An experimental approach to expectation formation in dynamic economic systems

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Chapter 6

A Simple Asset Pricing Model

6.1 Introduction

In this chapter we introduce a standard dynamic asset pricing model (for more details about this model see Cuthbertson (1996), Lucas (1978) and Brock and Hommes (1998)) which is the underlying price generating process in the experiments of Chapters 7 and 8. First we derive the market equilibrium price of the risky asset in a world where agents have to choose between investing in the risky asset and investing in a risk free asset. After this we consider the dynamics of the asset pricing model under the benchmark of rational expectations. The asset pricing model has multiple RE solutions, namely a (constant) fundamental RE price, given by the discounted sum of expected future dividends, as well as rational bubble solutions, growing at a rate equal to the risk free rate of return. We also discuss asset pricing dynamics under simple forecasting rules such as naive and adaptive expectations. Furthermore, we give an example of the asset pricing model with heterogeneous agents. In the asset pricing model evolutionary competition between two simple forecasting rules can lead to complicated, chaotic price dynamics. In Chapters 7 and 8 we report the results of experiments on expectation formation with this standard asset pricing model.

6.2 An asset pricing model

Consider an asset market with $H$ trader types, indexed by $h$. Each trader can invest his money in a risk free asset (e.g. a savings deposit) with a risk free gross rate of return $R = 1 + r$, where $r$ is the real interest rate, or he can invest his money in a risky asset with price $p_t$ in period $t$ paying an uncertain dividend $y_t$ in period $t$. We assume these
dividends to be independently and identically distributed (IID) with mean $\bar{y}$ and variance $\sigma^2_y$. Denote by $z_{ht}$ the number of shares of the risky asset purchased by trader type $h$ in period $t$. The trader's realized wealth in period $t+1$ is

$$W_{h,t+1} = RW_{h,t} + (p_{t+1} + y_{t+1} - R_{pt})z_{ht}. \quad (6.1)$$

Trader type $h$'s subjective beliefs about the evolution of wealth are characterized by their subjective conditional mean $E_{ht}$ and subjective conditional variance $V_{ht}$. Traders are mean-variance optimizers, that is, their demand of shares corresponds to the solution of

$$\max_{z_{ht}} \{E_{ht}(W_{t+1}) - \frac{1}{2} a V_{ht}(W_{t+1})\}
= \max_{z_{ht}} \{z_{ht} E_{ht}(p_{t+1} + y_{t+1} - R_{pt}) - \frac{1}{2} az_{ht}^2 V_{ht}(p_{t+1} + y_{t+1} - R_{pt})\},$$

where $a$ measures the degree of risk aversion (assumed to be the same for all traders). We assume $V_{ht}(p_{t+1} + y_{t+1} - R_{pt}) = \sigma^2$, for all $h$, that is, traders believe that the conditional variance of excess returns is constant (and the same for all traders). There is no harm in this assumption since we deal only with point predictions of traders and not with trader's beliefs about the distribution of returns. The actual demand for the risky asset by trader type $h$ is given by

$$z_{ht} = \frac{E_{ht}(p_{t+1} + y_{t+1} - R_{pt})}{a \sigma^2}.$$  \quad (6.2)

Notice that trader $h$ will supply the asset if he/she expects a (large) decrease in its price. Besides the demand or supply of the mean-variance optimizers there is a random outside supply (or demand) of $\mu$ shares per trader, which is independently and identically distributed with mean 0 and variance $\sigma^2_{\mu}$. The market equilibrium condition becomes

$$\sum_{h=1}^{H} z_{ht} = \frac{1}{a \sigma^2} \sum_{h=1}^{H} E_{ht}(p_{t+1} + y_{t+1} - R_{pt}) = \mu_t, \quad (6.3)$$

which can be written as

$$R_{pt} = \frac{1}{H} \sum_{h=1}^{H} E_{ht}(p_{t+1} + y_{t+1}) + \epsilon_t,$$  \quad (6.4)

where $\epsilon_t \equiv - \frac{a \sigma^2}{H} \mu_t$. The equilibrium condition (6.4) says that today's asset price is determined by beliefs about tomorrow's asset price and tomorrow's dividend and an extra noise term. Notice that, when traders have to make a prediction for the price in period $t+1$ they do not know the market price $p_t$ in period $t$ yet, and they can only use information up till time $t - 1$.

\footnote{In our experiments of Chapters 7 and 8, when forecasting $p_{t+1}$ subjects thus only have prices $p_1, p_2, \ldots, p_{t-1}$ in their information set.}
An important feature of the asset pricing model is its *self-confirming* nature: if all traders have a high/low prediction the realized price will also be high/low. This important feature is characteristic for a speculative asset market: if traders expect a high price, the demand for the risky asset will be high, and as a consequence the realized market price will be high assuming that the supply is fixed. For the cobweb model with a production lag we showed that a high prediction leads to a high supply which results in a low realized market price. Whereas the cobweb model is an expectations reversing feedback system, the asset pricing model is an expectations self-confirming feedback system.

### 6.3 Expectations in the asset pricing model

The basic equilibrium price equation of the asset pricing model is equation (6.4). The development of the asset price depends upon the (subjective) expectations of the traders. In this section we will consider the dynamics of the asset pricing model under rational expectations as well as under boundedly rational prediction strategies.

#### 6.3.1 The fundamental solution and rational bubbles

Under rational expectations the subjective expectation $E_{ht}$ of trader $h$ is equal to the objective mathematical conditional expectation $E_t$, for all $h$. Equation (6.4) then reduces to

$$Rp_t = E_t(p_{t+1} + y_{t+1}) + \epsilon_t. \quad (6.5)$$

After $K$ steps of repeated substitution we find

$$p_t = \frac{E_t(p_{t+k})}{R^K} + \sum_{k=1}^{K} \frac{E_t(y_{t+k})}{R^k} + \frac{1}{R} \epsilon_t, \quad (6.6)$$

where we have used $E_tE_{t+k}(\cdot) = E_t(\cdot)$ and $E_{t+k}(\epsilon_t) = 0$ for $k > 0$. In recent years, a number of contributions have established that bubbles can occur under rational expectations, see e.g. Tirole (1982, 1985). There are two types of RE solutions. Sometimes the solution paths are required to satisfy the transversality or *no-bubbles condition* $\lim_{K \to \infty} \frac{E_t(p_{t+k})}{R^K} = 0$. Given this condition we have

$$p_t = \sum_{k=1}^{\infty} \frac{E_t(y_{t+k})}{R^k} + \frac{1}{R} \epsilon_t, \quad (6.7)$$

which equals the present discounted value of expected future dividends, plus a noise term. The first part of this solution is usually called the *fundamental* price $p'$. For the IID
dividend process with mean $Ey_t = \bar{y}$, the fundamental price is constant and given by

$$p^f = \frac{\bar{y}}{R - 1} = \frac{\bar{y}}{r}. \quad (6.8)$$

Under rational expectations when the transversality condition is satisfied and dividends are IID, market equilibrium prices evolve according to $p_t = p^f + \frac{1}{R} \epsilon_t$, i.e. prices are random fluctuations around the constant fundamental price $p^f$. However, there is a priori no convincing reason why the no-bubbles condition should hold. In fact, it can easily be checked that, when there is no noise, i.e. $\epsilon_t \equiv 0$, any solution of the form

$$p^b_t = R^t c + p^f = R^t c + \frac{\bar{y}}{r}, \quad c \geq 0 \quad (6.9)$$

is an RE solution satisfying (6.4). These solutions are called rational bubbles. They grow with the gross rate of return $R$ and a solution exists for each $c \geq 0$, with actual prices evolving according to $p_t = p^b_t + \frac{1}{R} \epsilon_t$. Hence, under rational expectations there is a continuum of (exploding) solution orbits. These bubble solutions are often ignored and attention is given primarily to the fundamental solution $p^f$. Figure 6.1 (a) shows the fundamental solution while Figure 6.1 (b) shows four bubble solutions. The parameters $\bar{y}$ and $r$ correspond to the values of the parameters in the experiments of Chapter 7. Important questions we try to answer is whether the participants in the experiment of

![Graphs](image)

(a) fundamental solution  
(b) bubble solutions

Figure 6.1: (a): the RE fundamental solution, $p^f = 60$, with $\bar{y} = 3$ and $r = 0.05$. (b): four RE bubble solutions (6.9), for $c = 1, 5, 15$ and $40$, with $\bar{y} = 3$ and $r = 0.05$.

Chapter 7 will coordinate on the fundamental solution or on one of the rational bubbles solutions.

### 6.3.2 Boundedly rational prediction strategies

The rational expectations hypothesis is quite demanding. It requires that traders know the underlying asset pricing model and use this to compute the conditional expectation
for the future price and that they do not make structural forecast errors. In particular, in a heterogeneous world RE requires knowledge about the beliefs of all other traders. A RE solution will only prevail when traders coordinate on one of the possible rational expectations equilibria. In this section we investigate the price behavior when the agents in the model use simple forecasting rules. They do not have (exact) knowledge of the underlying model, but have their own beliefs about the development of asset prices and use this belief and the available time series observations to predict the price. The beliefs of the traders are sometimes called the perceived laws of motion. Given those perceived laws of motion the underlying model (6.4) is then referred to as the implied actual law of motion. An important objective for this thesis is to get some insights into the nature of the perceived laws of motion people actually use. We assume that, when traders have to predict a price for time $t + 1$, they know the interest rate $r$ (which is constant over time), the realized price up to time $t - 1$ and their own price predictions up to time $t$. We assume that the IID dividend process $y_t = \bar{y} + \delta_t$ is common knowledge and $E_{h,t}y_{t+1} = \bar{y}$ for all $h$. The market equilibrium price in (6.4) then simplifies to

$$p_t = \frac{1}{RH} \sum_{h=1}^{H} E_{ht}(p_{t+1}) + \frac{\bar{y}}{R} + \frac{1}{R} \epsilon_t. \quad (6.10)$$

A general form of traders forecasting rule or prediction strategy is

$$E_{ht}(p_{t+1}) = P_{h,t+1}^* = f_{ht}(p_{t-1}, p_{t-2}, \ldots, p_1, P_{ht}^*, p_{h,t-1}^*, \ldots, p_{h1}^*, \bar{y}, r), \quad (6.11)$$

where $f_{ht}$ can be any (possibly time-varying) function. There are no restrictions on the specification $f_{ht}$ and the possibilities are therefore unbounded. Given traders forecasting rules (6.11), the implied actual law of motion becomes

$$p_t = \frac{1}{RH} \sum_{h=1}^{H} f_{ht}(p_{t-1}, p_{t-2}, \ldots, p_1, P_{ht}^*, p_{h,t-1}^*, \ldots, p_{h1}^*, \bar{y}, r) + \frac{\bar{y}}{R} + \frac{1}{R} \epsilon_t. \quad (6.12)$$

The actual dynamics of prices is to a great extent characterized by the prediction strategies used by the traders. In particular, if all traders use linear prediction strategies, the (deterministic part of the) development of prices will also be linear. An important feature is that the asset pricing model is stable in the sense that if traders (on average) do not expect prices to diverge too fast the asset price will converge to a steady state. This can be seen by rewriting (6.4) as

$$p_t - \bar{y} = \frac{1}{1 + r} (p_{t+1}^* - \bar{y} \frac{r}{r}), \quad (6.13)$$

\footnote{Notice that traders have all necessary information (the risk free rate of return $r$ and the mean dividend $\bar{y}$) to calculate the fundamental price $p' = \frac{\bar{y}}{r}$ and use this as their forecast.}
where \( \hat{p}_{t+1} = \frac{1}{H} \sum_{h=1}^{H} p_{h,t+1} \) is the average prediction for period \( t + 1 \) and where we have taken \( \epsilon_t = 0 \) for convenience. It follows that the realized price will always lie between the average price prediction \( \hat{p}_{t+1} \) and the fundamental price \( p^f = \frac{\bar{u}}{r} \).

It is now easy to see that there are two types of perfect foresight solutions. The first arises when all traders predict the fundamental price.

\[
\hat{p}_{t+1} = p^f. \tag{6.14}
\]

Given the fundamental forecast (6.14) the realized market price (6.13) becomes

\[
p_t = p^f = \frac{\bar{u}}{r}. \tag{6.15}
\]

The second perfect foresight solution occurs when all traders believe that the price deviation from the fundamental grows by the gross rate of return each period, i.e.

\[
\hat{p}_{t+1} = p^f + (1 + r)^2(p_{t-1} - p^f). \tag{6.16}
\]

If all agents believe in exploding prices, (6.16), the implied actual law of motion (6.13) becomes

\[
p_t - p^f = (1 + r)(p_{t-1} - p^f). \tag{6.17}
\]

That is, if agents expect the price deviation from the fundamental to grow each period with the gross rate of return, the price will grow with the gross rate of return and this belief is self-fulfilling. Hence, both these perfect foresight solutions are self-fulfilling in the sense that what they expect will be realized.

Bubbles can only occur in this framework if traders expect them to occur. For example, if traders believe that

\[
\hat{p}_{h,t+1} = \beta p_{t-1}, \quad \beta > R \tag{6.18}
\]

prices grow (on average) with rate \( \frac{\beta}{R} > 1 \) and a speculative bubble emerges. This speculative bubble coincides with the rational bubble if and only if \( \beta = R^2 \). The asset pricing model therefore has a self-confirming nature: if traders expect prices to explode, it is likely that prices will indeed explode.

The simplest examples of this type of prediction strategies were already discussed in Chapter 2 for the cobweb model and correspond to naive expectations, where

\[
\hat{p}_{h,t+1} = p_{t-1}. \tag{6.19}
\]

Under the assumption that all traders have naive expectations the price dynamics (6.13) reduces to a linear difference equation with steady state \( p^f \) and slope \( \frac{1}{1+r} \),

\[
p_t = \frac{1}{1+r}(p_{t-1} + rp^f). \tag{6.20}
\]
Since the slope is \( 0 < \frac{1}{1+\tau} \) lies between 0 and 1 prices converge monotonically to the steady state. The asset pricing model is stable under naive expectations and there is (slow and monotonic) convergence to the steady state because the slope is close to 1.

Another simple prediction strategy is *adaptive expectations*, where

\[
p_{h,t+1}^\ast = w p_{t-1} + (1 - w) p_{ht}^e = p_{ht}^e + w(p_{t-1} - p_{ht}^e)
\] (6.21)

and \( 0 < w \leq 1 \) (notice that \( w = 1 \) corresponds to naive expectations). Intuitively, prediction strategies where people try to learn the equilibrium price, for example by some kind of recursive least squares learning process, will also converge monotonically to the fundamental price.

Now suppose that traders use forecast errors to update their prediction, that is, their current prediction is their previous prediction adapted in the direction of the last observation. If the price lies above (below) the fundamental value, the predictions are above (below) the price and on average the agents will adjust their predictions downward (upward). For this type of prediction strategy the price will also monotonically converge to the fundamental price.

We conclude this discussion on boundedly rational prediction strategies by looking at the class of linear autoregressive prediction strategies with 2 lags, that is

\[
p_{h,t+1}^\ast = \alpha_h + \beta_{h1} p_{t-1} + \beta_{h2} p_{t-2}.
\] (6.22)

We will refer to (6.22) as an \( AR(2) \) prediction rule. If all traders use an \( AR(2) \) prediction rule, the implied actual law of motion (6.4) becomes

\[
p_t = \frac{1}{R} (\alpha + \beta_1 p_{t-1} + \beta_2 p_{t-2}) + \frac{\bar{P}}{R} + \frac{1}{R} \varepsilon_t
\] (6.23)

where \( \alpha = \frac{1}{R} \sum_{h=1}^H \alpha_h \) and \( \beta_t = \frac{1}{R} \sum_{h=1}^H \beta_{ht} \). Depending on the values of \( \beta_1 \) and \( \beta_2 \) one can have different types of price dynamics. In particular, if \( \beta_1^2 + 4R\beta_2 < 0 \) the price will oscillate around the steady state price. These oscillations will converge to the steady state if \( -R < \beta_2 < -\frac{\beta_1^2}{4R} \), but they will explode when \( \beta_2 < -R \). On the other hand, if \( \beta_1^2 + 4R\beta_2 > 0 \), the prices move monotonically or jump up and down, one period below the steady state and the next period above the steady state. If \( |\beta_1| + |\beta_2| < R \), these price movements converge to the steady state, otherwise they explode. In particular, for \( \beta_1 + \beta_2 > R \), a speculative bubble occurs. Hence, this simple class of prediction strategies can already enforce many different types of dynamical behavior on the asset pricing model.

It will be useful that the \( AR(2) \) prediction strategy can be rewritten as

\[
p_{h,t+1}^\ast = \alpha + \beta p_{t-1} + \delta(p_{t-1} - p_{t-2}),
\] (6.24)
where $\beta \equiv \beta_1 + \beta_2$ and $\delta \equiv -\beta_2$. Expressed in this way the $AR(2)$ prediction strategy provides a nice intuition. Participants believe that the price will be determined by the last observation (the first two terms on the right-hand side) but they also try to follow the trend in the prices (expressed in the third term): if $\delta > 0$ they believe that an upward movement in prices will continue the next period, whereas if $\delta < 0$, they believe an upward movement in the prices will be (partially) offset by a downward movement in prices in the next period.

### 6.4 The fundamental robot trader

Brock and Hommes (1998) investigated evolutionary dynamics in the asset pricing model with heterogeneous beliefs. Each belief is used by a fraction $n_{ht}$ of the population and these fractions, $n_{ht}$ of trader type $h$, change over time. Market equilibrium price is given by:

$$R_t = \sum_{h=1}^{H} n_{ht} E_{ht}(p_{t+1} + y_{t+1}),$$

where there is a large population of traders but a finite set of prediction strategies $E_{ht}(p_{t+1})$, $h = 1, \ldots, H$. The fractions $n_{ht}$ of trader type $h$ changes over time according to differences in profits generated by the different prediction strategies. More precisely, the fractions are given by the discrete choice ‘probabilities’

$$n_{ht} = \frac{\exp(\beta\pi_{ht})}{Z_t}, \quad Z_t = \sum_{h=1}^{H} \exp(\beta\pi_{ht}),$$

where $Z_t$ is a normalization factor such that all fractions add up to one and $\pi_{ht}$ is the profit generated in the recent past by forecasting strategy $h$. Brock and Hommes (1998) consider some examples with only a few (2, 3 or 4) prediction strategies. They show that as evolutionary pressure, i.e. the intensity of choice parameter $\beta$, increases the interaction between heterogeneous forecasting rules and the evolutionary competition leads to complicated dynamics, possibly with asset prices and returns moving on a strange attractor. They distinguish between different types of traders, so-called fundamentalists and technical analysts. The fundamentalists believe that the price is determined by the fundamentals of the economy (in this case, the interest rate and the mean dividend). The technical analysts try to predict the price by looking carefully at the time series of the asset price and by extrapolating price trends. In general, the presence of fundamentalists has a stabilizing effect on the price, whereas technical analysts may destabilize the price by self-fulfilling speculative bubbles. In a market with two of these opposing forces fluctuations in the asset price arise naturally. It is interesting to note that the evolutionary
dynamics is characterized by switching between price fluctuations close to the RE fundamental price and temporary RE or speculative bubble solutions. Traders are boundedly rational in the sense that most agents switch to strategies that have performed well in the recent past.

In one of the treatments of the experiment of Chapter 7 we use a specification of the asset pricing model that incorporates evolving fractions. In this setup, besides the $H$ different traders, there is a fundamentalist, called a “robot trader”, who always expects the price to be equal to the fundamental price $p^f$. The robot trader acts as a ‘stabilizing force’, pushing prices in the direction of the fundamental price. See e.g. DeGrauwe, DeWachter and Embrechts (1993) for a discussion of a similar ‘stabilizing force’ in an exchange rate model with fundamentalists and chartists. Kyle and Xiong (2001) introduce a long-term investor that holds a risky asset proportionally to the spread between the asset price and its fundamental value. This is similar to our setup, where the amount the robot trader invests depends upon the distance between the price and the fundamental price. The larger the distance between the market price and the fundamental price the more the trader will invest. Market equilibrium price in the presence of such a robot trader is given by

$$p_t = \frac{1}{RH} \sum_{h=1}^{H} (1 - n_t) E_{h}(p_{t+1}) + \frac{1}{R} n_t p^f + \frac{1}{R} \hat{y} + \frac{1}{R} \epsilon_t,$$

where $n_t$ is given by

$$n_t = 1 - \exp(-\lambda|p_{t-1} - p^f|).$$

We will refer to this fundamentalist trader as the robot trader. Figure 6.2 shows a picture of the weight of the robot trader, as a function of $p_{t-1}$, for $\lambda = 0.005$ and $p^f = 60$, which are the values that are used in the experiments in Chapter 7. Notice that the fraction of robot traders $n_t$ equals 0 when the market price $p_t$ equals $p^f$. For prices $p_t$ close to the fundamental price $p^f$ the fraction $n_t$ of robot traders will be small. The fraction $n_t$ will increase when prices move further away from the fundamental. One interpretation of such a ‘stabilizing force’ is that when prices are far from the fundamental price it becomes more likely that they will return to the fundamental.

There are some important consequences of this robot trader. First, the system becomes nonlinear, even if all traders use linear prediction strategies. This implies that in the presence of a robot trader typical features such as endogenous fluctuations and multiplicity of equilibria may occur even when all other traders use linear forecasting rules. Second, speculative bubbles diverging to infinity are excluded in the presence of the robot trader, since exploding prices will lead to an increasing weight of the robot trader, which stabilizes the prices. Notice that in the presence of a robot trader, the RE fundamental solution
Figure 6.2: Weight of Robot trader as function of the price $p_{t-1}$.

seems more likely than a RE bubble solution due to the stabilizing force of the robot trader.

### 6.5 Concluding remarks

In this chapter we showed the most important features of the asset pricing model. We showed that there are multiple RE solutions, a constant RE fundamental price and infinitely many explosive rational bubble solutions. Furthermore, we show that even with very simple boundedly rational prediction rules complicated price behavior can occur. However, simple rules such as naive expectations and adaptive expectations lead to monotonic convergence to the steady state fundamental price. The asset pricing model is thus stable under naive expectations. An important difference with the cobweb model is that the asset pricing model has a self-confirming nature. That is, if all traders believe a high/low price a high/low price will emerge. In Chapters 7 and 8 we use this asset pricing model to investigate expectation formation in the laboratory.