On the inhomogeneous magnetised electron gas
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Chapter 2

Density profiles

Ever since Landau's original derivation [36] of diamagnetism in a magnetised free-electron gas, there has been interest in boundary effects. This is not surprising, since the diamagnetism of a finite sample is caused by currents flowing near the boundary. To get a deeper insight in the diamagnetic effect one needs to investigate the behaviour of these currents in the neighbourhood of a wall parallel to the external magnetic field. Of particular interest is the question how the current density decays in the bulk.

For high temperatures it is adequate to use Maxwell-Boltzmann statistics. In that approximation the precise form of the current profile in the neighbourhood of a hard wall has been studied by Ohtaka & Moriya [42] and by Jancovici [27] within the framework of linear response, and, more recently, by John & Suttorp [31] with the use of a Green function method. In both approaches a Gaussian decay of the current density in the bulk has been found: asymptotically the decay is proportional to \( \exp(-x^2) \), with \( x \) the distance from the wall in suitable units. A similar decay has been found [31, 30] for the excess charge density and the excess (kinetic) pressure.

For lower temperatures the effects of quantum statistics have to be taken into account. In that regime Macris, Martin and Pulé [40] have derived an exponential bound \( \sim \exp(-x) \) on the decay of the current density in the bulk, at least for non-zero temperature. For the strongly-degenerate case of vanishing \( T \), Ohtaka & Moriya [42] and Jancovici [27] have obtained a closed expression for the current density, via an inverse Laplace transform of the expression for Maxwell-Boltzmann statistics. Remarkably enough, their results exhibit a much slower algebraic decay proportional to \( x^{-1} \). Using the same method one easily derives similar expressions for the excess charge density and the excess pressure at \( T = 0 \). However, the expression for the excess pressure obtained along these lines shows the unphysical feature of an oscillatory behaviour that is no longer damped in the bulk.
The various findings for the asymptotic behaviour of physical quantities near the bulk, as described above, justify a closer look at the problem. In this chapter we will derive systematic asymptotic expansions for the charge and the current density near the bulk, by starting from exact integral expressions valid at \( T = 0 \), which will be established on the basis of a Green function formulation. The validity of these asymptotic expansions will be assessed by a comparison with the results of a numerical evaluation of the integral expressions.

### 2.1 Green functions; charge and current density

Consider the half-space \( x > 0 \), with a hard wall at \( x = 0 \). Choose the magnetic field in the \( z \)-direction, with vector potential \( \mathbf{A} = (0, Bx, 0) \). The transverse part of the Hamiltonian for a particle with charge \( e \) and mass \( m \) in this field is given by

\[
H_{\perp} = -\frac{\hbar^2}{2m} \Delta_{\perp} + i\hbar \omega_c \frac{\partial}{\partial y} + \frac{1}{2} m \omega_c^2 x^2
\]  

(2.1)

where \( \omega_c = eB/mc \) is the cyclotron frequency associated with the particle.

In order to simplify the notation we will choose units such that \( e = 1, m = 1, c = 1 \) and \( \hbar = 1 \) (which implies \( \omega_c = B \)), such that the Hamiltonian becomes

\[
H_{\perp} = -\frac{1}{2} \Delta_{\perp} + iBx \frac{\partial}{\partial y} + \frac{1}{2} B^2 x^2.
\]

(2.2)

The Green function for the eigenvalue equation \( H_{\perp} \psi_n(r) = E_n \psi_n(r) \) (\( r = (x, y) \)) is defined by

\[
(H_{\perp} - u)G_{\perp,u}(r, r') = -\delta(r - r')
\]

(2.3)

with \( u \) a complex energy variable and with boundary condition \( G_{\perp,u}(r, r') = 0 \) for \( x = 0 \) and/or \( x' = 0 \). This means we can express the Green function as

\[
G_{\perp,u}(r, r') = \sum_n \psi_n(r)\psi_n^*(r') \frac{1}{u - E_n}.
\]

(2.4)

The discontinuity of \( G_{\perp,u} \) at \( u = E \)

\[
G_{\perp,E}(r, r') = \frac{i}{2\pi} [G_{\perp,u=E+i0}(r, r') - G_{\perp,u=E-i0}(r, r')]
\]

(2.5)

will be referred to as the energy Green function.
2.1. Green functions; charge and current density

Due to the translation invariance in the y-direction of both the Hamiltonian and the boundary condition, a Fourier transform is appropriate. If we define the transform by

$$G_{\perp,\mathbf{u}}(\mathbf{r}, \mathbf{r}') = (2\pi)^{-1} \int_{-\infty}^{\infty} dk \exp[i\mathbf{k}(\mathbf{y}-\mathbf{y}')]G_{\perp,\mathbf{u}}(k, x, x'),$$

the Hamiltonian becomes

$$H_{\perp}(k) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} k^2 - Bkx + \frac{1}{2} B^2 x^2$$

(2.6)

and the equivalent of (2.3) is

$$[H_{\perp}(k) - \mathbf{u}]G_{\perp,\mathbf{u}}(k, x, x') = -\delta(x-x').$$

(2.7)

This means that we can write the Fourier transform of the energy Green function as

$$G_{\perp,E}(k, x, x') = \sum_n \psi_n(k, x)\psi_n^*(k, x') \delta(E_n(k) - E)$$

(2.8)

where the $\psi_n(k, x)$ are eigenfunctions of $H_{\perp}(k)$, with eigenvalues $E_n(k)$, normalised such that $\int_0^\infty dx |\psi_n(k, x)|^2 = 1$.

For high temperatures, we can use Maxwell-Boltzmann statistics to calculate the properties of the electron gas. The charge density for a gas of charged particles without mutual interaction at inverse temperature $\beta$ is given by

$$\rho_\beta(x) = \frac{\rho}{Z_\perp} \sum_n |\psi_n(x)|^2 e^{-\beta E_n}$$

(2.9)

where $\rho$ is the bulk density and $Z_\perp$ is the part of the one-particle bulk partition function (per unit volume) corresponding to the degrees of freedom in the directions perpendicular to the magnetic field. With the help of the Fourier transform of the energy Green function (2.8) we can write this as

$$\rho_\beta(x) = \frac{\rho}{Z_\perp} \int_0^\infty dE e^{-\beta E} \int_{-\infty}^\infty dk G_{\perp,E}(k, x, x).$$

(2.10)

At lower temperatures we will have to take into account the effects of quantum statistics. For fermions at $T = 0$ we can use the inverse Laplace transform technique from the introduction:

$$\rho_\mu(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta e^{\beta \mu} \frac{2Z}{\rho \beta} \rho_\beta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta e^{\beta \mu} \frac{2Z}{\beta} \sum_n |\psi_n(x)|^2 e^{-\beta E_n}$$

(2.11)

where the constant $c$ can be chosen arbitrarily as long as it is positive. Note the factor 2 that takes into account the spin degeneracy; we ignore Zeeman splitting here.
Chapter 2. Density profiles

Since $Z_y = (2\pi \beta)^{-1/2}$, the charge density for a gas of spin-$\frac{1}{2}$ fermions without mutual interaction at temperature $T = 0$ and chemical potential $\mu$ is given by

$$\rho_\mu(x) = \frac{2^{3/2}}{\pi} \sum_n |\psi_n(r)|^2 (\mu - E_n)^{1/2}. \quad (2.12)$$

Again, by using the Fourier transform of the energy Green function (2.8) we can write this as

$$\rho_\mu(x) = \frac{2^{1/2}}{\pi^2} \int_0^\mu dE (\mu - E)^{1/2} \int_{-\infty}^\infty dk G_{\perp,E}(k, x, x). \quad (2.13)$$

Likewise, the current density in the $y$-direction is given by

$$j_{y,\mu}(x) = \frac{2^{3/2}}{\pi} \sum_n \left\{ -\frac{i}{2} \left[ \psi_n^*(r) \frac{\partial \psi_n(r)}{\partial y} - \frac{\partial \psi_n^*(r)}{\partial y} \psi_n(r) \right] - Bx |\psi_n(r)|^2 \right\} (\mu - E_n)^{1/2} \quad (2.14)$$

or

$$j_{y,\mu}(x) = \frac{2^{1/2}}{\pi^2} \int_0^\mu dE (\mu - E)^{1/2} \int_{-\infty}^\infty dk (k - Bx) G_{\perp,E}(k, x, x). \quad (2.15)$$

2.2 Explicit form of Green function; parabolic cylinder functions

We now define dimensionless quantities by expressing the position $x$ in units $1/\sqrt{B}$ (or $\hbar/(m \omega_c)$)$^{1/2}$ in the original units) and the wavenumber $k$ in units $\sqrt{B}$ (or in the original units $(m \omega_c/\hbar)^{1/2}$). The relevant variables become $\xi = \sqrt{B}x$, $\kappa = k/\sqrt{B}$. We also express all energies in units $B$ (or $\hbar \omega_c$). Therefore we will use $\epsilon = E/B$ and $\nu = \mu/B$. Using these new variables, we get the following dimensionless Hamiltonian

$$H_\perp(\kappa) = -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2} (\xi - \kappa)^2. \quad (2.16)$$

The corresponding eigenfunctions are the parabolic cylinder functions $[41]$

$$\psi_n(\kappa, \xi) = \left[ \int_0^\infty d\xi D_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\xi - \kappa)) \right]^{-1/2} D_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\xi - \kappa)) \quad (2.17)$$

where we have applied a similar normalisation as before (but now with dimensionless $\xi$ and $\kappa$). The function $\epsilon_n(\kappa)$, which gives the eigenvalues, is defined by the boundary condition at $\xi = 0$

$$D_{\epsilon_n(\kappa) - 1/2}(-\sqrt{2}\kappa) = 0. \quad (2.18)$$

The function is plotted in figure 2.1. It has been studied before by MacDonald and

1. We use $\xi$ (with a bar) here to avoid confusion when we will use $\xi$, $\eta$ and $\zeta$ for differences between coordinates later on in chapter 5.
2.2. Explicit form of Green function; parabolic cylinder functions

Figure 2.1: The function $\epsilon_n(\kappa)$, for $n = 0$ up to 4.

Středa [39] and by Kunz [35]. As can be seen in figure 2.1, $\epsilon_n(\kappa)$ has the property that $\lim_{\kappa \to \infty} \epsilon_n(\kappa) = n + \frac{1}{2}$ and $\epsilon_n(0) = 2n + \frac{3}{2}$.

If we substitute (2.17) into (2.8) we get the following expression for the energy Green function

$$G_{\perp,E}(\kappa, x, x) = \frac{1}{\sqrt{B}} \sum_n \left[ \int_0^\infty d\tilde{\xi}' \, D^2_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\tilde{\xi}' - \kappa)) \right]^{-1} \times D^2_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\tilde{\xi} - \kappa)) \delta(\epsilon - \epsilon_n(\kappa)).$$

(2.19)

Inserting this expression into (2.13) and (2.15) we can carry out the integration over $\xi$. Defining $\kappa_n(\nu)$ by $\epsilon_n(\kappa_n(\nu)) = \nu$ we arrive at the following expressions for the charge density

$$\rho_\mu(x) = \frac{\sqrt{2}B^{3/2}}{\pi^2} \sum_n' \int_{\kappa_n(\nu)}^{\infty} \! d\kappa' \left[ \nu - \epsilon_n(\kappa) \right]^{1/2} \times \left[ \int_0^\infty d\tilde{\xi}' \, D^2_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\tilde{\xi}' - \kappa)) \right]^{-1} D^2_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\tilde{\xi} - \kappa)).$$

(2.20)
and the current density

\[ j_{y,u}(x) = -\frac{\sqrt{2B^2}}{\pi^2} \sum_n \int_{\kappa_n(y)}^\infty d\kappa \left[ \sqrt{\frac{2}{\pi}} (\xi - \kappa) \right]^{1/2} (\xi - \kappa) \]

\[ \times \left[ \int_0^\infty d\xi' D_{n(\xi')-1/2}^2 (\sqrt{2}(\xi' - \kappa)) \right]^{-1} D_{n(\xi)-1/2}^2 (\sqrt{2}(\xi - \kappa)). \] (2.21)

The summations are over all \( n < \nu - \frac{1}{2} \), which we indicate by the prime. From these expressions it is fairly easy to see that in the bulk the charge density is given by

\[ \rho_{y,u} = \lim_{x \to \infty} \rho_{y,u}(x) = \frac{\sqrt{2B^3/2}}{\pi^2} \sum_n \left[ \nu - (n + \frac{1}{2}) \right]^{1/2}. \] (2.22)

The current density in the bulk can be calculated in a very similar way:

\[ j_{y,u} = \lim_{x \to \infty} j_{y,u}(x) = -\frac{\sqrt{2B^3/2}}{\pi^2} \sum_n \left[ \nu - (n + \frac{1}{2}) \right]^{1/2} \frac{1}{\sqrt{\pi n!}} \int_{-\infty}^\infty d\lambda \lambda D_n^2(\sqrt{2}\lambda). \] (2.23)

Since the functions \( D_n(\sqrt{2}\lambda) \) and \( \lambda D_n(\sqrt{2}\lambda) \) are orthogonal, this means that there is no current in the bulk.

Alternative expressions for the charge and the current density can be found by writing \( G_{\perp,u}(k, x, x') \) as the sum of the Green function for an infinite domain and a correction due to the boundaries. The infinite-domain Green function is given by [21]

\[ G^0_{\perp,u}(k, x, x') = -\frac{1}{\sqrt{\pi B}} \Gamma (-u/B + \frac{1}{2}) D_{u/B-1/2}(\sqrt{2}(\sqrt{B}x - \kappa)) \]

\[ \times D_{u/B-1/2}(-\sqrt{2}(\sqrt{B}x' - \kappa)) \] (2.24)

for \( x > x' \), and an analogous expression for \( x < x' \). The correction for the chosen geometry is [31]

\[ G^c_{\perp,u}(k, x, x') = \frac{1}{\sqrt{\pi B}} \Gamma (-u/B + \frac{1}{2}) \frac{D_{u/B-1/2}(\sqrt{2}\kappa)}{D_{u/B-1/2}(-\sqrt{2}\kappa)} \]

\[ \times D_{u/B-1/2}(\sqrt{2}(\sqrt{B}x - \kappa)) D_{u/B-1/2}(\sqrt{2}(\sqrt{B}x' - \kappa)) \] (2.25)

for all \( x > 0 \) and \( x' > 0 \).

The energy Green function is determined by the poles of \( G^0_{\perp,u} + G^c_{\perp,u} \). Since the contributions from the gamma functions in \( G^0_{\perp,u} \) and \( G^c_{\perp,u} \) cancel, only the roots of
the denominator in (2.25) contribute. They give a residue proportional to the derivative \( \frac{\partial D_{\epsilon-1/2}(-\sqrt{2}\kappa)}{\partial \epsilon} \) in \( \epsilon = \epsilon_n(\kappa) \), which results in

\[
G_{\perp, \xi}(k, x, x) = \frac{1}{\sqrt{n\pi B}} \sum_n \Gamma(-\epsilon_n(\kappa) + \frac{1}{2}) D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi - \kappa)) \\
\times D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}\kappa) \left[ \frac{\partial D_{\epsilon-1/2}(-\sqrt{2}\kappa)}{\partial \epsilon} \right]_{\epsilon=\epsilon_n(\kappa)}^{-1} \delta(\epsilon - \epsilon_n(\kappa)).
\]

(2.26)

From (2.18) we see that

\[
\frac{\partial D_{\epsilon-1/2}(-\sqrt{2}\kappa)}{\partial \epsilon} \bigg|_{\epsilon=\epsilon_n(\kappa)} = -\frac{\partial D_{\epsilon-1/2}(-\sqrt{2}\kappa)}{\partial \kappa} \bigg|_{\epsilon=\epsilon_n(\kappa)} \left[ \frac{d\epsilon_n(\kappa)}{d\kappa} \right]^{-1}.
\]

(2.27)

With the help of the Wronskian \( W[D_{\lambda}(z), D_{\lambda}(-z)] = \sqrt{2\pi}/\Gamma(-\lambda) \) [41] we derive

\[
G_{\xi, \perp}(k, x, x) = -\frac{1}{2\pi\sqrt{B}} \sum_n \Gamma^2(-\epsilon_n(\kappa) + \frac{1}{2}) D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}\kappa) \\
\times D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi - \kappa)) \frac{d\epsilon_n(\kappa)}{d\kappa} \delta(\epsilon - \epsilon_n(\kappa)).
\]

(2.28)

By comparing this with (2.19) we find

\[
\left[ \int_0^\infty d\xi' D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi' - \kappa)) \right]^{-1} = \frac{1}{2\pi} \Gamma^2(-\epsilon_n(\kappa) + \frac{1}{2}) D_{\epsilon_n(\kappa)-1/2}(\sqrt{2}\kappa) \left[ \frac{d\epsilon_n(\kappa)}{d\kappa} \right]
\]

(2.29)

where we made use of the fact that \( d\epsilon_n(\kappa)/d\kappa < 0 \).

Plugging (2.29) into (2.20) and (2.21) gives alternative expressions for the charge density and the current density. Unfortunately neither these nor (2.20) and (2.21) allow us to evaluate the integrals over \( \kappa \) analytically. Both sets of formulas can be used for a numerical evaluation, although the expressions based on (2.28) are more convenient, since they involve a single integration only. Numerical results obtained along these lines are presented in figure 2.2. Both the charge and the current density decay to their bulk values within a distance of a few times the typical length scale of the system \( 1/\sqrt{B} \). Near the boundary, the current density exhibits a layered structure of currents flowing in alternate directions. The number of layers increases with the number of filled Landau levels.
2.3 Asymptotic expansions

The expressions (2.20) and (2.21) are a suitable starting-point to derive the asymptotic behaviour of the charge density and the current density for large $\xi$. We will start out with the latter since it is somewhat simpler; the current density vanishes in the bulk.

From (2.21) we see that for the current density, we need to determine asymptotic expansions of the integrals

$$ I_n(\xi) = \int_{\kappa_n(\nu)}^{\infty} d\kappa \left[ \nu - \epsilon_n(\kappa) \right]^{1/2} (\xi - \kappa) $$

$$ \times \left[ \int_0^{\infty} d\xi' D_{\epsilon_n(\kappa)-1/2}^2(\sqrt{2}(\xi' - \kappa)) \right]^{-1} D_{\epsilon_n(\kappa)-1/2}^2(\sqrt{2}(\xi - \kappa)). \quad (2.30) $$

It will turn out that $I_n(\xi)$ decays as $\exp(-\xi^2/2)$, so we can discard any terms that decay faster than that.

We split the integration interval at $\kappa' = \alpha'\xi$, with $\alpha' < 1 - \frac{1}{2}\sqrt{2}$. The contribution to
2.3. Asymptotic expansions

$I_n(\xi)$ from the interval $[\kappa_n(\nu), \kappa']$ can be estimated. Consider the normalisation factor

$$\int_0^\infty d\xi' D^2_\lambda(\sqrt{2}(\xi' - \kappa)) = \int_{-\kappa}^\infty d\xi' D^2_\lambda(\sqrt{2}\xi') \geq \int_{-\kappa_n(\nu)}^\infty d\xi' D^2_\lambda(\sqrt{2}\xi') \equiv c_n(\lambda)$$

(2.31)

with $\lambda = \epsilon_n(\kappa) - \frac{1}{2}$, which implies that $\lambda \in [n, \nu - \frac{1}{2}]$. Since $c_n(\lambda)$ is finite in the closed interval $[n, \nu - \frac{1}{2}]$, we conclude that $c_n(\lambda)$ is bounded from below by a certain $c_n$ independent of $K$. Now we use the following asymptotic series [41]

$$D_\lambda(z) \approx e^{-z^2/4}z^\lambda A_\lambda(z/\sqrt{2})$$

(2.32)

which is valid for large and positive $z$. Here we introduced

$$A_\lambda(z) = \sum_{m=0}^{\infty} \frac{(-\lambda/2)_m((1-\lambda)/2)_m(-z^2)^m}{m!}$$

(2.33)

where $(a)_n$ is Pochhammer’s symbol $a(a+1)\cdots(a+n-1)$. Note that $A_n(z)$ with $n$ integer has a finite number of terms only; it is related to the Hermite polynomials by $H_n(z) = (2z)^nA_n(z)$. From $\epsilon_n(\kappa) \leq \nu$ we conclude that

$$D^2_\epsilon_n(\kappa-1/2)(\sqrt{2}(\xi - \kappa)) \leq 2^{\nu-1/2}e^{-(\xi - \kappa)^2}(\xi - \kappa)^{2\nu-1} \left[ 1 + O((\xi - \kappa)^{-2}) \right]$$

(2.34)

for large positive $\xi - \kappa$. This means that the contribution of the interval $[\kappa_n(\nu), \kappa']$ to $I_n(\xi)$ is smaller than

$$\frac{2^{\nu-1/2}\nu^{1/2}}{c_n} \int_{\kappa_n(\nu)}^{\kappa'} d\kappa e^{-(\xi - \kappa)^2}(\xi - \kappa)^{2\nu} \left[ 1 + O((\xi - \kappa)^{-2}) \right].$$

(2.35)

Since we have chosen $\kappa' < (1 - \frac{1}{2}\sqrt{2})\xi$, this decays faster than $\exp(-\xi^2/2)$, so that it can be discarded.

For $\kappa > \kappa'$ we can use the asymptotic expansions of $\epsilon_n(\kappa)$

$$[\epsilon_n(\kappa) - (n + \frac{1}{2})] \approx \frac{1}{\sqrt{\pi n!}} 2^n e^{-\kappa^2} \kappa^{2n+1} A_n(\kappa)$$

(2.36)

(see appendix 2.A for a derivation) and of the normalisation factor

$$\left[ \int_0^\infty d\xi' D^2_{\epsilon_n(\kappa) - 1/2}(\sqrt{2}(\xi' - \kappa)) \right]^{-1} \approx \frac{1}{\sqrt{\pi n!}} - \frac{1}{\pi(n!)^2} 2^{n+1} e^{-\kappa^2} \kappa^{2n+1} C_n(\kappa)$$

(2.37)
(see appendix 2.B), both of which are valid for large $\kappa$. Since $[e_n(\kappa) - (n + \frac{1}{2})]$ is small, we can write

$$D_n^2(\sqrt{2}(\xi - \kappa)) =$$

$$D_n^2(\sqrt{2}(\xi - \kappa)) + \frac{\partial}{\partial \lambda} D_n^2(\sqrt{2}(\xi - \kappa)) \bigg|_{\lambda = n} [e_n(\kappa) - (n + \frac{1}{2})] + \text{h.o.t.} \quad (2.38)$$

With the help of these expressions we find

$$I_n(\xi) \approx \int_{\kappa'}^{\infty} d\kappa [\nu - (n + \frac{1}{2})]^{1/2} \frac{1}{\sqrt{\pi n!}} (\xi - \kappa) D_n^2(\sqrt{2}(\xi - \kappa))$$

$$- \int_{\kappa'}^{\infty} d\kappa \frac{1}{2} [\nu - (n + \frac{1}{2})]^{-1/2} \frac{1}{\pi(n!)^2} 2^n e^{-\kappa^2} \kappa^{2n+1} \frac{A_n(\kappa)}{B_n(\kappa)} (\xi - \kappa) D_n^2(\sqrt{2}(\xi - \kappa))$$

$$- \int_{\kappa'}^{\infty} d\kappa [\nu - (n + \frac{1}{2})]^{1/2} \frac{1}{\pi(n!)^2} 2^n e^{-\kappa^2} \kappa^{2n+1} C_n(\kappa)(\xi - \kappa) D_n^2(\sqrt{2}(\xi - \kappa))$$

$$+ \int_{\kappa'}^{\infty} d\kappa [\nu - (n + \frac{1}{2})]^{1/2} \frac{1}{\pi(n!)^2} 2^n e^{-\kappa^2} \kappa^{2n+1} \frac{A_n(\kappa)}{B_n(\kappa)} (\xi - \kappa) \frac{\partial}{\partial \lambda} D_n^2(\sqrt{2}(\xi - \kappa)) \bigg|_{\lambda = n}$$

+h.o.t. \quad (2.39)

The first term can be discarded. This can be seen by writing $D_n$ in terms of the Hermite polynomial $H_n$: \[D_n(z) = 2^{-n/2} e^{-z^2/4} H_n(z/\sqrt{2}).\] \quad (2.40)

As $H_n^2(z)$ is even in $z$, we have

$$\int_{\kappa'}^{\infty} d\kappa (\xi - \kappa) D_n^2(\sqrt{2}(\xi - \kappa)) = 2^{-n} \int_{2\xi - \kappa'}^{\infty} d\kappa e^{-(\xi - \kappa)^2} (\xi - \kappa) H_n^2(\xi - \kappa). \quad (2.41)$$

Since $\kappa'$ is less than $(1 - \frac{1}{2}\sqrt{2})\xi$, this decays faster than $\exp(-\xi^2/2)$.

In the remaining terms of (2.39) we split the integration interval once more, now at $\kappa'' = \alpha' \xi$, with $\alpha'' > \frac{1}{2}\sqrt{2}$. The contribution from $\kappa > \kappa''$ in the second and the third term is negligible. This can be shown in the same way as we did for the first term. For the fourth term we use the following integral representation of the parabolic cylinder function $[41]$

$$D_\lambda(z) = \sqrt{\frac{2}{\pi}} e^{z^2/4} \int_0^{\infty} dt e^{-t^2/2} \cos(\lambda \pi/2 - zt) t^\lambda \quad (2.42)$$

to show that

$$\frac{\partial}{\partial \lambda} D_n^2(\sqrt{2}(\xi - \kappa)) \bigg|_{\lambda = n} = \frac{2^{-n/2+3/2}}{\sqrt{\pi}} H_n(\xi - \kappa)$$

$$\times \int_0^{\infty} dt e^{-t^2/2} t^n \left\{ \cos[n\pi/2 - \sqrt{2}(\xi - \kappa)t] \ln t - \frac{\pi}{2} \sin[n\pi/2 - \sqrt{2}(\xi - \kappa)t] \right\}. \quad (2.43)$$
The absolute value of the part between curly brackets is smaller than $|\ln t| + \pi/2$, which implies that

$$
\left| \int_0^\infty dt \, e^{-t^2/2} t^n \left\{ \cos[n\pi/2 - \sqrt{2}(\xi - \kappa)t] \ln t - \frac{\pi}{2} \sin[n\pi/2 - \sqrt{2}(\xi - \kappa)t] \right\} \right| \leq c'_n
$$

where $c'_n$ is independent of $\kappa$ and $\xi$. As a consequence we find that

$$
\left| \int_{k''}^\infty d\kappa \, e^{-\kappa^2} \kappa^{2n+1} \frac{A_n(\kappa)}{B_n(\kappa)} (\xi - \kappa) \frac{\partial}{\partial \lambda} D_n^2(\sqrt{2}(\xi - \kappa)) \right|_{\lambda=n}^{\lambda=n} 
\leq c''_n \int_{k''}^\infty d\kappa \, e^{-\kappa^2} \kappa^{2n+1} \left| (\xi - \kappa) H_n(\xi - \kappa) \right| 
$$

where $c''_n$ is independent of $\kappa$ and $\xi$ as well. The right-hand side decays faster than $\exp(-\xi^2/2)$ since we have taken $\kappa'' > \frac{1}{2}\sqrt{2}\xi$.

We now collect all remaining terms, evaluating the quotient $A_n(\kappa)/B_n(\kappa)$ as a single series, writing $D_n$ in terms of Hermite polynomials, and using

$$
\frac{\partial}{\partial \lambda} D_\lambda(z) \bigg|_{\lambda=n} \approx e^{-z^2/4} z^n \left[ A_n(z/\sqrt{2}) \ln z + \frac{\partial}{\partial \lambda} A_\lambda(z/\sqrt{2}) \right]_{\lambda=n}. 
$$

This asymptotic relation is valid for large and positive $\lambda$ and follows by differentiating (2.32) with respect to $\lambda$. The result for $I_n(\xi)$ as defined in (2.30) is

$$
I_n(\xi) \approx \frac{2^{n+1}}{\pi (n!)^2} [\sqrt{\nu - (n + 1/2)}]^{1/2} \int_{k''}^\infty d\kappa \, e^{-\kappa^2} \kappa^{2n+1} (\xi - \kappa)^2 \frac{A_n(\kappa)}{B_n(\kappa)} \frac{\partial}{\partial \lambda} D_n^2(\sqrt{2}(\xi - \kappa)) 
$$

with

$$
P_n(\kappa, \xi - \kappa) = -\left\{ \frac{1}{4[\nu - (n + 1/2)]} + \sum_{m=1}^n \frac{1}{m} - \gamma - \ln(2\kappa(\xi - \kappa)) \right\} K_n(\kappa, \xi - \kappa) + L_n(\kappa, \xi - \kappa) 
$$

containing the asymptotic series

$$
K_n(\kappa, \xi - \kappa) = 1 - \frac{1 + n + n^2}{2} \kappa^{-2} + \frac{n - n^2}{2} (\xi - \kappa)^{-2} - \frac{4 + 9n - n^4}{8} \kappa^{-4} - \frac{n - n^4}{4} (\xi - \kappa)^{-2} - \frac{3n - 6n^2 + 4n^3 - n^4}{8} \kappa^{-4} + \ldots 
$$

$$
L_n(\kappa, \xi - \kappa) = -\frac{1 + 2n}{4} \kappa^{-2} + \frac{1 - 2n}{4} (\xi - \kappa)^{-2} - \frac{9 - 4n^3}{16} \kappa^{-4} - \frac{1 - 4n^3}{8} \kappa^{-2} (\xi - \kappa)^{-2} - \frac{3 - 12n + 12n^2 - 4n^3}{16} (\xi - \kappa)^{-4} + \ldots 
$$
We can now integrate over \( \kappa \). In order to do so we expand the integrand around \( \kappa = \frac{\bar{\xi}}{2} \). If we choose \( \kappa' \) and \( \kappa'' \) symmetrically around \( \frac{\bar{\xi}}{2} \) we can make use of

\[
\int_{\frac{\bar{\xi}}{2}-\alpha}^{\frac{\bar{\xi}}{2}+\alpha} d\kappa \, e^{-2(\kappa - \frac{\bar{\xi}}{2})^2} (\kappa - \frac{\bar{\xi}}{2})^n \approx \frac{(n-1)!!}{2^n} \sqrt{\frac{\pi}{2}} + O(e^{-2\alpha^2} \alpha^{2n-1}) \tag{2.51}
\]

which is valid for \( n \) even. The same integral yields 0 for \( n \) odd. Note that if we choose \( \kappa' \) and \( \kappa'' \) as indicated before the \( O(e^{-2\alpha^2} \alpha^{2n-1}) \) can be discarded when \( \bar{\xi} \gg 1 \). Term by term integration then gives us the following asymptotic expansion for \( I_n(\bar{\xi}) \)

\[
I_n(\bar{\xi}) \approx \frac{2^{-2n-3/2}}{\sqrt{\pi(n!)^2}} [\nu - (n + \frac{1}{2})]^{1/2} e^{-\bar{\xi}^2/2 \xi_4^{2n+2} R_n(\bar{\xi})} \tag{2.52}
\]

where the series \( R_n(\bar{\xi}) \) is given by

\[
R_n(\bar{\xi}) = - \left\{ \frac{1}{4[\nu - (n + 1/2)]} + \sum_{m=1}^{n} \frac{1}{m} - \gamma - \ln(\bar{\xi}^2/2) \right\} M_n(\bar{\xi}) + N_n(\bar{\xi}) \tag{2.53}
\]

with the asymptotic series

\[
M_n(\bar{\xi}) = 1 - (3 + 2n + 4n^2)\bar{\xi}^{-2} - (12 + 21n - 10n^2 - 8n^4)\bar{\xi}^{-4} + \ldots \tag{2.54}
\]

\[
N_n(\bar{\xi}) = -(1 + 4n)\bar{\xi}^{-2} - \frac{21 - 20n - 32n^3}{2} \bar{\xi}^{-4} + \ldots \tag{2.55}
\]

This result is independent of the particular choice of \( \kappa' \) and \( \kappa'' \) as it should be.

Finally, substitution of (2.52) in (2.21) yields the asymptotic expansion for the current density that we set out to establish. It has the form

\[
j_{y,\mu}(x) \approx -\frac{B^2}{2\pi^{5/2}} \sum_{n} \frac{2^{-2n}}{(n!)^2} [\nu - (n + \frac{1}{2})]^{1/2} e^{-\bar{\xi}^2/2 \xi_4^{2n+2} R_n(\bar{\xi})} \tag{2.56}
\]

with the asymptotic series \( R_n(\bar{\xi}) \) as given in (2.53).

The asymptotic behaviour of the charge density can be determined in a similar fashion. Instead of (2.30) we now have the integral

\[
I_n'(\bar{\xi}) = \int_{\kappa_n(\nu)}^{\infty} d\kappa [\nu - \epsilon_n(\kappa)]^{1/2} \times \left[ \int_{0}^{\infty} d\xi' D^2_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi' - \kappa)) \right]^{-1} D^2_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi - \kappa)). \tag{2.57}
\]
2.3. Asymptotic expansions

Contrary to the current, the charge density has a finite bulk value, so it is appropriate to subtract the bulk density

$$I_n'(\infty) = \int_{-\infty}^{\infty} d\kappa [\nu - (n + 1/2)]^{1/2} \frac{1}{\sqrt{\pi n!}} D_n^2(-\sqrt{2}\kappa)$$  \hspace{1cm} (2.58)

from $I_n'(\xi)$. Splitting the integral in the same way as before we arrive at:

$$I_n'(\xi) - I_n'(\infty) \approx \left[ -\int_{-\infty}^{\kappa'} d\kappa \frac{1}{2} [\nu - (n + 1/2)]^{1/2} \frac{1}{\pi(n!)^2} 2^n e^{-\kappa^2} \kappa^{2n+1} \frac{A_n(\kappa)}{B_n(\kappa)} D_n^2(\sqrt{2}(\xi - \kappa)) \right]$$

$$- \left[ \int_{\kappa'}^{\infty} d\kappa [\nu - (n + 1/2)]^{1/2} \frac{1}{\pi(n!)^2} 2^n e^{-\kappa^2} \kappa^{2n+1} C_n(\kappa) D_n^2(\sqrt{2}(\xi - \kappa)) \right]$$

$$+ \left[ \int_{-\infty}^{\infty} d\kappa [\nu - (n + 1/2)]^{1/2} \frac{1}{\sqrt{\pi n!}} \frac{D_n^2(\sqrt{2}(\xi - \kappa))}{\lambda = n} \right]$$

$$+ \text{h.o.t.} \hspace{1cm} (2.59)$$

The last term decays faster than $\exp(-\xi^2/2)$. This can be shown by expressing $D_n$ in terms of the Hermite polynomial $H_n$

$$\int_{-\infty}^{\kappa'} d\kappa D_n^2(\sqrt{2}(\xi - \kappa)) = 2^{-n} \int_{-\infty}^{\kappa'} d\kappa e^{-(\xi - \kappa)^2} H_n^2(\xi - \kappa) \hspace{1cm} (2.60)$$

and using that $\kappa'$ is less than $(1 - \frac{1}{2}\sqrt{2})\xi$.

The remaining terms in (2.59) can be handled in a similar way as we did for the current density (see (2.39)). The only difference is the absence of the factor $(\xi - \kappa)$. One finds

$$I_n'(\xi) - I_n'(\infty) \approx \frac{2^{2n-1/2}}{\sqrt{\pi(n!)^2}} [\nu - (n + 1/2)]^{1/2} e^{-\xi^2/2} \xi^{4n+1} R_n'(\xi)$$  \hspace{1cm} (2.61)

where we introduced the abbreviation

$$R_n'(\xi) = -\left\{ \frac{1}{4[\nu - (n + 1/2)]} + \sum_{m=1}^{n} \frac{1}{m} - \gamma - \ln(\xi^2/2) \right\} M_n'(\xi) + N_n'(\xi) \hspace{1cm} (2.62)$$

with the asymptotic series

$$M_n'(\xi) = 1 - (2 + 2n + 4n^2)\xi^{-2} - (10 + 19n - 6n^2 - 8n^4)\xi^{-4} + \ldots \hspace{1cm} (2.63)$$

$$N_n'(\xi) = -(1 + 4n)\xi^{-2} - \frac{19 - 12n - 32n^3}{2} \xi^{-4} + \ldots \hspace{1cm} (2.64)$$
Substitution of (2.61) in (2.20) gives us

$$\rho_n(x) - \rho_n(\infty) \approx \frac{B^{3/2}}{\pi^{5/2}} \sum_n \frac{2^{-2n}}{(n!)^2} [\nu - (n + \frac{1}{2})]^{1/2} e^{-\xi^2/2} \xi^{4n+1} R_n' (\xi).$$  \hspace{1cm} (2.65)

### 2.4 Discussion

To check the validity of our asymptotic expansions we have compare them with numerical results for the charge and the current density. In figure 2.3 we have plotted $I_0(x)$ for $\nu = 1.0$. For this value of $\nu$ there is only one (partially) filled Landau level, so $I_0(x)$ represents the complete current density. Because of its fast decay the pre-factor $\exp(-\xi^2/2)\xi^{4n+2}$ has been divided out. The solid line corresponds to the numerical results, the dotted line to the asymptotic expansion (2.52). As can be seen, the convergence is quite good.

As (2.56) and (2.65) show, the contribution of each Landau level $n$ to both the current density and the charge density has a Gaussian decay for large $x$ (in leading order
2.4. Discussion

proportional to \( \exp(-\xi^2/2)\xi^{4n+2}\ln(\xi^2/2) \) and \( \exp(-\xi^2/2)\xi^{4n+1}\ln(\xi^2/2) \), respectively. Hence, when only a limited number of Landau levels is filled, in other words for every finite magnetic field, both densities decay with a tail proportional to a Gaussian.

The decay found here is consistent with the bound derived by Macris, Martin and Pulé [40]. However, it disagrees with the results of Ohtaka & Moriya [42] and of Jancovici [27]. In the latter paper the current density at \( T = 0 \) is given as

\[
j_{\nu,\mu}(x) = \frac{\mu B}{16\pi^2} \left\{ 8\mu x^2 \left[ \frac{\pi}{2} - \text{Si}(2^{3/2}\mu^{1/2}x) \right] + \left( \frac{3}{4\mu^2} - 1 \right) \sin(2^{3/2}\mu^{1/2}x) \right. \\
\left. - \left( 2^{1/2}\mu^{1/2}x + 2^{3/2}\mu^{1/2}x \right) \cos(2^{3/2}\mu^{1/2}x) \right\} \quad (2.66)
\]

with \( \text{Si}(z) \) the sine integral. The right-hand side decays algebraically, with a tail proportional to \( x^{-1} \) for large \( x \). It is obtained via an inverse Laplace transform of the current density \( j_{\nu,\beta}(x) \) for a magnetised free-electron gas with Maxwell-Boltzmann statistics as explained in section 1.2.3. The Maxwell-Boltzmann form of the current density employed in [27] is obtained by a linear-response method valid for small magnetic field. In fact, the dimensionless parameter that has to be small is \( \beta B \). The integration in the inverse Laplace transform is taken over all values of \( \beta \), and thus in particular over all values of \( \beta B \). Hence, it is not justified \textit{a priori} to insert the linear-response expression for \( j_{\nu,\beta}(x) \) and to carry out the integration subsequently. As a consequence, the expression \( (2.66) \), and the ensuing algebraic decay is not guaranteed to be correct. As has been remarked already in the introductory section, the procedure of taking inverse Laplace transforms of Maxwell-Boltzmann expressions for small fields may even lead to weird effects like undamped oscillations, if it is applied to other physical quantities. Questions about the validity of \( (2.66) \) in the limit \( x \to \infty \) have been raised before by Shishido [46], who argues that the expression is not uniformly convergent, and is valid only for small \( x \) (and small \( B \)).

It should be noted here that our asymptotic expansions \((2.56)\) and \((2.65)\) are rather awkward when it comes to studying the limit \( B \to 0 \). In that limit the number of filled Landau levels goes to infinity. The coefficients in the expansion rapidly grow with the label \( n \) of the Landau level, as is clear from \((2.54)\), \((2.55)\), \((2.63)\) and \((2.64)\). Hence, the asymptotic region moves further and further away from the wall, as \( B \) goes to 0. This weak-field limit will be investigated in chapter 4.

Our approach to determine the asymptotic behaviour of profiles for finite magnetic fields can easily be generalised to other physical quantities, for instance the kinetic pressure. In general, the leading term is proportional to \( \exp(-\xi^2/2)\xi^m\ln(\xi^2/2) \), where \( m \)
increases with the number of filled Landau levels and with the number of particle momenta occurring as factors in the expression for the physical quantity being calculated.

2.A Appendix: Asymptotics of $\epsilon_n(\kappa)$

In section 2.2 we introduced the function $\epsilon_n(\kappa)$, which defines the eigenvalues of the Fourier-transformed Hamiltonian (2.16). It is defined by

$$
D_{\epsilon_n(\kappa)-1/2}(-\sqrt{2}\kappa) = 0. \quad (2.67)
$$

The asymptotic expansion of $D_\lambda(-\sqrt{2}\kappa)$ for large and positive $\kappa$ is given by [41]

$$
D_\lambda(-\sqrt{2}\kappa) \approx e^{im\lambda}e^{-\kappa^2/2(\sqrt{2}\kappa)}A_\lambda(\kappa) + \frac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{\kappa^2/2(\sqrt{2}\kappa)}^{-\lambda-1}B_\lambda(\kappa) \quad (2.68)
$$

with $A_\lambda$ as defined in (2.33) and with $B_\lambda$ given by

$$
B_\lambda(\kappa) = \sum_{m=0}^{\infty} \frac{((1+\lambda)/2)_m((2+\lambda)/2)_m(\kappa^2)^{-m}}{m!}. \quad (2.69)
$$

Setting (2.68) to zero and expanding around $\lambda = n$ we arrive at

$$
[e_n(\kappa) - (n + \frac{1}{2})] \approx \frac{1}{\sqrt{\pi n!}}2^n e^{-\kappa^2} \kappa^{2n+1} \frac{A_n(\kappa)}{B_n(\kappa)} \quad (2.70)
$$

for large positive $\kappa$. This is a generalisation of the expression given by Kunz [35].

2.B Appendix: Asymptotics of the normalisation factor

In section 2.3 we needed the asymptotic expansion of the normalisation factor

$$
\left[\int_0^\infty d\xi' D^2_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi' - \kappa))\right]^{-1} \quad (2.71)
$$

for large $\kappa$. In the previous appendix we have seen that for large $\kappa$ the function $[e_n(\kappa) - (n + \frac{1}{2})]$ is small. Therefore we can write

$$
\int_0^\infty d\xi' D^2_{\epsilon_n(\kappa)-1/2}(\sqrt{2}(\xi' - \kappa)) = \int_0^\infty d\xi' D^2_n(\sqrt{2}(\xi' - \kappa))
$$

$$
+ \frac{\partial}{\partial \lambda} \int_0^\infty d\xi' D^2_\lambda(\sqrt{2}(\xi' - \kappa)) \bigg|_{\lambda=n} [e_n(\kappa) - (n + \frac{1}{2})]
$$

$$
+ \frac{\partial^2}{\partial \lambda^2} \int_0^\infty d\xi' D^2_\lambda(\sqrt{2}(\xi' - \kappa)) \bigg|_{\lambda=n} \frac{1}{2} [e_n(\kappa) - (n + \frac{1}{2})]^2 + \text{h.o.t.} \quad (2.72)
$$
The first term in this expansion is given by

\[
\int_0^\infty d\xi' D_n^2(\sqrt{2}(\xi' - \kappa)) = \int_0^\infty d\xi' D_n^2(\sqrt{2}\xi') - \int_{-\infty}^{\kappa} d\xi' D_n^2(\sqrt{2}\xi')
\approx \sqrt{\pi} m! - 2^{n-1} e^{-\kappa^2} \kappa^{2n+1} \left(\kappa^{-2} - \frac{1 - 3n + n^2}{2} \kappa^{-4} + \ldots\right)
\] (2.73)

as can be derived by expressing \( D_n \) in terms of the Hermite polynomial \( H_n \), followed by term by term integration of the resulting series.

With the help of the integral representation (2.42) of \( D_\lambda \) the coefficient of the second term in (2.72) becomes

\[
\frac{\partial}{\partial \lambda} \int_0^\infty d\xi' D_\lambda^2(\sqrt{2}(\xi' - \kappa)) \bigg|_{\lambda = n} = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_{-\kappa}^{\infty} ds e^{s^2/2} D_n(\sqrt{2}s)
\times \int_0^\infty d\xi' e^{-\xi'^2/2} \xi^{m-1} \cos(n\pi/2 - \sqrt{2}s\xi') \ln \xi' - \frac{\pi}{2} \sin(n\pi/2 - \sqrt{2}s\xi')
\] (2.74)

Repeated partial integration yields

\[
\frac{\partial}{\partial \lambda} \int_0^\infty d\xi' D_\lambda^2(\sqrt{2}(\xi' - \kappa)) \bigg|_{\lambda = n} = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{n} (-1)^m 2^{-n/2+m/2+1} \frac{n!}{(n-m)!} H_{n-m}(s)
\times \int_0^\infty d\xi' e^{-\xi'^2/2} \xi^{n-1} e^{-i\kappa\xi'} \left[\cos((n + m + 1)\pi/2 - \sqrt{2}s\xi') \ln \xi' - \frac{\pi}{2} \sin((n + m + 1)\pi/2 - \sqrt{2}s\xi')\right]_{s=-\kappa}.
\] (2.75)

For all \( m < n \) we write the contribution at \( s = -\kappa \) of the sine term as

\[
-\frac{\pi}{2} \text{Im} \left[ i^{n+m+1} \int_0^\infty d\xi' e^{-\xi'^2/2} \xi^{n-m-1} e^{i\sqrt{2}\kappa\xi'} \right].
\] (2.76)

We now use a theorem [17] stating that for large \( x \) the Fourier integral

\[
\int_{-\alpha}^\beta dt \phi(t)e^{ixt}
\] (2.77)

has an asymptotic expansion to which the end point \( \alpha \) contributes as

\[
A = \sum_{n=0}^\infty i^n \frac{d^n \phi(\alpha)}{d\alpha^n} \chi^{-n-1} e^{ix\alpha}
\] (2.78)
if \( \phi(t) \) has no singularity in \([\alpha, \beta]\). With the help of this theorem we can show that

\[
-\frac{\pi}{2} \int_0^\infty d\xi' e^{-\xi'^{1/2}/2} \xi'^{m-1} \sin[(n+m+1)\pi/2 + \sqrt{2}\kappa \xi']
\approx -\frac{\pi}{2} (-1)^n (\sqrt{2}\kappa)^{-n+m} \sum_{l=0}^\infty \frac{(2l+n-m-1)!}{2^{2l} l!} \kappa^{-2l}.
\]  

(2.79)

The contribution of the cosine term at \( s = -\kappa \) in (2.75) can be written as

\[
\text{Re} \left[ \int_{0}^{\infty} d\xi' e^{-\xi^{1/2}/2} \xi^{m-1} e^{i\sqrt{2}\kappa \xi'} \ln \xi' \right].
\]

(2.80)

Because of the logarithm we need a generalisation of the previous theorem to Fourier integrals with logarithmic singularities. This generalisation can also be found in [17]. It states that for \( \phi(t) = \phi_1(t) \ln(t - \alpha) \) the asymptotic expansion of (2.77) contains a contribution from the lower end point which reads

\[
A = \sum_{n=0}^\infty i^{n+1} \frac{d^n \phi_1(\alpha)}{d\alpha^n} \left[ \psi(n+1) - \ln x + \frac{i\pi}{2} \right] x^{-n-1} e^{ix\alpha}.
\]

(2.81)

With the help of this theorem we see that the contribution from \( s = -\kappa \) of the cosine term is identical to the contribution of the sine term. Using the same method, we find that for \( s \to \infty \) the two terms in (2.75) cancel, at least for \( m < n \).

The contributions for \( m = n \) can be calculated in a similar fashion, although they need some extra attention because of the additional \( \xi'^{-1} \) singularity. They add up to

\[
\int_0^\infty d\xi' e^{-\xi^{1/2}/2} \left\{ \cos[(2n+1)\pi/2 - \sqrt{2}s \xi'] \ln \xi' - \frac{\pi}{2} \sin[(2n+1)\pi/2 - \sqrt{2}s \xi'] \right\}
\approx \pi(-1)^n \left[ -\gamma - \ln(\sqrt{2}\kappa) + \sum_{l=1}^\infty \frac{(2l-1)!}{2^{2l} l!} \kappa^{-2l} \right]
\]

(2.82)

where \( \gamma \) is Euler's constant. Collecting all these terms we get

\[
\frac{d}{d\lambda} \int_0^\infty d\xi' D_\lambda^2(\sqrt{2}(\xi' - \kappa)) \left|_{\lambda=n} \right. \approx 2\sqrt{\pi} n! \left[ \sum_{m=1}^n \frac{1}{m} - \gamma - \ln(\sqrt{2}\kappa)
+ \frac{1+2n}{4} \kappa^{-2} + \frac{3+6n+6n^2}{16} \kappa^{-4} + \ldots \right].
\]

(2.83)
Finally, we have for the third term in (2.72)
\[
\frac{1}{2} \frac{\partial^2}{\partial \lambda^2} \int_0^\infty d\xi' D^n_\lambda(\sqrt{2}(\xi' - \kappa)) \left|_{\lambda=n} \right. = \left. \int_0^\infty d\xi' D^n_\lambda(\sqrt{2}(\xi' - \kappa)) \frac{\partial^2}{\partial \lambda^2} D^n_\lambda(\sqrt{2}(\xi' - \kappa)) \right|_{\lambda=n} \\
+ \int_0^\infty d\xi' \left[ \frac{\partial}{\partial \lambda} D^n_\lambda(\sqrt{2}(\xi' - \kappa)) \right]^2. \tag{2.84}
\]

An asymptotic expansion of the first integral can be derived along the same route as above. The only new ingredient is a straightforward extension of the theorem by Erdélyi to Fourier integrals with squared logarithmic singularities. It states that if in (2.77) one takes \( \phi(t) = \phi_2(t) \ln^2(t - \alpha) \), the contribution of the lower boundary is given by the asymptotic expansion
\[
A = \sum_{n=0}^\infty i^{n+1} \frac{d^n\phi_2(\alpha)}{d\alpha^n} \left\{ [\psi(n+1) - \ln x]^2 + \zeta(2,n+1) \\
+ i\pi[\psi(n+1) - \ln x] - \frac{\pi^2}{4} \right\} x^{-n-1} e^{i\alpha}. \tag{2.85}
\]

While it is not very difficult to derive this extension of Erdélyi's theorem, a slightly different way to calculate the integral can be found in [50].

As a consequence of the results mentioned above, the asymptotics of the first integral in (2.84) is found to be of order \( \ln^2(\sqrt{2}\kappa) \). This implies that for large \( \kappa \) the first integral is negligible with respect to the second, as we shall see.

The asymptotic behaviour of the second integral in (2.84) follows by noting that for large \( \kappa \) the dominant contribution comes from the lower end of the integration interval. With the help of (2.42), (2.78) and (2.81) we can derive the following asymptotic expansion for the integrand
\[
\left. \frac{\partial}{\partial \lambda} D^n_\lambda(z) \right|_{\lambda=n} \approx \sqrt{2\pi n!} e^{z^2/4} z^{-n-1} B_n(z/\sqrt{2}) \tag{2.86}
\]
for large and negative \( z \). Term by term integration leads to
\[
\frac{1}{2} \frac{\partial^2}{\partial \lambda^2} \int_0^\infty d\xi' D^n_\lambda(\sqrt{2}(\xi' - \kappa)) \left|_{\lambda=n} \right. \\
\approx \pi(n!)^2 2^{-n-1} e^{\kappa^2} \kappa^{-2n-1} \left( \kappa^{-2} + \frac{5 + 5n + n^2}{2} \kappa^{-4} + \ldots \right). \tag{2.87}
\]
The right-hand side grows exponentially as \( \kappa \to \infty \), but this is compensated by the factor \( \exp(-2\kappa^2) \) in \( [\epsilon_n(\kappa) - (n + \frac{1}{2})]^2 \), resulting in an overall \( \exp(-\kappa^2) \) behaviour.
Chapter 2. Density profiles

That is the reason why we had to expand (2.72) up to second order in \([\varepsilon_n(\kappa) - (n + \frac{1}{2})]\). Higher-order derivatives of \(\int_0^\infty d\xi' D_n^2(\sqrt{2}(\xi' - \kappa))\) are also of order \(\exp(\kappa^2)\) or less, so that we do not have to go beyond second order.

Substitution of (2.70), (2.73), (2.83) and (2.87) in (2.72) gives

\[
\left[ \int_0^\infty d\xi' D_n^2(\varepsilon_n(\kappa) - 1/2(\sqrt{2}(\xi' - \kappa))) \right]^{-1} \approx \frac{1}{\sqrt{\pi n!}} - \frac{1}{\pi(n!)^2} e^{-\kappa^2} \kappa^{2n+1} C_n(\kappa) \tag{2.88}
\]

with the asymptotic series

\[
C_n(\kappa) = \sum_{m=1}^n \frac{1}{m} - \gamma - \ln(\sqrt{2}\kappa) \left( 1 - \frac{1 + n + n^2}{2} \kappa^{-2} - \frac{4 + 9n - n^4}{8} \kappa^{-4} + \ldots \right) + \left( \frac{1 + 2n}{4} \kappa^{-2} + \frac{9 - 4n^3}{16} \kappa^{-4} + \ldots \right). \tag{2.89}
\]