2}}[D_i], \quad (\text{A10})$$

$$p(i_1, \dots, i_n) \equiv p_n(D_{i_1}, \dots, D_{i_n}), \quad (\text{A11})$$

$$\alpha^2 \equiv \langle \psi_0 | E_{\lambda_1} | \psi_0 \rangle, \quad (\text{A12})$$

$$\beta^2 \equiv \langle \psi_0 | E_{\lambda_2} | \psi_0 \rangle. \quad (\text{A13})$$

All are non-negative, and can be easily shown to obey the following relations:

$$\sum_i A(i) = 1, \quad \sum_i B(i) = 1, \quad \alpha^2 + \beta^2 = 1,$$

$$p(i_1, \dots, i_n) = \alpha^2 A(i_1) \cdots A(i_n) + \beta^2 B(i_1) \cdots B(i_n),$$

$$\sum_{i_1, \dots, i_n} p(i_1, \dots, i_n) = 1.$$

Next, and this is the pivot of the proof, we define a *new* probability distribution function $P_n(z)$ of a real-valued variable z , which in turn is defined through the relation

$$e^z = \frac{\langle \psi_n | E_{\lambda_1} | \psi_n \rangle}{\langle \psi_n | E_{\lambda_2} | \psi_n \rangle}, \quad (\text{A14})$$

i.e., the probability that after the n th run the expectation value of E_{λ_1} is equal to that of E_{λ_2} times the exponent of z . The formal definition of $P_n(z)$ involves a Dirac δ -function,

$$P_n(z) = \sum_{i_1, \dots, i_n} p(i_1, \dots, i_n) \delta \left(z - \ln \frac{\langle \psi_n | E_{\lambda_1} | \psi_n \rangle}{\langle \psi_n | E_{\lambda_2} | \psi_n \rangle} \right). \quad (\text{A15})$$

As a probability distribution $P_n(z)$ should be normalized, as it is

$$\int_{-\infty}^{\infty} P_n(z) dz = 1. \quad (\text{A16})$$

Convergence, i.e., locking, is achieved when either $\langle \psi_n | E_{\lambda_1} | \psi_n \rangle \gg \langle \psi_n | E_{\lambda_2} | \psi_n \rangle$ or $\langle \psi_n | E_{\lambda_2} | \psi_n \rangle \gg \langle \psi_n | E_{\lambda_1} | \psi_n \rangle$, which means that $P_n(z)$ needs to become zero near $z=0$, because the area under $P_n(z)$ around $z=0$ stands for the probability that $\langle \psi_n | E_{\lambda_1} | \psi_n \rangle$ is approximately equal to $\langle \psi_n | E_{\lambda_2} | \psi_n \rangle$.

If we use that

$$\ln \frac{\langle \psi_n | E_{\lambda_1} | \psi_n \rangle}{\langle \psi_n | E_{\lambda_2} | \psi_n \rangle} = \ln \frac{\alpha^2}{\beta^2} + \ln \frac{A(i_1) \cdots A(i_n)}{B(i_1) \cdots B(i_n)},$$

then $P_n(z)$ can be conveniently written as the sum of two functions that are scaled by α^2 and β^2 :

$$P_n(z) = \alpha^2 P_n^A \left(z - \ln \frac{\alpha^2}{\beta^2} \right) + \beta^2 P_n^B \left(z - \ln \frac{\alpha^2}{\beta^2} \right). \quad (\text{A17})$$

These two functions $P_n^A(z)$ and $P_n^B(z)$ are independent of α^2 and β^2 :

$$P_n^A(z) = \sum_{i_1, \dots, i_n} A(i_1) \cdots A(i_n) \delta \left(z - \ln \frac{A(i_1) \cdots A(i_n)}{B(i_1) \cdots B(i_n)} \right), \quad (\text{A18})$$

$$P_n^B(z) = \sum_{i_1, \dots, i_n} B(i_1) \cdots B(i_n) \delta \left(z - \ln \frac{A(i_1) \cdots A(i_n)}{B(i_1) \cdots B(i_n)} \right), \quad (\text{A19})$$

and can both be regarded as probability distribution functions, since

$$\int_{-\infty}^{\infty} P_n^A(z) dz = 1, \quad \int_{-\infty}^{\infty} P_n^B(z) dz = 1. \quad (\text{A20})$$

Next, we will calculate the expectation values of z and z^2 with respect to $P_n^A(z)$ and $P_n^B(z)$.

The expectation value $\langle z \rangle_1^A$ of z for $P_1^A(z)$ is

$$\langle z \rangle_1^A = \int_{-\infty}^{\infty} P_1^A(z) z dz = \sum_i A(i) \ln \frac{A(i)}{B(i)} \equiv m_A, \quad (\text{A21})$$

where $m_A > 0$ because $A(i) \neq B(i)$ and¹¹

$$a \ln \frac{a}{b} + b \ln \frac{b}{a} = (a-b)(\ln a - \ln b) > 0, \quad a \neq b.$$

The expectation value of z^2 is given by:

$$\langle z^2 \rangle_1^A = \int_{-\infty}^{\infty} P_1^A(z) z^2 dz = \sum_i A(i) \ln^2 \frac{A(i)}{B(i)} \equiv s_A^2 + m_A^2, \quad (\text{A22})$$

which also defines s_A^2 . We can define m_B and s_B^2 in a similar way:

$$\langle z \rangle_1^B = \int_{-\infty}^{\infty} P_1^B(z) z dz = \sum_i B(i) \ln \frac{A(i)}{B(i)} \equiv -m_B < 0, \quad (\text{A23})$$

$$\langle z^2 \rangle_1^B = \int_{-\infty}^{\infty} P_1^B(z) z^2 dz = \sum_i B(i) \ln^2 \frac{A(i)}{B(i)} \equiv s_B^2 + m_B^2. \quad (\text{A24})$$

If we now calculate the expectation values $\langle z \rangle_n^A$, $\langle z^2 \rangle_n^B$, etc., for arbitrary n by plugging in Eqs. (A18) and (A19), we find these depend on n in the following way:

$$\langle z \rangle_n^A = \int_{-\infty}^{\infty} P_n^A(z) z dz = n \langle z \rangle_1^A = n m_A, \quad (\text{A25})$$

$$\langle z^2 \rangle_n^A = n s_A^2 + n^2 m_A^2, \quad (\text{A26})$$

$$\langle z \rangle_n^B = n m_B, \quad \langle z^2 \rangle_n^B = n s_B^2 + n^2 m_B^2. \quad (\text{A27})$$

¹¹ $A(i)$ unequal to $B(i)$ means $P_{e^{i\lambda_1}}[D_i] \neq P_{e^{i\lambda_2}}[D_i]$, which is usually true unless Q is zero, in which case there is no observable interference, and thus the whole experiment would be uninteresting.

The expectation values of z and z^2 tell a great deal over the general form of $P_n^A(z)$ and $P_n^B(z)$, by considering the mean $\mu = \langle z \rangle$ and the variance $\sigma^2 = \langle z^2 \rangle - \langle z \rangle^2$ of $P_n^A(z)$ and $P_n^B(z)$:

$$\mu_{An} = \langle z \rangle_n^A = n m_A, \quad (\text{A28})$$

$$\sigma_{An}^2 = \langle z^2 \rangle_n^A - (\langle z \rangle_n^A)^2 = n s_A^2, \quad (\text{A29})$$

$$\mu_{Bn} = -n m_B, \quad \sigma_{Bn}^2 = n s_B^2. \quad (\text{A30})$$

Both the mean μ and the variance σ^2 of $P_n^A(z)$ and $P_n^B(z)$ scale linearly with n . The standard width σ , however, is given by the square root of the variance σ^2 , and thus scales with $\sqrt{n}$. This means that if we compare $P_1^A(z)$ and $P_{100}^A(z)$, the mean of $P_{100}^A(z)$ is 100 times greater than that of $P_1^A(z)$, while the width has only increased by a factor of 10. The total area under both $P_1^A(z)$ and $P_{100}^A(z)$ remains equal to 1.

So, $P_n^A(z)$ and $P_n^B(z)$ are both probability distribution functions, of which the peak “runs away” from the origin when n increases, and these peaks run harder than they broaden. This behavior is in fact sufficient as proof. For let us look at $P_n(z)$, Eq. (A17), which is after all the probability distribution function we are physically interested in, for increasing n . The area under $P_n(z)$ for large values of z indicates the probability that after n runs $\langle \psi_n | E_{\lambda_1} | \psi_n \rangle = 1$; the area under $P_n(z)$ for large negative z is the probability that $\langle \psi_n | E_{\lambda_2} | \psi_n \rangle = 1$. By now, we know that $P_n(z)$ is completely dominated by the $P_n^A(z)$ part for large z and by the $P_n^B(z)$ part for large negative z , because of the way the means and widths scale with n .

Nevertheless, let us specify “large z ” by choosing a fixed $z' > 0$,¹² and calculate some z' -dependent areas under $P_n(z)$ when n tends to infinity:

$$\lim_{n \rightarrow \infty} \int_{-z'}^{z'} P_n(z) dz = 0, \quad (\text{A31})$$

$$\lim_{n \rightarrow \infty} \int_{z'}^{\infty} P_n(z) dz = \alpha^2, \quad (\text{A32})$$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{-z'} P_n(z) dz = \beta^2. \quad (\text{A33})$$

Clearly, the results of these integrals do not depend on our arbitrary choice of z' . And so, the projection is proved, i.e., at the end of the experiment the system is locked with complete certainty, including the correct probabilities for each possible outcome, since $\alpha^2 = p_{\lambda_1}$ and $\beta^2 = p_{\lambda_2}$.

The proof for more than two eigenvalues is a simple generalization. For the case of three eigenvalues, P_n depends on

¹²To be precise, if $z \geq z'$, this means that $\langle \psi_n | E_{\lambda_1} | \psi_n \rangle \geq 1/(1 + e^{-z'})$. So, if $z' = 25$, then $e^{z'} \approx 10^{10}$, and thus $\langle \psi_n | E_{\lambda_1} | \psi_n \rangle \geq 1 - 10^{-10}$: the state is pretty much projected, and we can regard $z' = 25$ as large.

two variables z and w , where $P_n(z, w)$ stands for the probability that at the same time both

$$e^z = \frac{\langle \psi_n | E_{\lambda_1} | \psi_n \rangle}{\langle \psi_n | E_{\lambda_2} | \psi_n \rangle}, \quad e^w = \frac{\langle \psi_n | E_{\lambda_1} | \psi_n \rangle}{\langle \psi_n | E_{\lambda_3} | \psi_n \rangle}. \quad (\text{A34})$$

$P_n(z, w)$ can now be written as a sum of three other scaled probability distribution functions $P_n^A(z, w)$, $P_n^B(z, w)$, and (because of the third eigenvalue) $P_n^C(z, w)$, which all “run away” from the origin with increasing n . This procedure can be easily generalized for an arbitrary, but finite, number of eigenvalues.

For the many-to-one experiment, the recipe of the proof is simply: replace the eigenvalues and projector operators by their appropriate counterparts, i.e., replace each $e^{i\lambda}$ by κ , each $P_{e^{i\lambda}[D_i]}$ by $P_\kappa[D_i]$, and each E_λ by E_κ . Although $|\psi_n\rangle$ in the many-to-one experiment is very different from $|\psi_n\rangle$ in the one-to-one experiment, i.e., an element of a very different vector space, and the same applies to E_λ and E_κ , we can nevertheless in the many-to-one experiment come up with an expression similar to Eq. (A3):

$$\langle \psi_n | E_\kappa | \psi_n \rangle = \frac{1}{K} P_\kappa[D_{i_1}] \cdots P_\kappa[D_{i_n}] \langle \psi_0 | E_\kappa | \psi_0 \rangle. \quad (\text{A35})$$

We arrive at Eq. (A35) by determining $\langle \psi_{j+1} | E_\kappa | \psi_{j+1} \rangle$ in terms of $\langle \psi_j | E_\kappa | \psi_j \rangle$, by doing some rewriting of $\langle \psi_{j+1} | \mathcal{R}_{j+2}^2 | \psi_{j+1} \rangle$:

$$\langle \psi_{j+1} | \mathcal{R}_{j+2}^2 | \psi_{j+1} \rangle = \sum_\kappa \kappa \langle \psi_{j+1} | E_\kappa | \psi_{j+1} \rangle \quad (\text{A36})$$

$$= \frac{1}{K'} \sum_\kappa \kappa P_\kappa[D_{i_j}] \langle \psi_j | E_\kappa | \psi_j \rangle. \quad (\text{A37})$$

From Eq. (A35) forward, the proof of the one-to-one and many-to-one experiments is exactly the same.

The functions $P_n^A(z)$ and $P_n^B(z)$ have other interesting properties as well, for instance $P_n^A(z) = e^z P_n^B(z)$, but also $P_2^A = P_1^A \circ P_1^A$ while $P_2 \neq P_1 \circ P_1$. Further, the sums over i , where i could be 1 or 2 corresponding to detector D_1 or D_2 , can easily be generalized to more detectors, i.e., sums where i can take on more values, and can also be replaced by integrals for setups with a “continuous spectrum of detectors.” These aspects we will not discuss here, as they lie outside the scope of the current paper; see Ref. [19] for some more details.

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