On the inhomogeneous magnetised electron gas

Kettenis, M.M.

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Chapter 4

Weak magnetic fields

In the previous two chapters, we have investigated the density profiles for a magnetised electron gas in the presence of a hard wall. We have seen that for the completely degenerate case, i.e. at zero temperature, the ultimate decay of the profiles towards their bulk values is Gaussian with algebraic and logarithmic pre-factors. At the end of chapter 3 we concluded that because of these pre-factors, this ultimate Gaussian decay shifts further away from the wall, towards the bulk, when the number of Landau levels increases. Since the number of Landau levels becomes large for weak magnetic fields, it would be interesting to know how the profiles behave before the ultimate Gaussian decay sets in.

In terms of the variables $\xi = \sqrt{B} x$ and $\nu = \mu/B$ for the (dimensionless) distance from the wall in magnetic lengths and dimensionless chemical potential, we have established that the Gaussian decay sets in for $\xi^2 \gg \nu$. However, in the case where $\xi^2 \approx \nu$, with $\nu, \xi$ large, we can still use the multiple-reflection expansion that we used in the previous chapter. Incidentally $\nu \gg 1$ corresponds precisely to the weak field limit that attracted our attention in the first place.

In order to investigate the situation where $\xi^2 \approx \nu$ (for large $\nu$), we will first look at the case where the magnetic field is perpendicular to the wall instead of parallel to it. This simplifies matters considerably, and it may be helpful in tackling the more difficult case where the magnetic field is parallel to the wall.

4.1 Magnetic field perpendicular to the wall

The situation with the magnetic field perpendicular to the wall has been investigated before by Horing and Yildiz [24, 25]. However, as we will see below, the analysis in
the first part of this chapter gives access to a slightly different regime, and provides us with more insight under which circumstances the approximations made in [25] are applicable.

We keep the magnetic field in the $z$-direction, which means that the wall is now in the $x-y$-plane, and the $\rho(x)$ of chapter 3 gets replaced by $\rho(z)$, and the (charge) density profile is given by

$$\rho_\beta(z) = \frac{\rho}{Z_{\parallel}} G_{\parallel,\beta}(z, z)$$

(4.1)

where $\rho$ is the bulk density. Calculation of this density profile for the non-degenerate case is almost trivial now since the magnetic field only couples directly to the $x$- and $y$-directions. The bulk Green function for the $z$-direction is simply

$$G_{\parallel,\beta}(z, z') = \frac{1}{\sqrt{2\pi\beta}} \exp \left[ -\frac{(z - z')^2}{2\beta} \right]$$

(4.2)

so if we use the reflection principle (see (3.10) in section 3.1) we see that

$$\delta\rho_\beta(z) = -\rho e^{-2z^2/\beta}.$$  

(4.3)

For the completely degenerate case we employ the inverse Laplace transform (3.46). Since

$$Z = \left( \frac{1}{2\pi\beta} \right)^{3/2} \frac{\beta B}{2 \sinh(\beta B/2)}$$

(4.4)

this means that we have to evaluate the integral

$$\delta\rho_\mu(z) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \frac{\beta e^{\beta \mu - 2z^2/\beta}}{2^{3/2} \pi^{3/2} \beta^{3/2} \sinh(\beta B/2)}.$$

(4.5)

Here the integration is in principle over a straight line $\Re \beta = c$ (with $c > 0$) parallel to the imaginary axis. However, we can always deform the contour, as long we do not pull it through any singularities.

In contrast to the case where the magnetic field is parallel to the wall, we can derive an exact expression (exact in the sense that it does not contain any integrals) for the density profile in the completely degenerate case. Using

$$\frac{1}{\sinh(\beta B/2)} = 2 \sum_{n=0}^{\infty} e^{-(n+1/2)\beta B}$$

(4.6)
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in (4.5) gives us

\[ \delta \rho_{\mu}(z) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dB \sum_{n=0}^{\infty} e^{\mu-(n+1/2)\beta} -2z^2/\beta \frac{2^{1/2} \pi^{3/2} \beta^{3/2}}{2^{1/2} \pi^{3/2} \beta^{3/2}}. \]  

(4.7)

If we set \( s = \beta B \) and use the dimensionless variables \( \nu = \mu/B \) and \( \zeta = Bz/\sqrt{\mu} \) instead of \( \mu \) and \( z \) we get

\[ \delta \rho_{\mu}(z) = -\frac{B^{3/2}}{\sqrt{2\pi}^{3/2}} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ e^{s[v-(n+1/2)]} s^{-3/2} e^{-2\nu \zeta^2/s}. \]  

(4.8)

The inverse Laplace transform in (4.8) is given by\(^1\)

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ e^{st} s^{-3/2} e^{-a/s} = \frac{1}{\sqrt{\pi a}} \sin(2\sqrt{at}) \theta(t) \]  

(4.9)

for \( a > 0 \) and \( c > 0 \). Use of this identity in (4.8) results in the following formula for the exact density profile:

\[ \delta \rho_{\mu}(z) = -\frac{B^{3/2}}{2\pi^2 \gamma^{1/2} \xi} \sum_{n} \sin(2\sqrt{2\nu \sqrt{\nu - (n + 1/2) \xi}) \]  

(4.10)

where the prime indicates that the summation is over all \( n \) with \( n < \nu - 1/2 \). This expression has the form of a sum over Landau levels. Therefore it is not very practical if the number of Landau levels becomes very large, i.e. for large \( \nu \). Even numerical evaluation becomes difficult since there can be quite a bit of cancellation between terms in the sum, which leads to loss of precision.

4.1.1 Density profiles for large \( \nu \)

In order to derive an expression for the density profile that behaves better for \( \nu \gg 1 \), we go back to (4.5). Taking \( \beta = it/B \), we have

\[ \delta \rho_{\mu}(z) = -\frac{B^{3/2}}{2^{5/2} \pi^{5/2}} \int_{C} dt \frac{\exp[i\nu(t + 2\xi^2/t)]}{(it)^{3/2} \sinh(it/2)} \]  

(4.11)

with \( C \) a suitable contour. The integrand has singularities at \( t = 2\pi n \) for all integers \( n \) (simple poles for \( n \neq 0 \), \( n = 0 \) is the endpoint of a branch cut). This allows us to choose a contour as pictured in figure 4.1. Note that the contribution from \( t < 0 \) is the

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1. [19], p. 245.
complex conjugate of the contribution from \( t > 0 \). Therefore we can focus on \( t > 0 \) and easily recover the full result afterwards.

To investigate the behaviour for large \( \nu \), we have to look closer at the points where the phase \( f(t) = t + 2\zeta^2/t \) is stationary, that is, where \( f'(t) = 0 \), thus \( t = \sqrt{2}\zeta \). In addition, it is useful to know that \( f'(t) < 0 \) when \( t < \sqrt{2}\zeta \) and \( f'(t) > 0 \) when \( t > \sqrt{2}\zeta \). The contribution arising from the point \( t = \sqrt{2}\zeta \) to the integral can be calculated using the method of steepest descent.

In order to apply this method, we have to find a path through \( t = \sqrt{2}\zeta \) where the real part of \( f(t) \) is constant. Setting \( \text{Re} \left[ f(t) - f(\sqrt{2}\zeta) \right] = 0 \) we see that for \( t \) close to \( \sqrt{2}\zeta \), we get \( \text{Re} \, t = \pm \text{Im}(t) + \sqrt{2}\zeta \). The minus sign corresponds to a path of steepest ascent, the plus sign to a path of steepest descent. Since we are looking for steepest descent we have to choose the plus sign, which is convenient, since that means that the path stays away from the cut along the positive imaginary axis (see figure 4.2). In the neighbourhood of \( t = \sqrt{2}\zeta \), we can now parametrise the path as \( t = \sqrt{2}\zeta + e^{i\pi/4}s \).

This means that the contribution arising from the point at \( t = \sqrt{2}\zeta \) is given by the asymptotic expansion of the integral

\[
-\frac{B^{3/2}}{2^{5/2}\pi^{5/2}} e^{i\pi/4} e^{i2\sqrt{2}\nu\zeta} \int_{-\infty}^{\infty} ds \frac{\exp(-\frac{\nu}{\sqrt{2}\zeta} s^2)}{(i\sqrt{2}\zeta)^{3/2} i \sin(\zeta/\sqrt{2} + e^{i\pi/4}s/2)}.
\]

(4.12)

It is necessary to keep the \( s \)-dependence in the argument of the sine in the denominator, even if \( s \) is small. The reason is that when \( \zeta/\sqrt{2} \) is close to a multiple of \( \pi \), small changes in \( s \) have a large impact on the overall value of the integrand. The \( s \)-dependence in the factor \((it)^{3/2}\) in the denominator on the other hand can be dropped immediately, since
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s will be small in comparison to $\sqrt{2}\xi$. It will prove to be convenient to write (4.12) as

$$
\frac{B^{3/2}}{2^{13/4}\pi^{5/2}} \frac{e^{2iv\sqrt{2}\xi}}{\xi^{3/2}} \int_{-\infty}^{\infty} ds \frac{\exp(-\frac{\gamma}{\sqrt{2}\xi}s^2)}{\sin(\xi/\sqrt{2})\cos(e^{i\pi/4}s/2) + \cos(\xi/\sqrt{2})\sin(e^{i\pi/4}s/2)}.
$$

(4.13)

Since $\gamma \gg 1$, the exponential in the numerator of the integrand suppresses contributions from s outside a small region around zero. In other words, we can indeed assume s to be small and write

$$
\int_{-\infty}^{\infty} ds \frac{\exp(-\frac{\gamma}{\sqrt{2}\xi}s^2)}{\sin(\xi/\sqrt{2}) + \cos(\xi/\sqrt{2})e^{i\pi/4}s/2} = \frac{4i\sin(\xi/\sqrt{2})}{\cos^2(\xi/\sqrt{2})} \int_{-\infty}^{\infty} ds \frac{\exp(-\frac{\gamma}{\sqrt{2}\xi}s^2)}{s^2 + 4i\tan^2(\xi/\sqrt{2})}
$$

(4.14)

for the integral in (4.13). With the help of the relation

$$
\int_0^\infty ds \frac{e^{-\alpha^2s^2}}{s^2 + b^2} = \text{Erfc}(ab) \frac{\pi}{2b} e^{ab^2} \quad |\arg a| < \frac{\pi}{4}, \quad \text{Re } b > 0
$$

(4.15)

for

$$
b = 2e^{i\pi/4} \left|\tan(\xi/\sqrt{2})\right|
$$

(4.16)

we can evaluate this integral. The result is

$$
2\pi e^{i\pi/4} \left[\frac{\tan(\xi/\sqrt{2})}{\sin(\xi/\sqrt{2})}\right] e^{23/4iv\tan^2(\xi/\sqrt{2})/\xi} \text{Erfc} \left(\frac{23/4v^{1/2}}{\xi^{1/2}}e^{i\pi/4} \left|\tan(\xi/\sqrt{2})\right|\right).
$$

(4.17)

Because of the factor $\exp(i\pi/4)$, the argument of the error function is complex. Expressing the error function in terms of the Fresnel integrals $S(x)$ and $C(x)$ [41]

$$
\text{Erf}(e^{i\pi/4}x) = \sqrt{2}e^{i\pi/4} \left[C \left(\sqrt{\frac{2}{\pi}}x\right) - iS \left(\sqrt{\frac{2}{\pi}}x\right)\right]
$$

(4.18)

allows us to write (4.17) in terms of functions with a real argument. The Fresnel integrals $S(x)$ and $C(x)$ are defined by

$$
S(x) = \int_0^x dt \sin \left(\frac{\pi}{2}t^2\right) \quad C(x) = \int_0^x dt \cos \left(\frac{\pi}{2}t^2\right).
$$

(4.19)

The arguments of these are still rather complicated. Therefore we introduce the shorthand

$$
X = \frac{25/4\gamma^{1/2}}{\pi^{1/2}\xi^{1/2}} \left|\tan(\xi/\sqrt{2})\right|.
$$

(4.20)

2. Eq. 3.466.1 in [23].
This gives the following result for the contribution of both stationary points to the density profile

$$\frac{B^{3/2}X}{4\pi v^{1/2} \zeta \sin(\zeta/2)} \left\{ \left[ C(X) - \frac{1}{2} \right] \sin \left( 2\sqrt{2}v \zeta + \frac{\pi}{2} X^2 \right) - \left[ S(X) - \frac{1}{2} \right] \cos \left( 2\sqrt{2}v \zeta + \frac{\pi}{2} X^2 \right) \right\}.$$ (4.21)

However, as we will see now, this is not the complete result.

In order to follow the path of steepest descent, we had to deform C in \((4.11)\) towards a contour as depicted in figure 4.2. In doing so, we have pulled it through the poles at \(\pm 2n\pi\) for all \(n > \frac{\zeta}{\sqrt{2}\pi}\). Therefore we will have to add the residue in those poles to \((4.21)\) in order to find the correct asymptotic expression. The sum of the relevant residues is given by

$$-\frac{B^{3/2}}{2\pi^3} \sum_{n=n_{\text{min}}}^{\infty} (-1)^n n^{3/2} \cos \left( 2\pi n v + \frac{\pi\zeta^2}{\pi n} - \frac{3}{4}\pi \right) \quad \text{(4.22)}$$

where \(n_{\text{min}}\) is the smallest integer larger than \(\frac{\zeta}{\sqrt{2}\pi}\).

Hence, including the contribution from the poles, the density profile is

$$\delta \rho_{\mu}(z) \approx \frac{B^{3/2}X}{4\pi v^{1/2} \zeta \sin(\zeta/\sqrt{2})} \left\{ \left[ C(X) - \frac{1}{2} \right] \sin \left( 2\sqrt{2}v \zeta + \frac{\pi}{2} X^2 \right) - \left[ S(X) - \frac{1}{2} \right] \cos \left( 2\sqrt{2}v \zeta + \frac{\pi}{2} X^2 \right) \right\}.$$
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\[- \left[ S(X) - \frac{1}{2} \right] \cos \left( 2\sqrt{2} \nu \xi + \frac{\pi}{2} X^2 \right) \]

\[- \frac{B^{3/2}}{2\pi^3} \sum_{n=n_{\text{min}}}^{\infty} \frac{(-1)^n \cos \left( 2\pi \nu n + \frac{\nu^2 m^2}{\pi^3} - \frac{3}{4} \pi \right)}{n^{3/2}} \]

(4.23)

for \( \nu \gg 1 \) and \( \xi \) not too close to zero. The dependence of the sum over \( n \) on \( n_{\text{min}} \) seems to suggest that there is a jump in the density when \( \xi \) goes through \( \sqrt{2} \pi m \). This would be rather unphysical, and therefore we can expect a cancellation of this jump by other terms in this expression. We will see below that this is indeed the case.

Of course the Fresnel integrals \( S(x) \) and \( C(x) \) are still special functions. It is therefore interesting to see if there are special cases in which (4.23) can be simplified. This is indeed possible if \( X \gg 1 \) or if \( X \ll 1 \).

The strong inequality \( X \gg 1 \) holds for \( \nu \gg 1 \) and finite \( \xi \) if \( |\tan(\xi/\sqrt{2})| \) is not close to zero, which is true if \( \xi \) is not close to \( \sqrt{2} \pi m \) (for \( m \) integer). In other words, the following approximation is valid if \( \xi \) is chosen such that the stationary point is not close to the poles. In that case

\[ S(X) - \frac{1}{2} \approx -\frac{1}{\pi X} \cos \left( \frac{\pi}{2} X^2 \right) \quad C(X) - \frac{1}{2} \approx \frac{1}{\pi X} \sin \left( \frac{\pi}{2} X^2 \right) \]

(4.24)

which leads to

\[ \frac{B^{3/2}}{4\pi^2 \nu^{1/2} \xi \sin(\xi/\sqrt{2})} \cos(2\sqrt{2} \nu \xi) \]

(4.25)

for the term containing the Fresnel integrals in (4.23). We would have obtained the same result by neglecting the \( s \)-dependence in the sine in the denominator of the integrand in (4.12).

Whereas \( X \gg 1 \) corresponded to \( \xi \) away from \( \sqrt{2} \pi m \), the case \( X \ll 1 \) corresponds to \( \xi \) close to \( \sqrt{2} \pi m \). So this case gives us information about the situation where the stationary point and a pole coalesce. We can now approximate the Fresnel integrals by

\[ S(X) \approx \frac{\pi}{6} X^3 \quad C(X) \approx X. \]

(4.26)

If we look at the full expression for \( \delta \rho_{\mu}(z) \) in this approximation

\[ \delta \rho_{\mu}(z) \approx -\frac{B^{3/2}(-1)^m}{4\pi^3 m^{3/2}} \text{sgn}(\xi - \sqrt{2} \pi m) \cos \left( 2\sqrt{2} \nu \xi - \frac{3}{4} \pi \right) \]

\[- \frac{B^{3/2}}{2\pi^3} \sum_{n=n_{\text{min}}}^{\infty} \frac{(-1)^n \cos \left( 2\pi \nu n + \frac{\nu^2 m^2}{\pi^3} - \frac{3}{4} \pi \right)}{n^{3/2}} \]

(4.27)
we see that there are now two jumps when \( \zeta \) reaches a multiple of \( \sqrt{2}\pi \): one in the first term, and the jump that we already observed in the last term of (4.23). Indeed these discontinuities compensate each other exactly, and the density is continuous in the points \( \zeta = \sqrt{2}\pi m \).

Horing and Yildiz [25] give their approximation for weak magnetic fields as the sum of two terms: \( \rho_T(z) \) and \( \rho_{dHV\text{A}}(z) \). The first term comes from the branch cut (which in our case lies along the imaginary axis) and is described by Horing and Yildiz as “monotonic in magnetic field dependence”. The second term comes from the poles (which in our case lie along the real axis) and is “oscillatory in the de Haas-van Alphen (dHvA) sense”.

We will give a comparison between their result in the limit \( T \to 0 \) and the results derived here. The \( T \to 0 \) limit of the contribution of the term \( \rho_{dHV\text{A}}(z) \) to \( \delta\rho_{\mu}(z) \) is given by

\[
\delta\rho_{dHV\text{A}}(z) = -\frac{B^{3/2}}{2\pi^3} \sum_{n=1}^{\infty} \frac{\cos\left(2\pi n v + \frac{\zeta^2}{\pi n} - \frac{3}{4}\pi\right)}{n^{3/2}}.
\]  

(4.28)

If we compare this with (4.22) one can spot two differences. The first is the absence of the factor \( (-1)^n \). This can be attributed to the fact that Horing and Yildiz do explicitly take into account the Zeeman-splitting, which we preferred not to do for consistency with the rest of this thesis. Incorporating the Zeeman-splitting in our result would amount to replacing the \( \sinh(\beta B/2) \) with \( \tanh(\beta B/2) \) in the denominator in (4.4). Indeed this does account for the absence of the factor \( (-1)^n \) from (4.28).

The second difference is that in (4.28) the summation over \( n \) starts at 1 instead of \( n_{\text{min}} \). This can be explained by the fact that the \( \rho_T(z) \) term in the result of Horing and Yildiz can only be valid for \( \zeta \ll \sqrt{2}\pi \) (although they do not mention this explicitly). So in a sense their result is complementary to the results derived here.

The contribution to \( \delta\rho_{\mu}(z) \) from the branch cut (which is called \( \rho_{2T}(z) \) in [25]) is given by

\[
\delta\rho_T(z) = -\frac{B^{3/2}}{2^{3/2}\pi^2v^{1/2}} \left[ \frac{\sin(2\sqrt{2}v\zeta)}{2^{3/2}v\zeta^3} - \frac{\cos(2\sqrt{2}v\zeta)}{\zeta^2} \right].
\]  

(4.29)

This result does not change if one ignores the Zeeman-splitting; its derivation is based on the approximation of \( 1/\tanh(\beta B/2) \) by the first term in its Laurent expansion, which is identical to the first term in of the Laurent expansion of \( 1/\sinh(\beta B/2) \). But in spite of this, (4.29) differs considerably from our result (4.21).
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4.1.4.1. Magnetic field perpendicular to the wall

Figure 4.3: Numerical evaluation of $\delta \rho_\mu(z)/B^{3/2}$ for $\nu = 5$: exact (4.10) (---), approximation according to Horing and Yildiz (4.30) (---) and our asymptotic approximation (4.23) (---). The latter is almost indistinguishable from the exact expression.

4.1.2 Numerical results

In order to check the quality of our asymptotic expression (4.23) for the density profile we would like to compare it numerically with the approximation given by Horing and Yildiz, and the exact result (4.10), for moderate values of $\nu$.

In the absence of Zeeman coupling the result of Horing and Yildiz would read

$$
\delta \rho_\mu(z) = -\frac{B^{3/2}}{2^{3/2/2} \pi^2 \nu^{1/2}} \left[ \frac{\sin(2\sqrt{2}\nu \zeta)}{2^{3/2/2} \nu \zeta^3} - \frac{\cos(2\sqrt{2}\nu \zeta)}{\zeta^2} \right]
- \frac{B^{3/2}}{2\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^n \cos \left( \frac{2\pi n \nu + \frac{\nu \zeta^2}{\pi n} - \frac{3\pi}{4}}{n^{3/2}} \right).
$$

(4.30)

Figure 4.3 shows that for large $\zeta$, our approximation is clearly better than the one given by Horing and Yildiz. In fact, the curve corresponding to (4.23) is almost indistinguishable from the exact result, even though $\nu = 5$ is not particularly large. We also see
that for $\xi$ smaller than $\sqrt{2}\pi$, (4.30) still seems to work reasonable well. But closer to $\sqrt{2}\pi$ and beyond, it makes no sense.

Of course it is not fair, not to discuss the situation with $\xi$ closer to zero. Here our approximation is not supposed to work. In fact, in the limit $\xi \to 0$ (4.23) diverges. Figure 4.4 shows that (4.30) behaves much better for smaller values of $\xi$, illustrating that our approximation and the approximation given by Horing and Yildiz are complementary.

4.2 Magnetic field parallel to the wall

After having treated the case where the magnetic field is perpendicular to the wall, let us return to the original geometry with the magnetic field parallel to the wall. As indicated before, we are interested in the regime where $\xi \gg 1$ and $\nu \gg 1$, with $\xi^2 \approx \nu$. In order to investigate this regime, it is useful to introduce the new variable $\bar{\xi} = \xi/\sqrt{\nu}$. We will
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start from (3.29) in the previous chapter, which, in terms of the new variable \( \bar{\xi} \), reads

\[
\delta \rho_{\beta}(x) \approx -\frac{\rho B v^{1/2} \bar{\xi}}{Z_\perp 8\sqrt{2\pi}^{3/2} q_0^{3/2}} \int_0^\infty \frac{dp}{\sqrt{p(1+p)}} \exp \left( \frac{-\gamma \xi^2}{2q_0} \right) \int_0^\infty dp \left( \frac{1-q_0^2}{2q_0} \right) \exp \left( \frac{-(1-q_0^2)p}{2q_0} \right). \tag{4.31}
\]

We will use once more the inverse Laplace transform (3.46). Choosing a contour slightly different from that in (3.47), by setting \( \beta = (i t + \bar{\xi})/B \), we arrive at

\[
\delta \rho_{\mu}(x) \approx -\frac{B^{3/2}v^{1/2} \bar{\xi}}{16\pi^3} \int_{-\infty}^\infty dt e^{v(it + \bar{\xi})} \frac{1-q_0^2}{q_0^{3/2}} \exp \left( \frac{-\gamma \xi^2}{2q_0} \right) J \left( \frac{\nu \xi^2}{2q_0}, 1-q_0^2 \right). \tag{4.32}
\]

Here \( J(w, u) \) is defined as

\[
J(w, u) = \int_0^\infty dy \sqrt{\frac{1+uy}{y(1+y)}} e^{-wyu}. \tag{4.33}
\]

In the new variables \( t \) and \( \bar{\xi} \), the parameter \( q_0 \) is given by \( \tanh[(it + \bar{\xi})/4] \). Since \( \bar{\xi} \) is \( O(1) \), \( q_0 \) is complex (and not in the vicinity of 1, as in (3.47)). Hence, the argument below (3.29), which justified rewriting the integral \( J \) in terms of a modified Bessel function, cannot be used here. Instead, we will have to determine the properties of \( J(w, u) \) itself, in particular for large values of \( \text{Re} w \). A suitable asymptotic expression for \( J(w, u) \) is derived in appendix 4.A.

Because of the asymptotic form (4.49) of \( J(w, u) \), it is useful to split the integration in (4.32) in intervals of length 2\( \pi \). We do this by setting \( t = \tau + 2\pi n \), with \( \tau \in [0, 2\pi] \).

This yields

\[
\delta \rho_{\mu}(x) \approx -\frac{B^{3/2}e^{v\bar{\xi}}}{8\sqrt{2\pi}^{3/2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{2\pi i n \nu} \int_0^{2\pi} d\tau \frac{1}{(i\tau + 2\pi n + \bar{\xi})^{3/2}} \times \left[ (n + 1) \cosech \left( \frac{i\tau + \bar{\xi}}{4} \right) \exp \left\{ iv \left[ \tau + \frac{i\bar{\xi}}{2} \cotanh \left( \frac{i\tau + \bar{\xi}}{4} \right) \right] \right\} \right] \tag{4.34}
\]

Since \( \nu \) is large, the integrand is in general a rapidly fluctuating function of \( \tau \), which allows us to employ the saddle-point method to calculate the integral over \( \tau \). The integral over the first term between the curly brackets is dominated by the saddle point
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that follows by setting the derivative of the function

\[ f(\tau) = i\tau - \frac{\xi^2}{2} \coth \left( \frac{i\tau + \xi}{4} \right) \]  

(4.35)

equal to zero. This leads to \( \tau = 2\psi + i\xi \), with \( \psi = \arccos(1 - \xi^2/4) \). Clearly, only values of \( \xi \) in the interval \( 0 < \xi < 2\sqrt{2} \) make sense here. They correspond to saddle points \( \tau \) with a real part between 0 and \( 2\pi \). If \( \xi \) approaches its lower or upper bound, the asymptotic behaviour for \( \delta \rho_\mu(x) \) as derived here will no longer be valid, as (4.49) then no longer holds.

Treating the second integral in (4.34) in a similar way, we arrive at the following asymptotic result for the excess particle density in the regime of large \( \nu \) and \( \xi \in (0, 2\sqrt{2}) \):

\[
\delta \rho_\mu(x) \approx \frac{B^{3/2}}{8\sqrt{2\pi^2}\nu^{1/2} \sin^{1/2} \psi} \exp[2i\nu(\psi + \sin \psi)] \\
\times \left[ \sum_{n=0}^{\infty} (-1)^n \frac{(n + 1)e^{2\sin \nu}}{(\pi n + \psi)^{3/2}} + i \sum_{n=2}^{\infty} (-1)^n \frac{(n - 1)e^{-2\sin \nu}}{(\pi n - \psi)^{3/2}} \right] + \text{c.c.} \tag{4.36}
\]

For small values of \( \xi \), the variable \( \psi \) is close to zero. As a consequence the \( n = 0 \) term in the first sum in (4.36) dominates. Hence, the asymptotic expression simplifies to:

\[
\delta \rho_\mu(x) \approx \frac{B^{3/2}}{4\sqrt{2\pi^2}\nu^{1/2} \sin^{1/2} \psi \psi^{3/2}} \cos[2\nu(\psi + \sin \psi)]. \tag{4.37}
\]

Numerically, it is found that this approximation is quite useful even if \( \xi \) is of order 1, as can be seen in figure 4.5.

The approximation (4.37) ceases to be accurate for values of \( \nu \) that are close to a half-odd integer. This can best be seen by writing the sums in (4.36) in terms of Lerch functions \( \Phi(z, s, \alpha) \) that are defined as [18]

\[
\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} (\alpha + n)^{-s} z^n \tag{4.38}
\]

for \( |z| < 1, s > 0 \) and \( \alpha \) non-integer. An analytical continuation as a function of \( z \) in the complex plane can be carried out, except for a cut from 1 to \( \infty \) along the positive real axis. Setting apart the first term in the first sum in (4.36) and defining \( \nu' = \nu + 1/2 - [\nu + 1/2] \) (that is, \( \nu' \) is the non-integer part of \( \nu + 1/2 \)), we get

\[
\delta \rho_\mu(x) \approx \frac{B^{3/2}}{8\sqrt{2\pi^2\nu^{1/2} \sin^{1/2} \psi}} \exp[2i\nu(\psi + \sin \psi)] \\
\times \left[ \sum_{n=0}^{\infty} (-1)^n \frac{(n + 1)e^{2\sin \nu}}{(\pi n + \psi)^{3/2}} + i \sum_{n=2}^{\infty} (-1)^n \frac{(n - 1)e^{-2\sin \nu}}{(\pi n - \psi)^{3/2}} \right] + \text{c.c.}
\]
4.2. Magnetic field parallel to the wall

Figure 4.5: Comparison between results from (4.36) (—) and the approximations (4.37) (---) and (4.44) (-----) for \( \delta \rho_\mu(x)/B^{3/2} \) at \( \nu = 10 \) as a function of \( \xi \).

\[
\begin{align*}
\times \left\{ \left( \frac{\pi}{\psi} \right)^{3/2} + \mathrm{e}^{2\pi i \nu'} \Phi \left( \mathrm{e}^{2\pi i \nu'}, \frac{1}{2}, 1 + \frac{\psi}{\pi} \right) \\
+ \mathrm{i} \mathrm{e}^{-2\pi i \nu'} \Phi \left( \mathrm{e}^{-2\pi i \nu'}, \frac{1}{2}, 1 - \frac{\psi}{\pi} \right) \\
+ \left( 1 - \frac{\psi}{\pi} \right) \left[ \mathrm{e}^{2\pi i \nu'} \Phi \left( \mathrm{e}^{2\pi i \nu'}, \frac{3}{2}, 1 + \frac{\psi}{\pi} \right) \\
- \mathrm{i} \mathrm{e}^{-2\pi i \nu'} \Phi \left( \mathrm{e}^{-2\pi i \nu'}, \frac{3}{2}, 1 - \frac{\psi}{\pi} \right) \right] \right\} + \text{c.c.} \\
(4.39)
\end{align*}
\]

If \( \nu' \) is close to an integer, the Lerch functions with \( s = 1/2 \) dominate. In fact, for \( \nu' \ll 1 \) one may write [18]

\[
\Phi \left( \mathrm{e}^{2\pi i \nu'}, \frac{1}{2}, \alpha \right) \approx \frac{\mathrm{e}^{i\pi/4}}{\sqrt{2\nu'}} \\
(4.40)
\]

whereas for \( 1 - \nu' \ll 1 \) one has:

\[
\Phi \left( \mathrm{e}^{2\pi i \nu'}, \frac{1}{2}, \alpha \right) \approx \frac{\mathrm{e}^{-i\pi/4}}{\sqrt{2(1 - \nu')}}. \\
(4.41)
\]

The sum of the Lerch functions with \( s = 1/2 \) in (4.39) becomes large for \( \nu' \) small, while it stays finite for \( \nu' \) near 1. Hence, a better approximation for small \( \nu' \) is obtained by
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Figure 4.6: Comparison between results from (4.36) (——) and the approximations (4.37) (- - - -) and (4.42) (--- ---) for $\delta \rho_\mu(x)/B^{3/2}$ at $\nu = 10.501$ as a function of $\xi$. 

incorporating the dominant contribution of the Lerch functions in (4.37), so that one gets for small $\nu'$:

$$
\delta \rho_\mu(x) \approx \frac{B^{3/2}}{4\sqrt{2}\pi^2 \nu^{1/2} \sin^{1/2}\psi} \left\{ \frac{1}{\nu^{3/2}} \cos[2\nu(\psi + \sin \psi)] 
+ \frac{\sqrt{2}}{\pi^{3/2} \nu^{1/2}} \cos[2\nu(\psi + \sin \psi) + \pi/4] \right\}. \quad (4.42)
$$

A numerical comparison of this expression with (4.36) shows that it indeed outperforms (4.37) for $\nu$ just above a half-odd integer, as is seen in figure 4.6. As remarked at the end of section 3.4, the singular behaviour near half-odd integer $\nu$ is associated with the de Haas-van Alphen effect.

Returning to the expression (4.37), which holds for $\mu$ not in the vicinity of a Landau level, we see that it can be simplified further if $\bar{\xi}$ is so small that the expressions for $\psi$ and $\sin \psi$ can be replaced by the first few terms in their power series expansion around $\bar{\xi} = 0$. In this way, one finds

$$
\delta \rho_\mu(x) \approx \frac{\sqrt{\mu}}{2\sqrt{2}\pi^2 x^2} \cos \left[ 2\sqrt{2}\mu x \left( 1 - \frac{B^2 x^2}{48\mu} + \ldots \right) \right] + \ldots, \quad (4.43)
$$

This approximation is valid for $x$ in the range $1/\sqrt{B} \ll x \ll \sqrt{\mu}/B$, so that $x$ is large compared to the Fermi wavelength $1/\sqrt{\mu}$. The cosine can be expanded if $x$ is limited
4.A. Appendix: Properties of $J(w, u)$

still further to the range $1/\sqrt{B} \ll x \ll \mu^{1/6}/B^{2/3} \ll \sqrt{\mu}/B$. In that case one finds:

$$\delta \rho_{\mu}(x) \approx \frac{\sqrt{\mu}}{2\sqrt{2}\pi^2 x^2} \cos(2\sqrt{2\mu}x) + \frac{x B^2}{48\pi^2} \sin(2\sqrt{2\mu}x) + \ldots.$$ (4.44)

According to (3.52) the current density in this regime is

$$j_{\mu, \mu}(x) \approx -\frac{B \sqrt{\mu}}{4\sqrt{2}\pi^2 x} \cos(2\sqrt{2\mu}x)$$ (4.45)

in leading order of $B$. These formulas describe the behaviour of the excess particle density and the current density for $1/\sqrt{B} \ll x \ll \mu^{1/6}/B^{2/3}$, in systems with $\mu \gg B$. As we see, the dependence on the distance to the wall shows an algebraic decay in this regime, with an oscillatory pre-factor. This type of decay, which agrees with that found from perturbation theory for small $B$ [42, 27], is much slower than the Gaussian decay in the far region $x \gg \sqrt{\mu}/B$, which cannot be obtained from perturbation theory. The validity of an algebraic decay in the far region was questioned before [46]. A numerical comparison of (4.44) with (4.36) and (4.37) is made in figure 4.5, which shows that (4.44) is useful for small $\xi$ only, as expected.

Finally we make a comparison of the asymptotic expansion in terms of Lerch functions for the density profile (4.39) with numerical results based on (2.20). The results are plotted in figure 4.7. It is clear that (4.39) reproduces the essential features of the exact result such as the oscillation frequency and overall (algebraic) decay. Nevertheless the difference between the two curves is considerably larger than in the case where the magnetic field is perpendicular to the wall (see figure 4.3).

4.A Appendix: Properties of $J(w, u)$

The function $J(w, u)$ is defined as

$$J(w, u) = \int_{0}^{\infty} dy \sqrt{\frac{1 + uy}{y(1 + y)}} e^{-wuy}. (4.46)$$

This integral representation is valid for $|\text{arg} wu| \leq \pi/2$ and $|\text{arg} u| \leq \pi$. In (4.32), both the variables $wu$ and $u$ encircle the origin repeatedly as $t$ varies. In fact, for $t \in (-2\pi, 2\pi)$ the arguments of both these variables are in the interval $(-\pi, \pi)$. When $t$ passes through $(2m + 1)2\pi$, for integer $m$, both $wu$ and $u$ cross the negative real axis in the clockwise direction. In contrast, the variable $w$ in (4.32) has a simpler behaviour:
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Figure 4.7: Numerical evaluation of $\delta \rho_\nu(x)/B^{3/2}$ for $\nu = 10$: exact (——) and our asymptotic approximation (4.36) (-----).

it stays in the right half-plane for all values of $t$. Hence, it is more convenient to switch to an integral representation for $J(w, u)$ in which the variables $w$ and $u$ are separated. This representation is found by considering (4.46) for real and positive $u$ and $w$ and rescaling $y$ by a factor $u$. In this way one gets the representation

$$
J(w, u) = \int_0^\infty dy \sqrt{\frac{1 + y}{y(y + u)}} e^{-wy}
$$

(4.47)

for which an analytical continuation to any $w$ and $u$ with $|\arg w| \leq \pi/2$ and $|\arg u| \leq \pi$ is obtained trivially. Hence, in this new form further analytical continuation is necessary for the variable $u$ only. That continuation is easily carried out by evaluating the discontinuity across the cut for negative $u$. Writing $J(w, u_m)$ for the analytical continuation of $J(w, u)$ that follows by letting $u$ encircle the origin $m$ times in the clockwise direction (corresponding to $t \in ((2m-1)2\pi, (2m+1)2\pi)$ in (4.32)), one finds:

$$
J(w, u_m) = \int_0^\infty dy \sqrt{\frac{1 + y}{y(y + u)}} e^{-wy} + 2im \int_0^1 dy \sqrt{\frac{1 - uy}{y(1-y)}} e^{wy}.
$$

(4.48)

The variables at the right-hand side may take any values corresponding to $t \in (-2\pi, 2\pi)$ in (4.32), that is, with $|\arg w| \leq \pi/2$, $|\arg u| \leq \pi$ and $|\arg wu| \leq \pi$. It should be noted that the second integral has a cut for $u \geq 1$. However, this poses no problem, since
the contour followed by \( u \) in (4.32) avoids this cut, as it crosses the positive real axis between 0 and 1.

Having performed the analytical continuation of the integral \( J(w, u) \), we can now determine its asymptotic behaviour for large \( \nu \) (and finite \( \xi \)). For these values of \( \nu \) and \( \xi \), the real part of \( w \) is large, whereas \( |u| \) is finite. Hence, the first integral in (4.48) is approximately equal to \( [\pi/(wu)]^{1/2} \). In the second integral in (4.48), the dominant contributions come from the end points of the integration interval, since \( |wu| \) is large. Evaluating the end-point contributions one finds an asymptotic expression that depends on the location of \( wu \) in the complex plane (with a cut at the negative real axis). In the right half-plane one finds \( \sqrt{1 - u} \exp(wu)\sqrt{\pi/(wu)} \). In the left half-plane the result depends on the quadrant: in the second quadrant, with \( \arg wu \in (\pi/2, \pi) \), the result is \( i\sqrt{\pi/(wu)} \), and in the third quadrant, with \( \arg wu \in (\pi, 3\pi/2) \) one gets \( -i\sqrt{\pi/(wu)} \). Finally, the asymptotic expressions along the imaginary axis follow by adding the limits of the expressions valid in the neighbouring quadrants. Collecting the results, we have found the following asymptotic expression for the analytic continuation of \( J(w, u) \) for large \( \text{Re} w \):

\[
J(w, u_m) \approx \sqrt{\frac{\pi}{wu}} \left[ (1 + 2m) + 2im\sqrt{1 - u} e^{wu} \right]
\]

(4.49)

with the restrictions \( |\arg w| \leq \pi/2 \), \( |\arg u| \leq \pi \) and \( |\arg wu| \leq \pi \). The upper or the lower sign apply to the cases of positive or negative \( \text{Im} wu \), respectively. Of course, in the right half-plane of \( wu \) the term with \( \exp(wu) \) dominates, and in the left half-plane the other terms.
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