Viscoelastic behavior of dental restorative composites during setting

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APPENDIX A
ANALYTICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATION ASSOCIATED TO LINEAR VISCOELASTIC MODELS

This appendix shows in detail the mathematics used for solving the differential equation of the Maxwell model and Standard Linear Solid model analytically. The content does not only show the analytical solutions as used in the modeling procedures in this research project, but also provide help for those who are not familiar with mathematics regarding solving differential equations. The analytical equation of the Kelvin model, which can be derived directly from the differential equation, is given in the last section of this appendix.

General solution of linear differential equations

Generally, linear differential equations can be written as [1]:

\[ y' + p(x)y = r(x) \quad \quad y' = \frac{dy}{dx} \] (A.1)

The characteristic feature of a linear differential equation is that it is linear in \( y \) and \( y' \), whereas \( p \) and \( r \) may be any given functions of one variable \( x \) only. In this research project, only first order (in \( y \)) differential equations where investigated. The variables obtained from the dynamic test method are the stress \((\sigma)\), strain \((\varepsilon)\), and time \((t)\). The function \( r(x) \) (input) on the right-hand side of Equation A.1 represent the strain \((\varepsilon)\) and the solution \( y(x) \) (output) the stress \((\sigma)\). The choice for making the strain as input, and therefore the independent variable, seems natural because the mechanics of the test method were such that the strain was externally controlled. Both stress and strain are functions of time only; i.e., \( \sigma(t) \) and \( \varepsilon(t) \) respectively.

The differential equation of the Maxwell and Standard Linear solid model written in the way of Equation A.1 reveals:

Maxwell:

\[ \sigma' + \frac{E}{\eta} \sigma = E \varepsilon' \] (A.2)

Standard Linear Solid

\[ \sigma' + \frac{E_i}{\eta} \sigma = (E_i + E_z) \varepsilon' + \frac{E_z}{\eta} \varepsilon \] (A.3)
Small time intervals \((t_0, t_0 + \Delta t)\) in the stress response (Fig. A.1) of the dental restorative material were analyzed, because the material parameters were assumed to be constant. Since the function of the applied strain for the interval is known; e.g. the stress in time interval \((t_0, t_0 + \Delta t)\) in Figure A.1 is the response on a sinusoidal and linear shrinkage strain, the differential equation associated with the model can be solved for the interval, allowing the stress to be expressed as a function of time and unknown material parameters.

The general solution for the differential equations of the mechanical models can be written accordingly Equation A.4, which presents the general solution of Equation A.1 in the form of an integral. The \(h\)-term and \(r\)-term of Equation A.4 for each model with a sinusoidal or linear strain function as input, are given in Table A.1.

\[
y(x) = e^{-h} \left[ e^{h} r dx + C \right] \quad h = \int p(x) dx \quad (A.4)
\]

In solving Equation A.4, the stress at \(t_0\), the so-called initial condition \((\sigma(t_0))\), is taken from experimental stress data. As the strain and its derivatives are known functions of time, the integral in Equation A.4 can be evaluated analytically for the differential equation for both models (Table A.1).
Appendix A | Analytical solution for linear viscoelastic models

Table A.1 Terms associated with Equation A.4 for two strain functions.

<table>
<thead>
<tr>
<th>Strain:</th>
<th>$\varepsilon(t) = A\sin(\omega t)$</th>
<th>$\varepsilon(t) = B + ct$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maxwell:</td>
<td>$h = \frac{E}{\eta}$</td>
<td>$h = \frac{E}{\eta}$</td>
</tr>
<tr>
<td></td>
<td>$r = E\omega \cos(\omega t)$</td>
<td>$r = Ec$</td>
</tr>
<tr>
<td>Standard Linear Solid:</td>
<td>$h = \frac{E_i}{\eta}$</td>
<td>$h = \frac{E_i}{\eta}$</td>
</tr>
<tr>
<td></td>
<td>$r = (E_1 + E_2)A\omega \cos(\omega t) + \frac{E_2 E_1}{\eta}A\sin(\omega t)$</td>
<td>$r = (E_1 + E_2)c + \frac{E_2 E_1}{\eta}(B + ct)$</td>
</tr>
</tbody>
</table>

To get the desired solutions of the differential equations for the mechanical models, first the integral in Equation A.4 is evaluated, and then the constant C in this equation is defined. The next section describes these two steps in detail for the Standard Linear Solid model. The analytical solution for the Maxwell model is obtained in the same way and is, therefore, only presented in Table A.2.

Analytical solution for Standard Linear Solid and Maxwell model

The general solution of the differential equation of the Standard Linear Solid model according to Equation A.4, with the terms in Table A.1 and a sinusoidal strain as input, can be represented as:

$$\sigma(t) = e^{-\frac{E_i}{\eta}t} \left[ \int e^{\frac{E_i}{\eta}t} \left( (E_1 + E_2)A\omega \cos(\omega t) + \left( \frac{E_2 E_1}{\eta} \right) A\sin(\omega t) \right) dt + C \right]$$

(A.5)

The integral can be split up in two parts:

Part A:

$$\left( E_1 + E_2 \right) A\omega \int e^{\frac{E_i}{\eta}t} \cos(\omega t) dt$$

(A.6a)

Part B:

$$\frac{E_2 E_1}{\eta} A \int e^{\frac{E_i}{\eta}t} \sin(\omega t) dt$$

(A.6b)
It can be proven that [1]:

\[ \int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \cos(bx) + b \sin(bx)) + C \] (A.7a)

\[ \int e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \sin(bx) - b \cos(bx)) + C \] (A.7b)

Applying Equation A.7a and A.7b on the integral in Equation A.6a and A.6b results in:

**Part A:**

\[ \frac{(E_1 + E_2)A \omega}{\eta} \left( \frac{E_1}{\eta} \cos(\omega \tau) + \omega \sin(\omega \tau) \right) + C \] (A.8a)

**Part B:**

\[ \frac{E_2 E_1}{\eta} \left( \frac{E_1}{\eta} \sin(\omega \tau) - \omega \cos(\omega \tau) \right) + C \] (A.8b)

By inserting Equation A.8a and A.8b back in Equation A.5, we get the general solution for the Standard Linear Solid model:

\[ \sigma(t) = C e^{-\frac{E_1}{\eta} t} + \frac{E_1^2 A \eta \omega \cos(\omega \tau) + A \left( E_1^2 E_2 + (E_1 + E_2) \eta^2 \omega^2 \right) \sin(\omega \tau)}{E_1^2 + \eta^2 \omega^2} \] (A.9)

At the begin of the interval (t_0 set at 0), the model stress is equal to the experimental stress (s(t_0)), and, therefore, the constant C will be:

\[ C = \sigma(t_0) - \frac{AE_1^2 \eta \omega}{E_1^2 + \eta^2 \omega^2} \] (A.10)

hence, the analytical solution of the differential equation A.3, the so-called particular solution, for the Standard Linear Solid model with a sinusoidal strain is:

\[ \sigma(t) = \left( \sigma(t_0) - \frac{AE_1^2 \eta \omega}{E_1^2 + \eta^2 \omega^2} \right) e^{-\frac{E_1}{\eta} t} + \frac{E_1^2 A \eta \omega \cos(\omega \tau) + A \left( E_1^2 E_2 + (E_1 + E_2) \eta^2 \omega^2 \right) \sin(\omega \tau)}{E_1^2 + \eta^2 \omega^2} \] (A.11)
Appendix A | Analytical solution for linear viscoelastic models

In Table A.2, the solutions for both strain inputs (linear or sinusoidal) are given for the differential equation of the Standard Linear Solid and Maxwell model. When the input is a combined function of linear and sinusoidal strain, then the analytical solution is simply the sum of the individual solutions. In the case of the Standard Linear Solid model, the final solution will be:

\[
\sigma(t) = \left(\sigma(t_0) - \eta c - BE_2 - \frac{AE_1^2 \eta \omega}{E_1^2 + \eta^2 \omega^2}\right) e^{-\eta t} + \eta c + BE_2 + \frac{E_1^2 A \eta \omega \cos(\omega t) + A(E_1^2 E_2 + (E_1 + E_2) \eta^2 \omega^2 \sin(\omega t))}{E_1^2 + \eta^2 \omega^2}
\]

(A.12)

This solution, together with those presented in Table A.2, have been used in the modeling procedures.

**Table A.2.** Particular solutions of the differential equations obtained by evaluating Eq. (A.4) for two strain functions.

<table>
<thead>
<tr>
<th>Strain:</th>
<th>( \varepsilon(t) = A \sin(\omega t) )</th>
</tr>
</thead>
</table>
| Maxwell: | \[
\sigma(t) = \left(\sigma(t_0) - \frac{AE_1^2 \eta \omega}{E_1^2 + \eta^2 \omega^2}\right) e^{-\eta t} + \frac{E_1^2 A \eta \omega \cos(\omega t) + A(E_1^2 E_2 + (E_1 + E_2) \eta^2 \omega^2 \sin(\omega t))}{E_1^2 + \eta^2 \omega^2}
\]
| Standard Linear Solid: | \[
\sigma(t) = \left(\sigma(t_0) - \frac{AE_1^2 \eta \omega}{E_1^2 + \eta^2 \omega^2}\right) e^{-\eta t} + \frac{E_1^2 A \eta \omega \cos(\omega t) + A(E_1^2 E_2 + (E_1 + E_2) \eta^2 \omega^2 \sin(\omega t))}{E_1^2 + \eta^2 \omega^2}
\]

<table>
<thead>
<tr>
<th>Strain:</th>
<th>( \varepsilon(t) = B + ct )</th>
</tr>
</thead>
</table>
| Maxwell: | \[
\sigma(t) = \left(\sigma(t_0) - c \eta\right) e^{-\eta t} + c \eta
\]
| Standard Linear Solid: | \[
\sigma(t) = \left(\sigma(t_0) - BE_2 - c \eta\right) e^{-\eta t} + BE_2 + c(\eta + E_2 t)
\]
The first (exponential) term in the analytical solutions will approach zero as time \( t \) approaches infinity. This means that after a sufficiently long time the stress response on an oscillatory strain executes practically harmonic oscillations (Fig. A.2). The time for reaching harmonic stress oscillations is represented by the exponential term. In oscillatory strain tests, the stress response is harmonic after approximately two sinusoid cross head cycles (Fig. 3.14). Since the data was collected after five cross head cycles, the stress response was modeled with the second term in the analytical solutions only (exponential term was skipped). In pulse sinusoidal strain tests, the time for reaching harmonic stress oscillations must be taken into account. For this type of measurements, the stress response was modeled with the analytical solutions as given in Table A.2.

\[
\sigma(t) = \sigma(t_0) + E\varepsilon + \eta\varepsilon' 
\]

Figure A.2 Stress response due to an oscillatory strain.

**Analytical solution for the Kelvin model**

In the preceding section, the Kelvin model was left out of consideration, because the analytical solution for this model can be derived directly from the differential equation. This section describes the analytical solution for several strain functions.

In the Kelvin model, the spring and dashpot are in parallel. As a result, the stress function for the isolated time interval \( [t_0, t_0 + \Delta t] \) is the sum of the stresses in each element individually:

\[
\sigma(t) = \sigma(t_0) + E\varepsilon + \eta\varepsilon' 
\]

where \( \sigma(t_0) \) is the stress at the beginning of the time interval.

For calculating the analytical solution for time interval \( [t_0, t_0 + \Delta t] \), the individual components in the Kelvin model have to be evaluated separately. The stress in the dashpot \( \eta\varepsilon' \) is straightforward: multiply
viscosity with the time derivative of the strain function. The stress in the spring ($E\varepsilon$) must be evaluated at the begin and end of the time interval:

**Sinusoidal strain:** $\varepsilon(t) = A\sin(\omega t)$

$$\sigma(t) = \sigma(t_0) + EA\sin(\omega t) + \eta A\omega \cos(\omega t)$$ (A.16)

**Linear strain:** $\varepsilon(t) = B + ct$

$$\sigma(t) = \sigma(t_0) + E(B + ct)$$ (A.17)

**Sinusoidal and linear strain:**

$$\sigma(t) = \sigma(t_0) + E(ct + A\sin(\omega t)) + \eta(c + A\omega \cos(\omega t))$$ (A.18)

The final analytical solutions for the Kelvin model for the isolated time interval $[t_0, t_0 + \Delta t]$ in the form of Equation A.13 can be written as:

**Sinusoidal strain:**

$$\sigma(t) = \sigma(t_0) + EA\sin(\omega t) + \eta A\omega \cos(\omega t)$$ (A.16)

**Linear strain:**

$$\sigma(t) = \sigma(t_0) + Ect + \eta c$$ (A.17)

**Sinusoidal and linear strain:**

$$\sigma(t) = \sigma(t_0) + E(ct + A\sin(\omega t)) + \eta(c + A\omega \cos(\omega t))$$ (A.18)

All these analytical solutions have been used in the modeling procedures throughout this research project.

**References**
