X-ray waveguiding studies of ordering phenomena in confined fluids
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Chapter 4

Transmission properties of the waveguide with air gap

We present a study of mode propagation in planar x-ray waveguide having an air gap as the guiding medium. Individual transverse electric modes were found to propagate through the waveguide essentially undisturbed and with negligible scattering losses to other modes. If different modes are excited simultaneously at the waveguide entrance, then the phase relation between these modes as given by their propagation constants is found to be preserved over the entire length of the waveguide.

4.1 Introduction

Almost twenty years ago, Spiller and Segmüller [37] made and tested the first planar waveguides for x rays. Their aim was to show that the emerging field of integrated-optics could be extended from the visible to x-ray wavelengths, so as to overcome the strong absorption in the ultraviolet. The devices they developed consisted of a boron nitride guiding layer sandwiched between Al₂O₃ claddings. Although the guiding layer material had a low Z-number, substantial absorption at x-ray wavelengths caused considerable power losses. In order to overcome these losses, they suggested that the ultimate guiding layer would be a vacuum (or air) gap. Six years later, such a waveguide for x rays was realized by Fischer and Ulrich [38]. They demonstrated self-imaging of the modes within the waveguide and
showed that the coherence of the incident field can be preserved upon propagation. This property plays an important role in the structure determination of confined fluids (see chapter 6).

With the advent of third-generation synchrotron x-ray sources, interest for x-ray waveguides increased [39, 40], because of their potential as nanometer-sized sources of extremely coherent x-ray beams. Low-loss transport of x-rays was achieved in hollow glass capillaries [41], which in tapered form are used as focusing devices. However, because of their large acceptance area and their varying cross section, such devices do not allow for propagation of single modes.

In this chapter, we demonstrate the excitation and propagation of a single mode in a planar x-ray waveguide (see Fig. 4.1). The waveguide is designed to confine a fluid within the gap between the plates, enabling the determination of the arrangement of the fluid's constituents by coherent scattering. In a first, essential, step towards achieving this goal, we discuss results obtained from waveguiding experiments in the absence of the fluid. Measurements of far-field diffraction patterns from the waveguide exit, and the transmitted intensity are compared with model calculations. Finally, we discuss measurements of the specular reflected intensity ($\theta_i = \theta_e$), showing that coherence is preserved upon propagation.
4.2 Mode excitation and propagation

The incident field is determined by the geometry of our experiment as shown in Fig. 4.1. Let \( z \) and \( x \) be the coordinates along the propagation direction and along the normal to the plates, respectively. A plane e.m. wave, with wave number \( k_0 \) and with the electric field polarized perpendicular to the plane of incidence, is incident onto the device at a grazing angle. In front of the waveguide entrance the wave is totally reflected from the large bottom plate. The resulting field across the entrance plane \( (z = 0) \) is a standing wave due to interference of the incident wave and the reflected wave. The standing wave field has a node at the surface of the bottom plate \( (x = 0) \) and is given by Eq. (3.1),

\[
\phi(x) \approx \sin(k_0 \theta_i x) \tag{4.1}
\]

where \( \theta_i \) is the angle of incidence, and we have neglected the evanescent wave in the plate. For a gap of width \( W \) the field \( \phi(x) \) will also have a node at the position of the upper plate if \( \theta_i = \theta_m \), with \( \theta_m = (m + 1)\pi/k_0 W \) and \( m = 0, 1, 2, \ldots \). This wave field, which excites the \( m \)th transverse-electric (TE) mode of the waveguide, will propagate through the waveguide undisturbed. The corresponding expression for the amplitude of the wave field within the waveguide is given by Eq. (3.3),

\[
\Psi_m(x, z) = \phi_m(x) \exp(-i\beta_m z), \tag{4.2}
\]

where \( \beta_m = n_1 k_0 \cos \theta_m \approx k_0 (1 - \theta_m^2/2) \) is the propagation constant of the mode (see section 2.2) with \( n_1 = 1 \) the refractive index of the guiding layer. The mode \( \phi_m(x) \) is given by

\[
\phi_m(x) = \sin(k_0 \theta_m x). \tag{4.3}
\]

In order to illustrate the effect of neglecting the evanescent waves, we have calculated the amplitude profiles of the nine lowest-order modes (see Fig. 4.2) by solving Eq. (2.5) for the experimental conditions considered here. The validity of the assumption is seen to hold up to the eight mode \( (\theta_i = \theta_8) \). In order to determine until which incidence angle \( \theta_i \) our assumption holds, we have calculated the penetration depth as a function of the mode angle (see Fig. 4.3) with

\[
\xi \approx \frac{1}{k_0 \sqrt{(\theta_c^2 - \theta_i^2)}}, \tag{4.4}
\]
where $\theta_c$ is the critical angle for total internal reflection [8]. Figure 4.3b shows the dependence of the penetration depth on the refractive index contrast between the guiding layer and the cladding, $\delta = n_1 - n_2 = 1 - n_2$. Clearly, for small angles $\theta_i$, large $\delta$ and a gap width of a few hundred nanometer, we have $\xi \ll W$. In this case, we may neglect the contribution of the evanescent field to the propagation of the modes.

For angles of incidence $\theta_i \neq \theta_m$, the wave field amplitude $\phi(x)$ at the entrance plane makes a sharp drop to zero at $x = W$. Therefore, a coherent superposition of guided modes is needed to match the incident field. The corresponding wave field is given by the Fourier expansion (3.9):

$$\Psi(x, z) = \sum_{m=0}^{m_{\text{max}}} c_m(\theta_i) \psi_m(x, z), \quad (4.5)$$

with

$$c_m(\theta_i) = \frac{2(-1)^m \theta_m \sin(k_0\theta_i W)}{k_0 W (\theta_m^2 - \theta_i^2)}. \quad (4.6)$$

Here, $\psi_m(x, z)$ is given by Eq. (4.2) and $m_{\text{max}} = k_0 W \theta_c / \pi - 1$ is the maximum mode number allowed up to the critical angle $\theta_c$. Modes with a mode number $m$
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Figure 4.3 (a) Penetration depth $\xi$ as a function of the angle of incidence (with $\theta_c = 0.18^\circ$) and (b) as a function of the refractive-index contrast between the air gap and the cladding, $\delta = 1 - n_2$. The wavelength $\lambda = 0.0751$ nm.

larger than $m_{\text{max}}$, i.e. the radiation modes [8], are not confined to the guiding layer and will be absorbed by the plate material. For an incident angle $\theta_i$ in between two consecutive guided-mode angles, the amplitude is distributed predominantly over the two neighboring modes.

4.3 Experimental

Our waveguide was made of two fused-silica plates with an optical flatness of $<\lambda/20$, coated with a thin metal film. Chromium layers of 30 nm thickness were sputter-deposited onto the bottom plate ($\varnothing 25$ mm) and thermally evaporated onto the upper plate ($\varnothing 5.2$ mm). The surfaces had an r.m.s. roughness of 0.3 nm and 0.4 nm, respectively. These values were determined from x-ray reflectivity measurements on each of the plates. The gap width was controlled by piezo-driven motors and measured by means of optical interferometry as discussed in chapter 3.

A photon energy of 16.5 keV ($\lambda = 0.0751$ nm) was selected using the (111) reflection of a diamond crystal monochromator. The monochromator is followed by a mirror for suppression of higher harmonics from the undulator. The intensity of the beam of 0.1 mm width passing through a vertical gap of 500 nm was typically
2.4 \times 10^7 \text{ photons/s}. The transverse coherence length $\xi_v$ of the beam in the vertical plane along the coordinate $x$ is determined by $\xi_v = \lambda D/\sigma_v = 144 \mu m$, where $D \approx 45 \text{ m}$ is the distance from the source to the sample and $\sigma_v = 23.5 \mu m$ the vertical source size (full-width-at-half-maximum) of the beam. The incident field is fully coherent across the gap since $\xi_v$ is much larger than the typical gap sizes used in the experiment. In the horizontal plane the source size equals 928 $\mu m$, which yields a transverse coherence length $\xi_h = 3.6 \mu m$. As $\xi_h$ is much smaller than the horizontal beam width of 0.1 mm, the beam has incoherent properties in this direction. The longitudinal coherence length equals $\xi_l = \lambda^2/\Delta \lambda = 1.5 \mu m$, with $\Delta \lambda/\lambda = 5 \times 10^{-5}$ the monochromator bandwidth [24]. $\xi_l$ is to be compared with the maximum path length difference $PLD_{\text{max}} \approx L(\theta_c^2 - \theta_0^2)/2$ between the highest and lowest modes after travelling over the length of the waveguide $L$. For $L = 5.2 \text{ mm}$ we find $PLD_{\text{max}} \approx 26 \text{ nm}$ and we conclude that $\xi_l \gg PLD_{\text{max}}$. Hence, the non-zero bandwidth of the monochromator does not affect the coherent phase relation between different guided modes.

Given an angular spacing $\Delta \theta_m = \pi/k_0 W$ between modes of typically 0.005° and a vertical beam divergence being much smaller than this value, it is possible to excite only one mode at a time. The total number of guided modes $m_{\text{max}}$ is determined by the critical angle for total reflection from the chromium layer which equals $\theta_c = 0.18^\circ$ ($n_2 = 1 - 5.0 \times 10^{-6}$). For a gap width of 400 nm we find $m_{\text{max}} = 33$.

### 4.4 Results and discussion

#### 4.4.1 Far-field diffraction patterns

The modes propagating through the waveguide for a given incidence angle $\theta_i$ are identified by measurement of the far-field angular distribution of intensity exiting the waveguide. The diffracted intensity was recorded as a function of the exit angle $\theta_e$ by a NaI scintillating detector which can be rotated in the vertical plane. A slit in front of the detector fixes the vertical opening angle at 0.0005°. The presence of the reflecting bottom plate behind the exit plane causes interference between the direct and speculatively reflected waves emerging from the exit, making it the time-reversed case of the interference occurring at the front of the waveguide.
We measured the diffraction patterns for a range of fixed incidence angles $\theta_i$ up to a value corresponding to excitation of the $11^{th}$ TE mode. Fig. 4.4 shows a logarithmic contour plot of the intensity over a mesh of angle pairs $(\theta_i, \theta_e)$, in steps of $\Delta \theta_i = 0.001^\circ$ and $\Delta \theta_e = 0.0005^\circ$. The peaks along the diagonal at mode angles $\theta_i = \theta_m$, are the unperturbed guided modes. Their angular spacing ($\Delta \theta_m \approx 0.0044^\circ$) corresponds to a gap width $W$ of 486 nm, which confirms the interferometrically measured gap width. The off-diagonal peaks in between mode angles are subsidiary diffraction maxima associated with the finite width of the gap. For $\theta_i$ in between mode angles $\theta_m$ and $\theta_{m+1}$, the field amplitude within the waveguide is distributed over a complete set of modes, but mainly the neighboring ones. Figure 4.5 illustrates this for patterns along the vertical lines in Fig. 4.4, which were taken at $\theta_i$ values equal to $\theta_7$, $(\theta_7 + \theta_8)/2$ and $\theta_8$. The corresponding amplitude distributions within the entrance plane are shown as well.

The measured diffraction patterns are compared with the predictions of a mode
propagation model based on the wave field amplitude as given by Eq. (4.5). The diffracted intensity in the far-field limit, including the post-reflection at the lower plate, is given by Eq. (3.13b):

\[ I(\theta_i, \theta_e) = \frac{W^2}{R \lambda} \left| \sum_{m=0}^{m_{\text{max}}} e^{-i \beta_m L} c_m(\theta_i) \cdot c_m(\theta_e) \right|^2 , \]  

(4.7)

where \( R \) is the distance from the waveguide exit to the detector. The calculated positions and heights of the diffraction minima are in good agreement with the measurements (dashed curves in Fig. 4.5). However, the observed phase contrast is smaller than calculated. This probably relates to a partial incoherence of the beam in the vertical plane, which is caused by the optical elements along the incident beam path [42]. We find a better fit to the measured diffraction patterns if \( I(\theta_i, \theta_e) \) as given by Eq. (4.7) is convoluted with a Gaussian distribution in \( \theta_i \) having a full-width-at-half-maximum of 0.0033° (solid curves in Fig. 4.5). It is as yet unclear which optical elements are responsible for the reduced phase contrast.

Deviations between measurements and calculations are found at higher subsidiary maxima of the diffraction pattern for \( \theta_i = (\theta_\gamma + \theta_\delta)/2 \) (Fig. 4.5b). This part of the spectrum is sensitive to the change of the field amplitude at \( x = W \), where it has to drop sharply to zero. Given the good fits, there is no indication that modes are excited by surface imperfections as was previously found in solid waveguiding structures [43].

### 4.4.2 Transmission measurements

The power transmitted by the waveguide is determined by opening the detector so as to capture all of the intensity emerging from the exit. This is equivalent to integrating over vertical lines in the contour plot of Fig. 4.4. In this way, we have measured the transmitted power as a function of \( \theta_i \). The measurements shown in Fig. 4.6a were performed on a waveguide with SiO\(_2\) plates (see chapter 3), set at gaps of 727 nm and 371 nm. The wavelength was \( \lambda = 0.0931 \) nm.

As expected, the transmitted power decreases proportionally with decreasing gap width. For both measurements, the period of the oscillations equals the mode spacing \( \Delta \theta_m \). The shoulder at \( \theta_i \approx 0.0035^\circ \) in the top curve (\( W = 727 \) nm) is a suppressed peak. This is a geometrical effect, caused by the finite size of the lower plate. For very small angles of incidence, the footprint of the incident beam
Figure 4.5 Diffraction patterns from the exit of the waveguide for different angles of incidence $\theta_i$ corresponding with the vertical lines in Fig. 4.4. The values of $\theta_i$ are (a) $\theta_7$, (b) $(\theta_7+\theta_8)/2$, and (c) $\theta_8$. The dashed curves are patterns calculated with the use of Eq. (4.7). The solid curves have been obtained by convoluting $I(\theta_i, \theta_e)$ in Eq. (4.7) with a Gaussian intensity distribution in $\theta_i$, see text.
becomes larger than the length \( L_e \) of the lower plate in front of the waveguide entrance. This reduces the flux of the incident beam at the waveguide entrance for angles \( \theta_i < W/L_e \). Inserting the parameter values for our geometry, \( L_e \approx 11.5 \text{ mm} \) and \( W = 727 \text{ nm} \), we indeed find \( \theta_i < 0.0036^\circ \). Note that the same happens when the x rays exit the waveguide.

Let us now compare the transmission curve measured for \( W = 371 \text{ nm} \), with a calculation of the transmitted power. The latter is given by

\[
P(\theta_i) = \int_0^\infty I(\theta_i, \theta_e) R d\theta_e. \tag{4.8}
\]

It can be readily verified, by substituting Eq. (3.12) into Eq. (3.13a), that

\[
\int_0^\infty I(\theta_i, \theta_e) R d\theta_e = \int_0^W |\Psi(x, L)|^2 dx. \tag{4.9}
\]

If we now substitute Eq. (4.5) into the right-hand side of Eq. (4.9), we obtain Eq. (3.10b):

\[
P(\theta_i) = \frac{W}{2} \sum_{m=0}^{m_{\text{max}}} c_m(\theta_i)^2. \tag{4.10}
\]

Here it is assumed that the radiation modes, with \( m > m_{\text{max}} \), are absorbed by the cladding before they reach the exit of the waveguide. The calculated transmitted power is depicted by the thick solid curve shown in Fig. 4.6. The powers \( P(\theta_i) \), multiplied with a constant scaling factor, match well with the data. At angles \( \theta_i \geq 0.02^\circ \), the transmitted power becomes lower than the calculated one. The difference arises from absorption in the cladding, which becomes non-negligible as the penetration depths increase for larger angles.

The oscillations observed in the transmitted power are dominated by the fact that the incident field profile is clipped by the top plate at the waveguide entrance for \( \theta_i \neq \theta_m \), i.e. the field drops sharply to zero at \( x = W \). Besides being clipped, the incident field is filtered. The waveguide works as a low-pass spatial filter for the incident field since only the guided modes (\( m < m_{\text{max}} \)) contribute to the wave field at the waveguide exit. Even if \( m_{\text{max}} \to \infty \), \( P(\theta_i) \) still oscillates with a substantial amplitude as can be seen from

\[
\lim_{m_{\text{max}} \to \infty} P(\theta_i) = \frac{W}{2} \sum_{m=0}^{\infty} c_m(\theta_i)^2 = \int_0^W \sin^2(k_0 \theta_i x) dx = P_i(\theta_i) \tag{4.11a}
\]

\[
= \frac{W}{2} - \frac{\sin(2k_0 \theta_i W)}{4k_0 \theta_i}. \tag{4.11b}
\]
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Figure 4.6 Transmitted power as a function of the incidence angle $\theta_i$, measured for two different gap sizes (top to bottom): $W = 727$ and 371 nm. The line through the data for $W = 727$ nm is to guide the eye. The thick solid and dashed curves are the powers calculated with the use of Eq. (4.10) and Eq. (4.11b), respectively.

Here, $P_i(\theta_i)$ is the power of the incident field across the gap, which is shown as the dashed curve in Fig. 4.6. Comparing the curves for $P_i(\theta_i)$ and $P(\theta_i)$ we conclude that the effect of filtering is much smaller than the effect of clipping of the incident field.

4.4.3 Multi-mode interference

The specular reflectivity of the waveguide, i.e. the diffracted intensity at $\theta \equiv \theta_o = \theta_i$, has its maximum value at each mode angle $\theta_m$. At other angles the reflectivity is smaller because of destructive interference between modes, see the intensity along the diagonal of the contour plot in Fig. 4.4. Angle-dependent reflectivity curves are shown in Fig. 4.7 for three different values of $W$. For the mode spacing, the relation $\Delta \theta_m = \pi/k_0 W$ is confirmed. Also a longer-period variation of the reflectivity is present, which is due to multi-mode interference. We have calculated the reflectivity using Eq. (4.7), see dashed curves. The function $I(\theta, \theta)$ multiplied
Figure 4.7 Diffracted intensities measured along the diagonal $\theta = \theta_0 = \theta_i$ in Fig. 4.4, for $L = 5.2$ mm and gap widths $W$ of (a) 391 nm, (b) 478 nm, and (c) 506 nm. The dashed curves are intensities calculated with the use of Eq. (4.7). The solid curves have been obtained by convoluting Eq. (4.7) with a gaussian intensity distribution, see text.

by a constant scaling factor, reproduces the measured reflectivity curves very well, except at the lowest angles where surface irregularities at the entrance may have affected the measurements.

Again, the observed interferences are weaker than calculated and the convolution as described above provides a better fit (solid curves). Our observation that the slow periodic variations in the specular reflectivity are in good agreement with the multi-mode propagation theory [44], is direct proof that the coherence is preserved over the entire length of the waveguide. The latter is quite remarkable.

While the number of oscillations made by the e.m. field of a single mode over the
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Fig. 4.8 Intensity of the field within a waveguide, calculated with the use of Eq. (4.7), with $\lambda = 0.0751$ nm, $W = 442$ nm, $L = 10.4$ mm and $\theta_0 = 1.5\theta_0 = 0.0091^\circ$.

distance $L$ is of the order $\beta_n L/2\pi \sim 10^8$, the difference $\left(\beta_m - \beta_n\right)L/2\pi \sim 1$ in the number of oscillations between (not too distant) modes $m$ and $n$ is found to remain well-defined.

The minima of the reflected intensities shown in Fig. 4.7 exhibit a strong dependence on the angle $\theta$. However, for certain combinations of $\lambda$, $W$ and $L$, there may be no dependence on $\theta$ at all. In the following it is shown that for these parameters, the field profile across the incidence plane is directly imaged onto the exit plane of the waveguide, i.e. $\Psi(x, L) = \Psi(x, 0)$. Alternatively, an inverted image with respect to the line $x = W/2$ may be formed, i.e. $\Psi(x, L) = \Psi(W-x, 0)$.

In order to derive the conditions for which this self-imaging occurs, we take the phase of the $TE_0$ mode as a common factor out of the sum in Eq. (4.5) [44], so that

$$\Psi(x, L) = \sum_{m=0}^{m_{\text{max}}} c_m \phi_m e^{i(\beta_0 - \beta_m)L}. \quad (4.12)$$

The phase factor in Eq. (4.12) may be worked out as follows:

$$\begin{align*}
(\beta_0 - \beta_m)L & \approx k_0 \left(1 - \frac{\theta_0^2}{2}\right) L - k_0 \left(1 - \frac{\theta_m^2}{2}\right) L, \\
& = \frac{\pi \lambda}{4W^2} m(m+2)L. \quad (4.13)
\end{align*}$$
By defining the 'self-imaging' length as

$$L_s = \frac{4W^2}{\lambda},$$

we rewrite the phase factor as

$$(\beta_0 - \beta_m)L = \frac{m(m + 2)\pi L}{L_s}.$$  \hspace{1cm} (4.15)

Now, we make a distinction between the even and odd modes in the summation in Eq. (4.12). For the even modes, \(m(m + 2)\) is also even. This results in

$$e^{i(\beta_0 - \beta_m)L} = 1 \text{ for } L = pL_s,$$  \hspace{1cm} (4.16)

with \(p = 0, 1, 2, \ldots\). For the odd modes, \(m(m + 2)\) is odd and

$$e^{i(\beta_0 - \beta_m)L} = \begin{cases} 1 & \text{for } L = 2pL_s \\ -1 & \text{for } L = (2p + 1)L_s. \end{cases}$$  \hspace{1cm} (4.17)

Hence, if \(L\) is an even multiple of \(L_s\), then the phase changes of all the modes along \(L\) differ by multiples of \(2\pi\). In this case, all the guided modes interfere with the same relative phases as in \(z = 0\). This results in a direct image of the incident field, i.e., \(\Psi(x, L) = \Psi(x, 0)\). If on the other hand \(L\) is an odd multiple of \(L_s\), the even and odd modes will be in antiphase. Because in a symmetric waveguide \(\phi_m(x) = \phi_m(W - x)\) for the even modes and \(\phi_m(x) = -\phi_m(W - x)\) for the odd modes, it is readily verified that in that case the resulting field is an inverted image of the incident field, i.e., \(\Psi(x, L) = \Psi(W - x, 0)\). For illustration, we calculated the intensity of the field within a waveguide \(|\Psi(x, z)|^2\) having a length \(L = L_s = 10.4\) mm, see Fig. 4.8. Indeed, at \(z = 10.4\) mm an inverted image of the incident field is formed.

For values of \(L\) half-way in between multiples of \(L_s\), multiple images can be found. For example, for \(L = (p + \frac{1}{2})L_s\) with \(p = 0, 1, 2, \ldots\), the total field is given by

$$\Psi(x, (p + \frac{1}{2})L_s) = \sum_{m=0}^{m_{\max}} c_m \phi_m e^{im(m+2)(p+\frac{1}{2})\pi}.$$  \hspace{1cm} (4.18)

For even and odd modes, the phase factor assumes two different values:

$$e^{im(m+2)(p+\frac{1}{2})\pi} = \begin{cases} 1 & \text{for } m = 2k \\ (-i)^{p+1} & \text{for } m = 2k + 1, \end{cases}$$  \hspace{1cm} (4.19)
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with \( k = 0, 1, 2, \ldots \). This results in

\[
\Psi(x, (p + \frac{1}{2})L_s) = \sum_k c_{2k}\phi_{2k}(x) - (-i)^{p+1} \sum_k c_{2k+1}\phi_{2k+1}(W - x),
\]

(4.20)

where we have used \( \phi_{2k+1}(x) = -\phi_{2k+1}(W - x) \). It is easy to show that

\[
\sum_k c_{2k}\phi_{2k} = \frac{1}{2} \left[ \Psi(x, 0) + \Psi(W - x, 0) \right]
\]

(4.21)

and

\[
\sum_k c_{2k+1}\phi_{2k+1} = \frac{1}{2} \left[ \Psi(W - x, 0) - \Psi(x, 0) \right].
\]

(4.22)

Hence,

\[
\Psi(x, (p + \frac{1}{2})L_s) = \frac{1 + (-i)^{p+1}}{2} \Psi(x, 0) + \frac{1 - (-i)^{p+1}}{2} \Psi(W - x, 0).
\]

(4.23)

This equation represents a pair of images of \( \Psi(x, 0) \). One is erect, the other is inverted. In Fig. 4.8 we can distinguish both images at \( z = L_s/2 \), although they overlap.

Now, we derive an approximate expression for the intensity of the minima in \( I(\theta, \theta) \). These occur for angles \( \theta \) in between mode angles:

\[
I_{\text{min}} = I(\theta_{m+1/2}, \theta_{m+1/2})
\]

(4.24a)

\[
\approx \frac{W^2}{RA} \left| c_m^2(\theta_{m+1/2})e^{-i\beta_mL} + c_{m+1}^2(\theta_{m+1/2})e^{-i\beta_{m+1}L} \right|^2,
\]

(4.24b)

where we have taken into account only the contribution of nearest-neighbor modes.

By making use of Eq. (4.6), it is readily verified that:

\[
c_m(\theta_{m+1/2}) = -\frac{2}{\pi} \frac{m + \frac{1}{4}}{m + \frac{3}{4}}
\]

(4.25)

and

\[
c_{m+1}(\theta_{m+1/2}) = \frac{2}{\pi} \frac{m + \frac{1}{4}}{m + \frac{3}{4}}
\]

(4.26)

If we assume \( c_m^2(\theta_{m+1/2}) \approx c_{m+1}^2(\theta_{m+1/2}) = 4/\pi^2 \) for \( m \geq 4 \), which is valid to within 14%, we may rewrite Eq. (4.24b) as

\[
I_{\text{min}} \approx I_0 \frac{32}{\pi^4} \left\{ 1 + \cos \left[ (\beta_{m+1} - \beta_m)L \right] \right\},
\]

(4.27)
Figure 4.9 Specular intensity measured for \( \lambda = 0.0751 \) nm, \( L = 5.2 \) mm and a gap width of \( W = 442 \) nm. The solid curve are intensities calculated with the use of Eq. (4.7).

with \( I_0 = I(\theta_m, \theta_m) = W^2/R\lambda \). Equation (4.27) is valid to within 11\% for mode numbers \( m \geq 4 \). The difference in propagation constants is approximated as follows:

\[
\beta_{m+1} - \beta_m = k_0 \cos \theta_{m+1} - k_0 \cos \theta_m, \\
\approx -(2m + 3) \frac{\lambda \pi}{4W^2}, \\
= -(2m + 3) \frac{\pi}{L_s}. 
\]  

Substitution into Eq. (4.24a) leads to

\[
I_{\text{min}} = I_0 \frac{32}{\pi^4} \left\{ 1 + \cos \left[ \frac{(2m + 3)\pi L}{L_s} \right] \right\}. 
\]  

From this expression it becomes clear that the intensities of the minima are constant as a function of \( m \) whenever the parameters \( L, W \) and \( \lambda \) fulfil the self-imaging conditions discussed previously. It is readily verified that

\[
I_{\text{min}} = \begin{cases} 
64I_0/\pi^4 & \text{for } L = 2pL_s \\
0 & \text{for } L = (2p + 1)L_s \\
32I_0/\pi^4 & \text{for } L = (p + \frac{1}{2})L_s. 
\end{cases} 
\]  

(4.31)
The last case is illustrated in Fig. 4.9. It shows intensities $I(\theta, \theta)$ which were measured and calculated with the parameter values $W = 442$ nm, $\lambda = 0.0751$ nm and $L = 5.2$ mm $\sim L_s/2$. Indeed, the minima are constant as a function of $\theta$. The measured intensities agree very well with the calculated intensities. Once again, we conclude that the coherent properties of the incident wave field are preserved upon propagation through the waveguide.

4.5 Conclusions

The measured far-field diffraction patterns show that, due to the high spatial coherence of the incident wave field, single TE modes are excited at the entrance of the waveguide for angles of incidence $\theta_i = \theta_m$. For other incidence angles we observed the excitation of multiple modes. These were found to interfere continuously upon propagation. The preservation of longitudinal coherence is indicated by the good agreement between the measured and calculated reflected intensities. Measurements of the transmitted power as a function of $\theta_i$ showed oscillations with a period $\Delta \theta_m$. These are due to the clipping of the incident field profile by the top plate. The clipping was found to have a bigger effect on the transmitted power than the loss of power due to the excitation of radiation modes.