Bose-Einstein condensation with high atom number in a deep magnetic trap

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Chapter 2

Theoretical basics

In this Chapter 2 theoretical principles and results are briefly summarized to an extend as is needed for the description of the experiments. The atomic properties of rubidium, which are relevant for laser cooling and magnetic trapping are discussed. The principle of magnetic trapping is introduced, and the Ioffe-quadrupole magnetic trapping configuration [Pritchard, 1983] is explained. In this thesis the magnetic trap is referred to as a ‘Ioffe-quadrupole’ trap. Bose-Einstein condensation of the trapped ideal gas is described followed by the main theoretical results for the case of repulsive interactions between the particles and large atom numbers. As of great importance for the achievement of Bose-Einstein condensation the principle evaporative cooling is briefly introduced.

2.1 Atomic properties of rubidium

Because of their advantageous atomic properties alkali atoms are used in many experiments on trapping and cooling of neutral atoms. In the following a brief description is given of atomic properties of rubidium, which are of relevance for the present work.

2.1.1 Hyperfine structure and laser cooling

The optical transitions from the ground state of rubidium are shown in Figure 2.1. Trapping and cooling is done with a ‘trapping laser’ operated at the hyperfine transition $5S_{1/2}, F = 2 \rightarrow 5P_{3/2}, F = 3$ of the D-2 line at a wavelength of $\lambda = 780\text{nm}$. Light from this laser also gives rise to non-resonant excitation to the $5P_{3/2}, F = 2$ state, from which $3/5$ decay back to the $F = 2$ (upper) hyperfine level of the ground state, but $2/5$ decay to the $F = 1$ (lower) hyperfine level. Once the atoms are in this state, they can no longer be exited by the trapping light, as the 6.8 GHz energy splitting of the ground state is much bigger than the $\Gamma/(2\pi) = 6\text{MHz}$ natural width of the trapping transition. Considering the hyperfine splitting of 267 MHz of the exited state the relative probability of ending up in the $F = 1$ ground state level, after excitation by the trapping laser is $5 \times 10^{-5}$, small but not negligible. By an additional so called ‘repumping laser’ the atoms can be
CHAPTER 2 : THEORETICAL BASICS

FIGURE 2.1: Hyperfine structure of $^{87}$Rb: The D-2 line in the near infra-red regime is commonly used for laser cooling. The atoms can be magnetically trapped in both the $|F = 1\rangle$ and $|F = 2\rangle$ hyperfine states of the ground state $5S_{1/2}$. The data were taken from [Arimondo et al., 1977] and [Radzig and Smirnov, 1985].

Resonantly exited to the state $5P_{3/2}, F = 2$, from where they can spontaneously decay back into the state $5S_{1/2}, F = 2$ again. If the trapping transition is driven resonantly with a saturation parameter $S_0 = 1$ it is sufficient to drive the repumping transition resonantly with a saturation parameter of $S_0 \approx 5 \times 10^{-5}$. The saturation intensity of the D-2 line is 1.65 mW/cm$^2$. For optical pumping of the atoms to the $5S_{1/2}, F = 2, m_F = 2$ Zeeman state an ‘optical pumping’ laser resonant with the transition $5S_{1/2}, F = 2 \rightarrow 5P_{3/2}, F = 2$ is applied with $\sigma^+$-circular polarization.

### 2.1.2 Zeeman effect on the hyperfine ground states

Rubidium atoms can be magnetically trapped in either of the two hyperfine ground states. The Zeeman energy of these states in presence of a static magnetic field $B$ can be expressed by the Breit-Rabi formula [Breit and Rabi, 1931, Ramsey, 1956], for the case of vanishing orbital angular momentum,

$$E_{F,m_F}(B) = (-1)^F \frac{1}{2} \hbar \omega_{hf} \sqrt{1 + \frac{4m_F}{2I + 1}} x + x^2 + \text{const.} \quad (2.1)$$

$$x = \frac{(g_I + g_s) \mu_B B}{\hbar \omega_{hf}} \quad (2.2)$$
FIGURE 2.2: Zeeman-effect of the hyperfine ground state: The energies of the magnetic sublevels and the parameter \((g_I + g_s) \mu_B B\) representing the magnetic field are normalized to the hyperfine splitting \(E_{hf}\) of the ground state. It is possible to magnetically trap the four ‘low field seeking’ states \(|F = 2, m_F = 2\rangle, |F = 2, m_F = 1\rangle, |F = 2, m_F = 0\rangle,\) and \(|F = 1, m_F = -1\rangle\), of which the first one is used in this experiment.

The values of the gyromagnetic factors \(g_I\) and \(g_s\), the nuclear spin \(I\), and the frequency of the hyperfine transition between the ground state levels \(\omega_{hf}\) are summarized in Table 2.1 and were taken from [Vanier and Audoin, 1988]. The contribution of the nuclear angular momentum can be neglected, as \(g_I \ll g_s\). The resulting energies of the magnetic sublevels \(m_F\) are shown in Figure 2.2.

In typical magneto-static traps the Zeeman splitting \(g_s \mu_B B\) is smaller than the hyperfine splitting \(h \omega_{hf}\) of the ground state, and Equation (2.1) can be approximated by

\[
E_{F,m_F}(B) = (-1)^F \left( \frac{1}{2} h \omega_{hf} + m_F \mu_B B + \frac{1}{16} (4 - m^2) \frac{(g_s \mu_B B)^2}{h \omega_{hf}} \right) + \text{const.} \quad (2.3)
\]

\[
g_F = (-1)^F \frac{1}{2I + 2} g_s \quad (2.4)
\]

The first term in Equation (2.3) gives the 6.8 GHz hyperfine splitting of the rubidium ground state. The second term describes the linear Zeeman effect. The third term describes the quadratic dependency of the energy on the magnetic field strength, the so-called ‘quadratic Zeeman effect’. The states \(|F = 2, m_F = \pm 2\rangle\) are pure spin states, for
which the quadratic Zeeman effect is absent. For the trapped state $|F = 2, m_F = 1\rangle$ a significant shift of the Zeeman energy (10% relative to the linear Zeeman effect) occurs for a magnetic field strength of 300G.

### 2.2 Magnetic trapping and Ioffe-quadrupole configuration

The principle of magnetic trapping is based on the interaction of the magnetic moment $\mu$ of the atom with an external magnetic field $B(r)$. The interaction energy is given by

$$ E = -\mu \cdot B(r), $$

where $\mu = m_F g_F \mu_B F / \hbar$ and $F$ is the total angular momentum of the atom. The orientation of the magnetic moment relative to the direction of the magnetic field is quantized. Once the atom is polarized in a certain Zeeman-state $m_F$, it is driven towards either high (‘high field seekers’) or low (‘low field seekers’) magnetic fields depending on the sign of the gyromagnetic factor $g_F$, if $m_F \neq 0$. As has been proven by [Wing, 1983], in a spatial region of an inhomogeneous magnetic field free of electric currents only magnetic field minima but no maxima can exist. Thus, only atoms in low field seeking states can be trapped in a magnetic field minimum. In the low magnetic field limit the three trapped states of rubidium are $|F = 1, m_F = -1\rangle$, $|F = 2, m_F = 1\rangle$ and $|F = 2, m_F = 2\rangle$. Due to the quadratic Zeeman effect also the state $|F = 2, m_F = 0\rangle$ can be trapped in a low magnetic field. For this thesis the experiments were done with atoms in the $|F = 2, m_F = 2\rangle$ state. The simplest way to create a trapping magnetic field geometry is to use a pair of anti-Helmholtz coils producing a static spherical quadrupolar field. This geometry was used for the first demonstration of magnetic trapping of neutral atoms (sodium) [Migdall et al., 1985]. However, in spherical quadrupolar traps the life time of the thermal atomic sample is limited by atomic loss at the trap center where the magnetic field vanishes and the polarization of the gas is not conserved [Petrich et al., 1995]. At the magnetic field zero in the trap center the atoms undergo transitions to non-trapped Zeeman states. In order to circumvent this so called ‘Majorana loss’, different trapping geometries have been developed. One solution is the so called ‘TOP-trap’ [Petrich et al., 1995] employing a time varying magnetic field to create a finite magnetic field value in the center. This was used for the first observation of Bose-Einstein condensation in rubidium atoms [Anderson et al., 1995]. For this thesis a ‘Ioffe-quadrupole’ (IQ) magnetic trapping configuration is used. As shown in Figure 2.3, this is

<table>
<thead>
<tr>
<th>g-factor nucleus</th>
<th>$g_I$</th>
<th>$0.995 \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>g-factor electron</td>
<td>$g_s$</td>
<td>2.0023</td>
</tr>
<tr>
<td>nuclear spin</td>
<td>$I$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>hyperfine energy</td>
<td>$\omega_{hf}$</td>
<td>$2\pi \cdot 6.8346826128(5)$ GHz</td>
</tr>
</tbody>
</table>

**TABLE 2.1:** Some atomic properties of $^{87}$Rb.
2.2. MAGNETIC TRAPPING AND IOFFE-QUADRUPOLE CONFIGURATION

FIGURE 2.3: The Ioffe-quadrupole magnetic trap geometry: The coils (top) produce a potential with axial symmetry. The direction of the currents $I$ is indicated by the small arrows. Four ‘race-track’ shaped coils are combined to form four bars generating a linear quadrupole field (bottom right) for transverse confinement. The two pinch coils create a magnetic field minimum for confinement along the z-direction (bottom left). The large magnetic field of the pinch coils is reduced by the compensation coils to a small value of about 1 G, preventing Majorana loss.

A magneto-static magnetic with non-zero minimum magnetic field for avoiding Majorana depolarization, as described by [Pritchard, 1983]. Often this trapping configuration is called ‘Ioffe-Pritchard’ configuration. The trapping potential has a cylindrical symmetry. For confinement along the radial direction a two-dimensional quadrupole magnetic field is used. The quadrupole field is produced by four ‘race-track’ shaped so called ‘Ioffe coils’. The straight parts of these coils form four current bars, the so called ‘Ioffe bars’,...
carrying currents in alternating directions and producing the radial quadrupole field. The absolute value of the quadrupole field increases linearly in radial direction with gradient $\alpha$. The magnetic field produced by the curved sections of the Ioffe coils can be neglected, as these parts are positioned far from the trap center and cancel each other on the symmetry axis of the trap. The confinement along the direction of the symmetry axis is provided by the field in the $z$-direction produced by two pinch coils. As the currents in these coils have the same orientation and their separation exceeds the Helmholtz distance, a field minimum is created. The magnetic field along the symmetry axis is to a large extend quadratic with a curvature $\beta$. The large central magnetic field produced by the pinch coils is largely compensated by two so called compensation coils, placed at large distance from the trap center, to yield a small but finite offset value $B_0$. If $B_0$ is large enough, Majorana loss is prevented and a long life time of the trapped sample can be anticipated. In order to estimate the offset magnetic field needed to prevent transitions to untrapped Zeeman states, consider an atom moving with the thermal velocity $v_T$ from outside towards the trap center along the radial direction. The atom is in a trapped Zeeman state with respect to the local direction of the magnetic field as a quantization axis, which is pointing in the radial plane. When passing through the center, the trapped Zeeman state is coupled to untrapped states by the offset magnetic field pointing in the axial direction. During crossing through the center, the direction of the atomic magnetic moment will adiabatically follow the local direction of the magnetic field, and the atom will remain in a trapped state, if the following condition for the Landau-Zener parameter $\Gamma_{1z}$ [Zener, 1932, Rubbmark et al., 1981], is fulfilled:

$$\Gamma_{1z} = \frac{(g_F \mu_B B_0)^2}{\hbar m_F g_F \mu_B \alpha v_T} \gg 1 \quad ,$$

where $v_T = \sqrt{8k_B T/\pi m}$ is the thermal velocity of the gas. For the trapped gases described in this thesis, the temperature ranges up to 1 mK, in which case an offset magnetic field of $B_0 > 0.3$ G should be sufficient to suppress Majorana loss to the $10^{-5}$ level in a single pass. The magnetic fields inside the coil assembly can be represented in cylindrical coordinates by [Luiten, 1993]†, [Bergeman et al., 1987]

$$B_\rho(\rho, \phi, z) = -\alpha \rho \sin(2\phi) - \frac{1}{2} \beta \rho (z - z_0)$$
$$B_\phi(\rho, \phi, z) = -\alpha \rho \cos(2\phi)$$
$$B_z(\rho, \phi, z) = B_0 + \frac{1}{2} \beta (z - z_0)^2 - \frac{1}{4} \beta \rho^2 \quad .$$

Therefore,

$$|B| = \sqrt{\left(B_0 + \frac{\beta}{2} (z - z_0)^2\right)^2 + \left(\alpha^2 - \frac{\beta}{2} B_0 + \alpha \beta (z - z_0) \sin(2\phi)\right) \rho^2 + \frac{\beta^2 \rho^4}{16}} \quad .$$

†Please note the difference in the definition of the magnetic field curvature $\beta$ used here in contrast to the definition used in the cited publication.
Close to the center for \((B(\rho, \phi, z) - B_0)/B_0 \ll 1\) the magnetic field can be approximated by the harmonic field profile

\[
B(\rho, z) = B_0 + \frac{1}{2} \left( \frac{\alpha^2}{B_0} - \frac{\beta}{2} \right) \rho^2 + \frac{1}{2} \beta z^2 .
\]  

(2.9)

Along the radial direction, for \(z = 0\) the harmonic approximation is valid over a radius of

\[
\rho_h = \frac{B_0}{\alpha} \left( 1 - \frac{\beta B_0}{2\alpha} \right)^{-\frac{1}{2}}
\]  

(2.10)

from the trap center. Within this radius the corrections due to anharmonic orders are less than 25%. From the harmonic potential expressed by Equation (2.5) with the magnetic field of Equation (2.9) the harmonic trap frequencies can be identified:

\[
\omega_z = \sqrt{\frac{\mu_B g_F m_F}{m} \beta} ,
\]  

(2.11)

\[
\omega_\rho = \sqrt{\frac{\mu_B g_F m_F}{m} \left( \frac{\alpha^2}{B_0} - \frac{\beta}{2} \right)} .
\]  

(2.12)

For a strong radial confinement the radial frequency is dominated by the term \(\alpha^2/B_0\), and thus can be adjusted by changing \(B_0\) with the help of the compensation coils. This adjustment is used for the final stage of adiabatic compression of the trapped cloud as will be described in Section 6.2. In order to evaluate the depth of the magnetic trap it is necessary to calculate the exact field at a distance from the trap center near the magnetic field coils (compare Section 3.3). Exact expressions of the magnetic fields generated by closed loops and straight bars of zero thickness can be found in [Bergeman et al., 1987]. For this thesis similar expressions have been derived for the magnetic field of a ‘half-loop’ representing the turns of the ‘race-track’ coils. With the help of these expressions the field was calculated allowing for the finite size of real wires by numerical integration over the cross section. It was found that in the region of interest the result was not significantly deviating from the magnetic field as calculated by Equation (2.8) for the same trapping parameters \(\alpha, \beta, \) and \(B_0\).

2.3 The ideal Bose-Gas

2.3.1 Bose-distribution function and Bose-Einstein condensation

In the following the theoretical results, which are necessary to describe trapped Bose-gases within this work, are briefly introduced. Expressions for the density distribution of the trapped gas and the transition temperature are obtained. A more detailed description can be found in [Bagnato et al., 1987, Walraven, 1996, Huang, 1963]. The trapped gas of bosons is described within the framework of the grand canonical ensemble. The atoms are trapped in an external potential \(U(r)\) and occupy the single-particle motional states of the trap with the eigenenergies \(\epsilon_k\). The occupation number of these states is given by
the Bose-distribution

$$\langle n_k \rangle = \frac{1}{e^{(\epsilon_k - \mu)/k_B T} - 1} = \sum_{i=1}^{\infty} \exp(-l (\epsilon_k - \mu)/k_B T) \ . \ (2.13)$$

For a given temperature the chemical potential is fixed by

$$N = \sum_k \langle n_k \rangle \ . \ (2.14)$$

At large temperature $\mu$ is large and negative. As the gas is cooled $\mu$ approaches the ground state energy $\epsilon_0$ from negative values, until at the critical temperature $T_C$ it becomes equal to $\epsilon_0$. For given $N$ the occupation number of the ground state $N_0$ becomes at $T_C$ a macroscopic fraction of the total number of atoms, while $\mu$ and the number of atoms in the exited states become fixed. In the limit $T \to 0$ all atoms will end up in the ground state. This phenomenon is called Bose-Einstein condensation (BEC). In typical BEC experiments with ultra-cold gases in harmonic traps, the number of atoms is large and the energy associated to the temperature is big compared with the level spacing, $k_B T \gg \hbar \omega$. The total number of atoms in exited states can be written as

$$N - N_0 = \int_0^{\infty} d\epsilon \; \rho(\epsilon) \frac{1}{e^{(\epsilon - \mu)/k_B T} - 1} \ , \ (2.15)$$

where $N_0$ is the occupation of the ground state and $\rho(\epsilon)$ is the density of states of the gas trapped in the external potential $U(r)$, given by

$$\rho(\epsilon) = \frac{1}{(2\pi \hbar)^3} \int \rho^+_{\epsilon} \rho^+_{\epsilon} \delta\left(\epsilon - U(r) - \frac{p^2}{2m}\right) \ (2.16)$$

$$= \frac{2\pi (2m)^{3/2}}{(2\pi \hbar)^3} \int_{U < \epsilon} d\epsilon \sqrt{\epsilon - U(r)} \ .$$

By changing from the sum of Equation (2.14) to the integration $\epsilon_0 = 0$ can be assumed. Above the phase transition the density distribution $n(r)$ of the cloud can be found with the normalization relation $N = \int dr \ n(r)$ and Equations (2.15) and (2.16) as

$$n(r) = \frac{1}{\Lambda_T^3} \ g_{3/2} \left(\tilde{z} e^{-U(r)/k_B T} \right) \ , \ (2.17)$$

with the thermal de Broglie wavelength

$$\Lambda_T = \sqrt{\frac{2\pi \hbar^2}{m k_B T}} \ , \ (2.18)$$

the fugacity $\tilde{z} = e^{\mu/k_B T}$ and the poly-logarithm function $g_{3/2}(x) = \sum_{k=1}^{\infty} x^k/k^3$. Independently of the trap geometry one finds, that BEC occurs, if in the center of the trap the so called degeneracy parameter $n(0)\Lambda_T^3$ is reaching the critical value ($\mu = 0$)

$$n(0)\Lambda_T^3 = g_{3/2}(1) = 2.612 \ldots \ . \ (2.19)$$
Here \( g_\eta(1) = \zeta(n) \), where \( \zeta(n) \) is Riemann's \( \zeta \)-function. Equation (2.19) reflects the feature that BEC occurs, if the thermal wavelength of the atoms in the gas is becoming on the order of the separation between the atoms. In case of an harmonic potential, e.g. in the center of a Ioffe-quadrupole trap, the density of states (2.16) is found to be

\[
\rho_\eta(\epsilon) = \frac{1}{2} (\hbar \omega)^{-3} \epsilon^2 ,
\]

(2.20)

with the mean trap frequency \( \bar{\omega} \) defined as

\[
\bar{\omega} = (\omega_p^2 \omega_z)^{\frac{1}{3}}.
\]

(2.21)

The number of atoms in excited states follows by substituting Equation (2.20) into Equation (2.15) and evaluates to

\[
N - N_0 = g_3(1) \left( \frac{k_B T}{\hbar \omega} \right)^3.
\]

(2.22)

At the phase transition \( N_0 = 0 \) and one can solve for the critical temperature

\[
T_C = \frac{\hbar \omega}{k_B} \left( \frac{N}{g_3(1)} \right)^{\frac{1}{3}} \approx 0.94 \frac{\hbar \omega}{k_B} N^{\frac{1}{3}}.
\]

(2.23)

### 2.3.2 The trapped gas in the classical regime

In the classical regime, where \( |\mu| \gg k_B T \) and \( \mu < 0 \), and thus \( \tilde{z} \ll 1 \), one finds with Equation (2.17) the density in the trap center

\[
n(0) = \frac{1}{\Lambda_T^3} g_\frac{3}{2}(\tilde{z}) \approx \frac{1}{\Lambda_T^3} \tilde{z}.
\]

(2.24)

The thermal wavelength is in the classical regime much smaller than the separation between the atoms. From Equations (2.15), (2.16), and (2.24) it can be seen, that in the classical regime the number of atoms can be written as

\[
N = n(0) \Lambda_T^3 Z_1
\]

(2.25)

with the classical canonical partition function for a single atom

\[
Z_1 = \frac{1}{(2\pi \hbar)^3} \int_{-\infty}^{\infty} d\mathbf{p} \int_{-\infty}^{\infty} d\mathbf{r} \ e^{-(U(r) + p^2/2m)/k_B T}.
\]

(2.26)

Defining the reference volume of the gas \( V_e \equiv N/n(0) \) one finds

\[
V_e = \Lambda_T^3 Z_1
\]

(2.27)

and the degeneracy parameter

\[
n(0) \Lambda_T^3 = \frac{N}{Z_1}.
\]

(2.28)
In many cases the magnetic trapping potential belongs to the class of so called ‘power-law’ potentials, which can be written in the form

\[ U(x, y, z) \propto |x|^{1/\delta_1} + |y|^{1/\delta_2} + |z|^{1/\delta_3} \]

with \( \delta = \sum_i \delta_i \) [Bagnato et al., 1987]. For example, \( \delta = 3/2 \) describes the harmonic trap and \( \delta = 3 \) the spherical-quadrupole geometry. In case of power-law potentials a simple form of the single-atom partition function can be found

\[ Z_1^\delta = A_{PL}^\delta (k_B T)^{\delta + \frac{3}{2}} \Gamma \left( \frac{3}{2} + \delta \right) , \tag{2.29} \]

where \( A_{PL}^\delta \) is a constant, which depends on the strength of the potential. \( \Gamma(x) = \int_0^\infty dt \ t^{x-1} e^{-t} \) is the Euler gamma function. In case of the Ioffe-quadrupole trap with \( B_0 = 0 \), the single-atom partition function can be expressed by Equation (2.29) for \( \delta = 5/2 \). In the general case of the Ioffe-quadrupole trap the single-atom partition function can be expressed by the sum of two power-law contributions [Luiten et al., 1996, Pinkse et al., 1997, 1998]

\[ Z_1^{IQ} = Z_1^{\delta=\frac{3}{2}} + Z_1^{\delta=\frac{5}{2}} = 6 A_{IQ} (k_B T)^4 \left( 1 + \frac{2U_0}{3k_BT} \right) , \tag{2.30} \]

where \( U_0 \equiv \mu_B g_F m_F B_0 \), and where the trap dependent constant \( A_{IQ} \) is related to the parameters \( \alpha \) and \( \beta \), of the Ioffe-quadrupole trap through

\[ A_{IQ} = \frac{(2\pi^2 m)^{\frac{3}{2}}}{(2\pi\hbar)^{3/2}(\mu_B g_F m_F \alpha)^2 \sqrt{\mu_B g_F m_F \beta}} . \tag{2.31} \]

In case of the harmonic trap in the classical regime it follows from Equations (2.17) and (2.28)-(2.31) that the density distribution has a Gaussian shape

\[ n(r) = \frac{N}{(\pi)^{3/2}} \prod_i r_{0,i} \ e^{-\sum_i \left( \frac{r_i}{r_{0,i}} \right)^2} , \tag{2.32} \]

with a 1/e-radius of the cloud in the \( i \)-direction

\[ r_{0,i} = \frac{1}{\omega_i} \sqrt{\frac{2k_BT}{m}} , \tag{2.33} \]

depending on the harmonic frequencies \( \omega_i \).

### 2.3.3 Adiabatic compression

Thermodynamic properties of the gas can be evaluated from statistical properties with the relation between the canonical partition function \( Z \) and the free energy \( F = E - TS \)

\[ Z = \frac{Z_1^N}{N!} = e^{-F/k_BT} . \tag{2.34} \]
The internal energy $E$ can be expressed as

$$ E = N k_B T \left( \frac{3}{2} + \gamma \right) ,$$

(2.35)

where

$$ \gamma \equiv \left( \frac{T}{V_i} \right) \frac{\partial V_i}{\partial T} .$$

(2.36)

The first term in Equation (2.35) represents the kinetic energy, whereas the second term, which is proportional to $\gamma$, represents the contribution of the potential energy. Starting from Equation (2.34) and using Equations (2.28), (2.35), and Stirling's formula $\ln(N!) \approx N \ln(N) - N$ one finds that the degeneracy parameter can be written as [Pinkse et al., 1997]

$$ n(0) \Lambda^3_f = e^{\frac{3}{2} + \gamma - \frac{5}{2} k_B T} .$$

(2.37)

For power-law traps in the classical regime the relation $\gamma = \delta$ holds. In case of the Ioffe-quadrupole trap $\gamma$ can be found by evaluating the Equations (2.26), (2.27), and (2.30)

$$ \gamma Q = \frac{3}{2} + \frac{3 k_B T_5}{2 v_0} .$$

(2.38)

During an adiabatic change of the trapping potential, where the entropy $S$ and the atom number $N$ are conserved, temperature and density are changing. The degeneracy parameter stays constant unless $\gamma$ is varied by changing the 'shape' of the potential. The reversible change of the degeneracy parameter by changing the shape, i.e. the $\delta$ parameter, of the power law potential, and thus $\gamma$, was first experimentally demonstrated for magnetically trapped hydrogen by [Pinkse et al., 1997]. Adiabatic compression of the gas in the Ioffe-quadrupole trap, as discussed in Chapter 6.2, will result in a change of the temperature as well as the degeneracy parameter of the gas. From Equation (2.37) it follows that for the compression from some initial (index $i$) to final (index $f$) trapping parameters

$$ \frac{n_f(0) \Lambda^3_f}{n_i(0) \Lambda^3_i} = e^{\gamma Q} .$$

(2.39)

The final temperature and density has to be determined by solving (2.39) numerically. In the special case of adiabatic compression in a harmonic trap the simple relation

$$ T_f = T_i \frac{\omega_f}{\omega_i}$$

(2.40)

can be obtained. Equations (2.39) and (2.40) are valid, if the change of the trapping potential is done adiabatically. During the compression of a harmonic trap the density of states remains unchanged. The adiabaticity is then given, if the condition

$$ \frac{d\omega}{dt} \ll \omega^2$$

(2.41)

is fulfilled. A more detailed discussion of the adiabaticity criterium in this case is given in [Ketterle et al., 1999].
2.4 The weakly interacting Bose-gas

In the physics of ultra-cold, dilute alkali gases the elastic interaction between the atoms play an important role. For example, the density profile \( n_c(r) \) of the Bose-Einstein condensate is deviating from the Gaussian shape of the ground state of the harmonic potential because of elastic interaction. At low energies elastic collisions happen in the s-wave scattering limit. Scattering can be described by the use of the pseudo potential [Huang, 1963]

\[
V(r - r') = g \delta(r - r') ,
\]
with the interaction parameter

\[
g = \frac{4\pi\hbar^2a}{m} .
\]

In the case of rubidium the interaction is repulsive and the positive value of the triplet scattering length is \( a = (106 \pm 4) \times a_0 = 57.7 \times 10^{-10} \) m [Roberts et al., 1998, Boesten et al., 1997] with Bohr's radius \( a_0 = 0.529 \times 10^{-10} \) m. The use of the s-wave approximation is justified as long as the condition \( |R_0/\Lambda| < 1 \) is fulfilled, where \( R_0 = (C_6 m/(2\hbar^2))^{1/4} \) is the range of the potential. As in case of \(^{87}\)Rb the van der Waals coefficient is \( C_6 = 4700 \pm 50 \) a.u. = \( 4.3 \times 10^{-76} \) J m\(^6\) [Roberts et al., 1998] and hence \( R_0 = 73 \times 10^{-10} \) m the s-wave scattering approximation is justified for temperatures much smaller than 1 mK. It is noted here that for temperatures around 400\(\mu\)K the elastic collisional cross section of \(^{87}\)Rb is enhanced due to the presence of a d-wave resonance [Burke et al., 1998]. At typical densities of \( n_c = 10^{15} \) cm\(^{-3}\) the so called gas parameter is small \( n a^3 \ll 1 \), and only binary elastic collisions have to be considered. If the correlations between the atoms manifest themselves only over a short range, i.e. at a range much smaller than the size of the gas cloud, the gas can be described in terms of a mean field theory [Bogoliubov, 1947]. For \( T < T_C \) the macroscopic wave function \( \phi(r) \) of the condensed part of the trapped gas obeys in the equilibrium case the so called time-independent Gross-Pitaevskii (GP) Equation [Goldman and Silvera, 1981, Huse and Siggia, 1982]

\[
\left(-\frac{\hbar^2 \nabla^2}{2m} + U(r) + g |\phi(r)|^2 \right) \phi(r) = \mu \phi(r) .
\]

(2.44)

At high densities \( n_c(r) = |\phi(r)|^2 \) and if repulsive interaction \( (a > 0) \) is considered, the mean field interaction \( g n_c(r) \) becomes dominant over the kinetic energy, which is on the order of \( \hbar \omega_\rho \). In the case that

\[
g n_c(r) \gg \hbar \omega_\rho
\]

(2.45)

the kinetic term in the GP Equation (2.44) can be neglected. This is known as the Thomas-Fermi approximation. In this case the GP equation can easily be solved, and the density distribution of the condensate is a result of the balance between the external potential and the repulsive mean field interaction:

\[
n_c(r) = \frac{1}{g} (\mu - U(r)) .
\]

(2.46)

In the case of a harmonic potential with trap frequencies \( \omega_i \), the density profile of the cloud has a parabolic shape. The density vanishes where the potential energy surpasses
2.5.2.5. EVAPORATIVE COOLING

Thus, the ‘Thomas-Fermi’ radius \( R_i \) of the cloud in the i-direction is given by

\[
R_i = \frac{1}{\omega_i} \sqrt{\frac{2\mu}{m}} .
\]  

(2.47)

By integrating over the density distribution the number of condensed atoms is calculated to be

\[
N_0 = \int dr \, n_c(r) = \left( \frac{2\mu}{\hbar\omega} \right)^{\frac{3}{2}} \frac{a_{ho}}{15a} ,
\]  

(2.48)

where \( a_{ho} = \sqrt{\hbar/m\omega} \) is the harmonic oscillator length.

2.5 Evaporative cooling

Evaporative cooling is a powerful cooling method that does not suffer from the limitations encountered with optical cooling methods, such as the Doppler-limit and the recoil-limit. For all experiments in which BEC was achieved evaporative cooling was used as the final cooling stage. Evaporative cooling was first realized in an experiment with trapped atomic hydrogen [Hess et al., 1987]. A review on evaporative cooling is given by [Walraven, 1996, Ketterle and van Druten, 1996].

2.5.1 Evaporative cooling model

In the following the principle of evaporative cooling is briefly sketched. The description is based on the model of evaporative cooling introduced by [Luiten et al., 1996, Walraven, 1996]. Evaporative cooling is based on the preferential removal of atoms above a certain truncation energy \( \epsilon_t \) from the trap and subsequent thermalization by elastic collisions. For constant truncation barrier \( \epsilon_t \) (plain evaporation) elastic collisions between atoms in a trap produce atoms of energies higher than \( \epsilon_t \), which are ‘evaporated’ from the trap. The evaporation rate per atom can be expressed as

\[
\tau_{ev}^{-1} = \frac{\dot{N}_{ev}}{N} = n(0) v_T \sigma e^{-\eta \frac{V_{ev}}{V_e}} ,
\]  

(2.49)

where \( \sigma = 8\pi a^2 = 7.9 \times 10^{-16} \text{ m}^2 \) is the elastic collisional cross section, and

\[
\eta = \frac{\epsilon_t}{k_B T} ,
\]  

(2.50)

is the truncation parameter. Here \( V_{ev} \approx V_e \) is the effective volume for evaporation [Walraven, 1996]. As atoms are evaporated from the trap, the mean energy per atom is decreased and hence the gas cools. As the temperature decreases, \( \eta \) becomes larger and the evaporation rate is exponentially suppressed. A continuous cooling process can be realized by reducing \( \epsilon_t \) (forced evaporative cooling) for instance in a way that \( \eta \) remains constant. It can be shown that for forced evaporative cooling at constant \( \eta \)

\[
T \propto N^{\alpha_{ev}} ,
\]  

(2.51)
when the efficiency parameter

\[ \alpha_{\text{ev}} \equiv \frac{d \ln(T)}{d \ln(N)} \]  

only depends on the value of the truncation parameter \( \eta \). During cooling also the effective volume is decreasing. For power-law traps this can be expressed by

\[ V_e \sim T^\delta , \]  

where \( \delta = 5/3 \) describes the Ioffe-quadrupole trap at high temperature (linear limit) and \( \delta = 3/2 \) describes the Ioffe-quadrupole trap at low temperature (harmonic limit). Importantly, in spite of the massive loss of atoms due to the evaporation process, the density \( n(0) \) in the trap center can remain constant or even increase provided \( \alpha_{\text{ev}} \geq 1/\delta \), as can be seen by combining Equations (2.51) and (2.53) as well as the definition \( V_e = N/n(0) \). In order to realize such a density increase, large \( \eta \) and thus slow evaporation is necessary. If the increase of the density is so strong that the elastic collisional rate \( \tau_{\text{el}}^{-1} \sim n(0) v_T \sim n(0) T^{1/2} \) increases, although the temperature decreases, the so called ‘runaway’ evaporative cooling regime is reached: The evaporation process proceeds faster and faster for decreasing temperature. The runaway condition can be expressed as

\[ \frac{d \ln(n\sqrt{T})}{d \ln(N)} = 1 - 2 \alpha_{\text{ev}} < 0 \]  

In practice, the efficiency of evaporative cooling is limited by atomic loss from the trap, which in case of rubidium is caused by collisions with particles from the vacuum background (see Section 6.3) and at high densities is caused by three-body recombination. In general the atomic loss rate \( \tau_i^{-1} \) for an i-body loss process can be expressed by the rate constants \( G_i \),

\[ \tau_i^{-1} \equiv \frac{\dot{N}_i}{N} = -G_i n(0)^{i-1} \frac{V_{ie}}{V_e} , \]  

where the reference volumes for i-body collisions are defined as

\[ V_{ie} \equiv \int dr \left( \frac{n(r)}{n(0)} \right)^i . \]  

For two- and three-body loss processes \( (i = 2, 3) \) the reference volume can be calculated numerically, whereas for background collisions this volume is the reference volume as introduced in Equation (2.27). Evaporative cooling is most efficient for a large ratio \( R_i \) of the evaporation rate to the dominant loss rate, the so called ‘ratio of good to bad collisions’, which is given by

\[ R_i \equiv \frac{\dot{N}_{\text{ev}}}{\dot{N}_i} = \frac{n(0) v_T \sigma V_{ie} e^{-\eta}}{n^{-1}(0) G_i \frac{V_{ie}}{V_e}} \equiv \frac{1}{\lambda_i} \frac{V_{ev}}{V_e} e^{-\eta} . \]  

Here,

\[ \lambda_1 \equiv \frac{\tau_{\text{el}}^{-1}}{\tau_i^{-1}} = \frac{n^{i-2}(0) G_i V_{ie}}{v_T \sigma \frac{V_{ie}}{V_e}} , \]
where the elastic collisional rate $\tau^{-1}_{\text{el}}$ is defined as
\[ \tau^{-1}_{\text{el}} \equiv n(0) v_T \sigma . \] (2.59)

In typical BEC experiments with alkali gases at the beginning of evaporative cooling only background collisional loss has to be considered. In case of the Ioffe-quadrupole trap a numerical calculation of the efficiency parameter $\alpha_{\text{ev}}$ as a function of $\eta$ and $\lambda_i$ is presented in [Valkering, 1999]. From the calculation follows that for an optimal choice of $\eta$ and for
\[ \lambda_i \leq 10^{-2} \] (2.60)
forced evaporative cooling in the run-away regime, i.e. $\alpha_{\text{ev}} > 1/2$, can be achieved (see Section 7.1. Once the threshold condition for run-away evaporative cooling is fulfilled, it is desirable to perform the cooling in such a way, that at the BEC transition the maximum number of atoms remains. This can be achieved by choosing an optimized time dependence of $\eta(t)$ in order to maximize $\alpha_{\text{ev}}$. A simulation of forced evaporative cooling with the goal to find the optimized trajectory $\eta(t)$ for the conditions of this experiment was performed by [Valkering, 1999]. The calculation was based on the kinetic model of [Luiten et al., 1996, Walraven, 1996]. As a result it was found that evaporative cooling to the BEC transition within a few seconds is possible for typical initial conditions as used for the present experiment (compare Section 7.1). The optimal time dependence of the truncation energy $\epsilon_t(t)$ is almost linear. As the analysis shows, only close to the BEC transition it is advantageous to evaporate faster, because the density has increased to a value where 3-body losses are important.

### 2.5.2 Rf-evaporative cooling

The energy selective removal of the atoms from the trap is done by driving transitions to untrapped Zeeman states by applying an oscillating magnetic field of angular frequency $\omega_{\text{rf}}$. The covered frequency range lies typically in the radio frequency regime from 500 kHz to 50 MHz. The atoms undergo the transitions only at positions where the resonance condition
\[ \mu_B g_F |B(r)| = \hbar \omega_{\text{rf}} \] (2.61)
is fulfilled. Therefore, the truncation energy $\epsilon_t$ at which atoms in the state $m_F$ are removed from the trap is related by Equation (2.5) to the rf-frequency $\omega_{\text{rf}}$ as
\[ \epsilon_t = m_F \hbar (\omega_{\text{rf}} - \omega_0) , \] (2.62)
where $\omega_0 = \mu_B g_F |B(0)|/\hbar$ is the resonance frequency at the center of the trap. Due to their thermal motion the atoms are passing with different velocities through the trap. This can equivalently be seen as an atom at rest experiencing a time varying magnetic field. In which state the atoms leave the resonance region depends on the rate of change of the magnetic field, e.g. the velocity of the atoms, and on the strength of the oscillating rf-magnetic field. In case of low velocity and high rf-magnetic field amplitude the atom adiabatically undergoes the transition to the untrapped state. In the opposite case the atom remains in a trapped state after a non-adiabatic passage through the region of resonance. For a two-level atom the transition probability has been calculated in a
FIGURE 2.4: For the adiabaticity parameter $\Gamma_{lz} \equiv 1$ the transition probability to an untrapped state of the five level atom is about 97%.

Landau-Zener picture by [Zener, 1932, Rubbmark et al., 1981]. In case of the $F = 2$ state of rubidium the 5 Zeeman levels $|F = 2, m_F = -2, \ldots, +2 \rangle$ have to be considered. Atoms are originally trapped in the state $|F = 2, m_F = +2 \rangle$. Assuming a linearly polarized oscillatory field with amplitude $B_{rf}$ along the x-axis perpendicular to the trapping field $B(t)$, the magnetic field can be expressed as

$$B(t) = B(t) \hat{e}_z + B_{rf} \cos(\omega_{rf} t) \hat{e}_x \ .$$

The time dependent Hamiltonian (2.5) becomes then

$$\mathcal{H}(t) = \frac{\mu_B g_F}{\hbar} B(t) F_z + 2\omega_R \cos(\omega_{rf} t) F_x \ ,$$

with the Rabi frequency $\omega_R = \mu_B g_F B_{rf}/(2\hbar)$. The set of time dependent Schrödinger equations for this Hamiltonian has been solved numerically for an atom passing through the resonance region being initially in the $m_F = +2$ state. After the passage the atom is in a superposition of all Zeeman states with the amplitudes $c_i, i \in \{-2, \ldots, +2\}$. From the result of the numerical calculation [Valkering, 1999] the probability of the atom being in an untrapped state $P = |c_{-2}|^2 + |c_{-1}|^2 + |c_0|^2$ can be calculated. The probability is dependent on the Landau-Zener parameter

$$\Gamma_{lz} = \frac{\hbar \omega_R^2}{\mu_B g_F 2v_T (dB/dr)} \ .$$

As shown in Figure 2.4 for $\Gamma_{lz} \equiv 1$ the atom is removed from the trap with a probability of about 97%. For a given temperature and rf-magnetic potential a minimum magnetic field
amplitude has to be applied in order to ensure an adiabatic passage. If one assumes, that the atom which is to be removed from the trap moves with twice the thermal velocity \( v_T \), it follows from (2.65) that for constant transition probability \( B_{\text{rf}} \) decreases as \( \sqrt{T} \). It is noted here, that [Vitanov and Suominen, 1997] the Landau-Zener transition probability of a multi-level system is related to that of a two-level system by recursion relations. At high truncation energies, e.g. for large magnetic fields, the quadratic Zeeman effect leads to incomplete evaporation [Desruelle et al., 1999] as will be further discussed in Section 8.5. A strong rf-magnetic field leads to an effective reduction of the oscillation frequencies of the atoms in the trap. An upper boundary on the rf-magnetic field amplitude, which may be applied, is imposed by the energy shift of the magnetic sublevels caused by the oscillatory magnetic field. By diagonalizing the Hamiltonian (2.64) the energy eigenstates ('dressed' states) of the trapped atom in presence of the rf-field can be found. Writing the energy of the trapped state up to second order in the static magnetic field for a given \( \omega_{\text{rf}} \) and \( \omega_R \), yields the effective trap frequency of the trap. In case of the Ioffe-quadrupole trap the effective trap frequencies \( \omega_{\text{eff}} \) are given by

\[
\omega_{\text{eff}}^i = \omega_i \left( \left( \frac{\omega_R}{\omega_0 - \omega_{\text{rf}}} \right)^2 + 1 \right)^{-1/4}.
\]  

(2.66)

In case of low detuning from the trap center and large Rabi- frequency the effective trap frequency can be strongly reduced, even to the point that the confinement is lost.