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Worst VaR Scenarios: A Remark

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Abstract

Theorem 15 of Embrechts, Höing & Puccetti (2005) proves that the comonotonic dependence structure gives rise to the on-average-most-adverse Value-at-Risk scenario for a function of dependent risks, when the marginal distributions are known but the dependence structure between the risks is unknown. This note extends this result to the case where, instead of no information, partial information on the dependence structure between the risks is available.

Keywords: Dependent Risks, Value-at-Risk, Copulas, Worst Case Scenarios, Comonotonicity

JEL-Classification: D81, G10, G20

MSC-Classification: 60E05, 60E15, 62P05

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1 Introduction

The problem of finding the best-possible upper bound on the Value-at-Risk (VaR) of a function of dependent risks when the marginal distributions of the risks are known but the dependence structure between the risks is not (or not completely) known has been an important research topic for many years; see, for example, Makarov (1981), Rüschendorf (1982), Frank, Nelsen & Schweizer (1987), Denuit, Genest & Marceau (1999), Denuit *et al.* (2005) and Embrechts & Puccetti (2006).

The above-cited works have shown that the comonotonic dependence structure does in general not lead to the worst VaR scenario. At first sight, this may seem surprising: comonotonicity, under which every risk is a non-decreasing function of the other, is generally considered as the strongest dependence notion. It is the worst possible dependence scenario in the stop-loss and supermodular order sense. The interested reader is referred to Dhaene *et al.* (2002a) for an elaborate study of the concept of comonotonicity and its applications in insurance and finance. Nevertheless, as becomes clear, for example, from Theorem 6 of Embrechts, Höing & Puccetti (2005), there exists in general for any probability level a copula that yields a VaR that is larger than the VaR under comonotonicity.

Though the comonotonic dependence structure may not lead to the worst VaR scenario for a given probability level, Theorem 15 of Embrechts, Höing & Puccetti (2005) proves that the comonotonic dependence structure gives rise to the on-average-most-adverse VaR scenario for a function of dependent risks; a precise statement of this result is deferred until Section 2. This result supports the use of comonotonicity also in VaR-based risk management.

The aim of this note is to extend this result to the case where, instead of no information, partial information on the dependence structure between the risks is available. In particular, we assume that there exists a common risk factor with a given distribution function conditionally upon which the marginal distribution functions of the risks are available. We then prove that the improved comonotonic dependence structure as introduced in Kaas, Dhaene & Goovaerts (2000) and Dhaene *et al.* (2002a) arises as the on-average-most-adverse VaR scenario.

The outline of this note is as follows: In Section 2, we introduce some preliminaries on worst VaR scenarios. In Section 3, we recall the improved comonotonic dependence structure and derive some new results on it. Section 4 states the main results and Section 5 concludes the note.

2 Worst VaR Scenarios: Preliminaries

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a random vector $\underline{X} := (X_1, \dots, X_n)$ defined on it. For a given measurable function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$, we consider the problem of finding the best-possible lower bound on the probability

$$\mathbb{P}[\psi(\underline{X}) < s], \quad s \in \mathbb{R}, \quad (1)$$

when the marginal distribution functions (d.f.'s) $F_{X_1}, \dots, F_{X_n}$ of $\underline{X}$ are known but the dependence structure is unknown.

Equivalently, given the marginal d.f.'s of $\underline{X}$, we can consider the problem of finding the best-possible upper bound on

$$\text{VaR}_\alpha[\psi(\underline{X})], \quad \alpha \in (0, 1). \quad (2)$$

Here, as usual, the VaR at probability level α is defined by $\text{VaR}_\alpha[X] := F_X^{-1}(\alpha)$ with

$$F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq \alpha\}.$$

Remark 2.1 *Note that $\text{VaR}_\alpha[X]$ is left-continuous.*

We denote by $C : [0, 1]^n \rightarrow [0, 1]$ an n -copula and by $\mathcal{C}^n$ the family of all n -copulas. Furthermore, we denote by $M(u_1, \dots, u_n) := \min\{u_1, \dots, u_n\}$, the Fréchet upper bound. We refer to Nelsen (1999) for further details on copulas. In the following, given the marginals $F_1, \dots, F_n$, we let $\underline{X}^C$ denote the random vector induced by the n -copula C .

For $n = 2$, the copula that gives rise to the best-possible lower bound on the probability in (1) is known; see, for example, Theorem 3.1 of Embrechts & Puccetti (2006). It is known that in general this copula depends on the value of s and hence, in VaR terms, on the probability level α . Clearly, this is highly inconvenient from a practical point of view.

Noting this, Embrechts, Höing & Puccetti (2005) introduce, for a given copula C , the loss function

$$e_{C,\psi}(s) := \mathbb{P}[\psi(\underline{X}^C) < s] - \inf_{\tilde{C} \in \mathcal{C}^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}, \quad (3)$$

and consider the following optimization problem:

$$\inf_{C \in \mathcal{C}^n} \left\{ \int_d^{+\infty} e_{C,\psi}(s) ds \right\}. \quad (4)$$

Now, we restate Theorem 15 of Embrechts, Höing & Puccetti (2005); for a definition of supermodularity, we refer to Denuit *et al.* (2005), p. 179.

Lemma 2.1 *For every real number d and every non-decreasing supermodular function ψ satisfying $\mathbb{E}[\psi(\underline{X}^M)] < +\infty$, M is a minimizer of (4).*

Remark 2.2 *Note that (4) can be solved for any $n \geq 2$ even though when $n > 2$, the copula that gives rise to the best-possible lower bound on the probability in the second term on the right-hand side of (3) is not (yet) known.*

3 Additional Information on the Dependence Structure

In the remainder of this paper we assume that there exists a random variable (r.v.) Λ with a given d.f. such that, given $\Lambda = \lambda$, the conditional marginal d.f.'s of $\underline{X}$ are available; this for all possible values of λ . Notice that, unless even more information on the dependence structure is available, conditionally upon $\Lambda = \lambda$ any joint d.f. with marginals $F_{X_1|\Lambda}(x_1|\lambda), \dots, F_{X_n|\Lambda}(x_n|\lambda)$ is possible.

Consider the following simple example:

Example 3.1 *Let $S = X_1 + \dots + X_n$ represent the stochastically discounted sum of running year losses of n non-life portfolios that an insurer holds. We assume that any X_i can be decomposed into an (unhedgeable) “insurance risk” Y_i and a common (hedgeable) “financial risk” Λ , such that for all $i = 1, \dots, n$, $X_i = \Lambda Y_i$. Suppose that Λ with given d.f. F_Λ is independent of $\underline{Y}$ with given marginal d.f.'s $F_{Y_1}, \dots, F_{Y_n}$, and that the dependence structure within $\underline{Y}$ is unknown. We aim to determine a lower bound on $\mathbb{P}[S < s]$.*

We introduce the notions of *stop-loss* and *supermodular* order. We say that a r.v. X is smaller than a r.v. Y in *stop-loss* order if for any non-decreasing and convex function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)],$$

provided that the expectations exist. Furthermore, we say that a random vector $\underline{X}$ is smaller than a random vector $\underline{Y}$ in *supermodular* order if for any supermodular function $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\mathbb{E}[f(\underline{X})] \leq \mathbb{E}[f(\underline{Y})],$$

provided that the expectations exist. We write $X \leq_{\text{sl}} Y$ and $\underline{X} \leq_{\text{sm}} \underline{Y}$. As is well-known (see, for example, Proposition 6.3.7 of Denuit *et al.* (2005))

$$\underline{X}^C \leq_{\text{sm}} \underline{X}^M. \tag{5}$$

Then, we state the following theorem, which extends Proposition 2 of Kaas, Dhaene & Goovaerts (2000) where only the case of $\psi = +$ is considered:

Theorem 3.1 *Let U be a r.v. uniformly distributed on $(0, 1)$ and independent of Λ . Then, for any non-decreasing supermodular function ψ*

$$\begin{aligned}\psi(X_1, \dots, X_n) &\leq_{\text{sl}} \psi(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda)) \\ &\leq_{\text{sl}} \psi(F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U)).\end{aligned}\tag{6}$$

Proof: It is well-known that for any non-decreasing convex function f , $f \circ \psi$ is supermodular; see, for example, Proposition 3.4.67 of Denuit *et al.* (2005). Hence, recalling (5),

$$\begin{aligned}\mathbb{E}[f \circ \psi(X_1, \dots, X_n)] &= \int_{\Lambda} \mathbb{E}[f \circ \psi(X_1, \dots, X_n) | \Lambda = \lambda] dF_{\Lambda}(\lambda) \\ &\leq \int_{\Lambda} \mathbb{E}[f \circ \psi(F_{X_1|\Lambda}^{-1}(U|\lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\lambda))] dF_{\Lambda}(\lambda) \\ &= \mathbb{E}[f \circ \psi(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))].\end{aligned}$$

This proves the first inequality in (6). Furthermore, since $(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))$ has marginals $F_{X_1}, \dots, F_{X_n}$ it also follows from (5) and the supermodularity of $f \circ \psi$ that

$$\mathbb{E}[f \circ \psi(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))] \leq \mathbb{E}[f \circ \psi(F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U))].$$

This proves the second inequality in (6). □

Remark 3.1 *The random vector $(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))$ is often referred to as the improved comonotonic dependence structure. Note that given Λ , for any non-decreasing supermodular function ψ , $\psi(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))$ is the best-possible upper bound on $\psi(\underline{X})$ in stop-loss order sense.*

Remark 3.2 *Notice that if $\underline{X}$ is comonotonic, then for any Λ , it holds that $\underline{X} =^d (F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda)) =^d \underline{X}^M$.*

Henceforth, we denote by M_{Λ} the (or, in case of non-uniqueness, an) n -copula for which $M_{\Lambda}(F_{X_1}(x_1), \dots, F_{X_n}(x_n))$ is the joint distribution function of the random vector $(F_{X_1|\Lambda}^{-1}(U|\Lambda), \dots, F_{X_n|\Lambda}^{-1}(U|\Lambda))$. Note that contrary to M , M_{Λ} may change when $\underline{X}$ changes.

4 Main Results

This section extends Lemma 2.1 to the case of additional information on the dependence structure in the sense as formalized in the previous section. Given $F_{X_1}, \dots, F_{X_n}, F_\Lambda$, and $F_{X_1|\Lambda}, \dots, F_{X_n|\Lambda}$, we define the family $\mathcal{C}_\Lambda^n$ of n -copulas as follows:

$$\mathcal{C}_\Lambda^n := \left\{ C \in \mathcal{C}_n \mid C(F_{X_1}(x_1), \dots, F_{X_n}(x_n)) = \int_\Lambda C_\lambda(F_{X_1|\Lambda}(x_1|\lambda), \dots, F_{X_n|\Lambda}(x_n|\lambda)) dF_\Lambda(\lambda) \right. \\ \left. \text{for every } \underline{x} \in \mathbb{R}^n \text{ and with } C_\lambda \in \mathcal{C}^n \right\}. \quad (7)$$

Consider the following example:

Example 4.1 *Let Λ be a Bernoulli r.v. defined by*

$$\Lambda = \begin{cases} 0, & \text{with probability } 1/2; \\ 1, & \text{with probability } 1/2. \end{cases}$$

Furthermore, let $X_1 = \Lambda$ and $X_2 = 1_{\{\Lambda=0\}}$ so that conditionally upon Λ , X_1 and X_2 are degenerate. One easily verifies that in this case $M \notin \mathcal{C}_\Lambda^n$, since $C_0(F_{X_1|\Lambda}(0|0), F_{X_2|\Lambda}(0|0)) = C_1(F_{X_1|\Lambda}(0|1), F_{X_2|\Lambda}(0|1)) = 0$ for any $C_0, C_1 \in \mathcal{C}^2$, while $M(F_{X_1}(0), F_{X_2}(0)) = 1/2$.

For a given copula C , we introduce the loss function

$$e_{C,\psi,\Lambda}(s) := \mathbb{P}[\psi(\underline{X}^C) < s] - \inf_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}. \quad (8)$$

Clearly, because $\mathcal{C}_\Lambda^n \subset \mathcal{C}^n$,

$$\inf_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\} \geq \inf_{\tilde{C} \in \mathcal{C}^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\}.$$

We consider the following optimization problem:

$$\inf_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} e_{C,\psi,\Lambda}(s) ds \right\}. \quad (9)$$

Then, we state the main theorem:

Theorem 4.1 *For every real number d and every non-decreasing supermodular function ψ satisfying $\mathbb{E}[\psi(\underline{X}^{M_\Lambda})] < +\infty$, M_Λ is a minimizer of (9).*

Proof: Following the proof of Theorem 15 of Embrechts, Höing & Puccetti (2005), we write:

$$\begin{aligned}
\inf_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} e_{C, \psi, \Lambda}(s) ds \right\} &= \inf_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} \left(\mathbb{P}[\psi(\underline{X}^C) < s] - \inf_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) < s] \right\} \right) ds \right\} \\
&= - \sup_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} \left(\mathbb{P}[\psi(\underline{X}^C) \geq s] - \sup_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) \geq s] \right\} \right) ds \right\} \\
&= \int_d^{+\infty} \sup_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) \geq s] \right\} ds - \sup_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} \mathbb{P}[\psi(\underline{X}^C) \geq s] ds \right\} \\
&= \int_d^{+\infty} \sup_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) \geq s] \right\} ds - \sup_{C \in \mathcal{C}_\Lambda^n} \left\{ \int_d^{+\infty} \mathbb{P}[\psi(\underline{X}^C) > s] ds \right\} \\
&= \int_d^{+\infty} \sup_{\tilde{C} \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{P}[\psi(\underline{X}^{\tilde{C}}) \geq s] \right\} ds - \sup_{C \in \mathcal{C}_\Lambda^n} \left\{ \mathbb{E}[(\psi(\underline{X}^C) - d)_+] \right\}.
\end{aligned}$$

Since $(\psi(\underline{x}) - d)_+$ is supermodular, recalling Theorem 3.1 presented above,

$$\mathbb{E}[(\psi(\underline{X}^C) - d)_+] \leq \mathbb{E}[(\psi(\underline{X}^{M_\Lambda}) - d)_+],$$

which proves the stated result. $\square$

Remark 4.1 M_Λ is best-possible in (9).

5 Conclusion

Consider a vector of risks of which the marginal distributions are known but the dependence structure is unknown. Suppose that there exists a common risk factor, with a given distribution function, conditionally upon which the marginal distributions of the vector of risks are known. Then there are two possible reasons to use the improved comonotonic dependence structure as introduced by Kaas, Dhaene & Goovaerts (2000) and Dhaene *et al.* (2002a):

1. as an approximation of the real dependence structure (see Dhaene *et al.* (2002b) and many subsequent papers), also when the dependence structure is in fact known but the real d.f. under study is of a complicated form, by choosing the common risk factor as “smart” as possible. For particular (light-tailed¹) cases encountered in practice this has proven to work well.

¹Laeven, Goovaerts & Hoedemakers (2005) show that in the presence of heavy tails, other approximation methods may be preferred.

2. as a worst case scenario since, as we prove in this note, it is the most adverse dependence structure in stop-loss and supermodular order and the on-average-most-adverse dependence structure in VaR-based risk management.

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