Quadratic Forms, Generalized Hamming Weights of Codes and Curves with Many Points

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Words in certain binary trace codes correspond to algebraic curves over finite fields of characteristic 2 and correspond to quadratic forms over $\mathbb{F}_2$. Via the relation with quadratic forms we construct subcodes in which all non-zero words have minimum weight. We show how the relation between quadratic forms, codes and curves can be used to construct curves with many rational points.

INTRODUCTION

In the interaction between algebraic geometry and classical coding theory the correspondence between words in trace codes $\mathcal{C}$ and Artin–Schreier curves over finite fields is a central theme. This correspondence can be extended to a relation between subcodes of $\mathcal{C}$ and fibre products of Artin–Schreier curves. As a consequence weights of words and subcodes are related to numbers of rational points on curves, where low weight words and subcodes yield curves with many rational points. Subcodes in which all non-zero words have minimum weight lead to curves with a large number of points compared to known upper bounds.

The most general upper bound for the number of rational points $\# C(\mathbb{F}_q)$ on a smooth projective curve $C$ of genus $g$ over a finite field with $q$ elements $\mathbb{F}_q$, is

$$\# C(\mathbb{F}_q) \leq q + 1 + g[2 \sqrt{q}]$$  \hspace{1cm} (1)
which is Serre’s improved form of the Hasse–Weil upper bound (see [S]).

Trace codes which are subcodes of (binary) second-order Reed–Muller
codes have proved to be successful in the construction of curves with many
points. The words in these trace codes correspond not only to algebraic
curves but also to quadratic forms over \( \mathbb{F}_2 \). Using the well known theory
of quadratic forms we shall develop a simple tool to construct subcodes in
such trace codes consisting of minimum weight words, by which we obtain
curves carrying many rational points.

A slightly modified construction could be applied to trace codes over \( \mathbb{F}_p \)
with odd prime \( p \), but we will restrict to the binary codes.

We shall compare our curves with the curves in the table of Wirtz [W].
This table lists for many small fields \( \mathbb{F}_q \) and genera \( g \) an interval \([A, B]\)
where \( A \) means that there exists a curve \( C \) of genus \( g \) over \( \mathbb{F}_q \) with \( A \)
rational points, while \( B \) is an upper bound. In many cases we can improve
the known lower bound “\( A \)” substantially, and in some cases we even
attain the theoretical optimum “\( B \)”. For instance for \( q \) a power of 4 and
certain genera between \( \sqrt{q} \) and \( \sqrt{q} - 1 \) we exhibit curves attaining the
Hasse–Weil upper bound (Prop. 5.2 (ii)).

The paper is organized as follows. In Section 1 we recall elementary facts
on fibre products of curves and generalized Hamming weights of codes and
we introduce the codes we shall consider. In Section 2 the relation between
the codes and quadratic forms is explained. Then we describe the construc-
tion of low weight subcodes in case \( q = 2^m \) with \( m \) odd in Section 3 and in
Section 4 we give applications of the construction. In the last Section we
discuss the case \( q = 2^m \) with \( m \) even.

1. Fibre Products of Curves and Generalized Hamming Weights

In a series of papers [GV-2, 3, 4] we showed that there is a relation
between generalized Hamming weights of trace codes and the number of
rational points on fibre products of curves associated to these trace codes.
We used that relation to construct curves with many rational points. In this
section we recall some of the ingredients.

Let \( \mathbb{F}_q \) be a finite field of cardinality \( q = 2^m \). For \( 0 < h \leq \left\lfloor m/2 \right\rfloor \) we
consider the finite dimensional \( \mathbb{F}_q \)-vector space of \( 2 \)-linearized polynomials

\[
R_h = \left\{ R = \sum_{i=0}^{h} a_i x^{2^i} : a_i \in \mathbb{F}_q \right\}.
\]
The vector space $R_h$ defines a binary linear code $C_h$ of length $q - 1$ and dimension $m(h + 1)$ (resp. $(m + 1)/2$) for $h < [m/2]$ (resp. $h = [m/2]$) by applying the trace map $\text{Tr}$ from $\mathbb{F}_q$ to $\mathbb{F}_2$:

$$C_h = \{ c_R = (\text{Tr}(xR(x)))_{x \in \mathbb{F}_q^*} : R \in R_h \}.$$

The code $C_h$ is a subcode of the punctured binary second-order Reed-Muller code and for $h = 1, 2$ the codes $C_h$ are the duals of the 2- and 3-error correcting BCH codes of length $q - 1$.

To a non-zero word $c_R$ of $C_h$ we associate a non-singular projective curve $C_R$ with affine equation:

$$y^2 + y = xR(x).$$

It is an Artin-Schreier cover of $\mathbb{P}^1$. If $a_0 \neq 0$ the curve $C_R$ has genus $2^h - 1$ and the weight $w(c_R)$ of the word $c_R$ is related to the number of $\mathbb{F}_q$-rational points on $C_R$ by the equation

$$w(c_R) = q - (\# C_R(\mathbb{F}_q) - 1)/2.$$

If $\mathcal{D} \subset C_h$ is an $r$-dimensional subcode we associate a curve $C^{(\mathcal{D})}$ to $\mathcal{D}$ as follows. Choose an $\mathbb{F}_2$-basis $c_R_1, \ldots, c_R_r$ of $\mathcal{D}$ and set

$$C^{(\mathcal{D})} = \text{Normalization of } C_{R_1} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 C_{R_r},$$

where the fibre product is taken with respect to the canonical projections $\varphi_R : C_{R_i} \to \mathbb{P}^1$ with $(x, y) \mapsto x$. Up to isomorphism this does not depend on the chosen basis.

In [G-V2] we proved the relation

$$t_{\text{Frob}}(C^{(\mathcal{D})}) = \sum_{c_R \in \mathcal{D} - \{0\}} t_{\text{Frob}}(C_R),$$

(2)

between the traces of Frobenius of the curves involved. Here the trace of Frobenius for a curve $C$ over $\mathbb{F}_q$ is defined as $t_{\text{Frob}} = q + 1 - \# C(\mathbb{F}_q)$. Furthermore, we have the relation for the genera $g$ of the curves involved:

$$g(C^{(\mathcal{D})}) = \sum_{c_R \in \mathcal{D} - \{0\}} g(C_R).$$

(3)

An important parameter of a subcode $\mathcal{D}$ of a code $\mathcal{C}$ is its weight $w(\mathcal{D})$, by which we mean the number of coordinate places for which at least one word of $\mathcal{D}$ has a non-zero coordinate. In the binary case we have the relation

$$w(\mathcal{D}) = \frac{1}{2^{r-1}} \sum_{d \in \mathcal{D}} w(d).$$
The \( r \)th generalized Hamming weight \( d_r(C) \) of \( C \) is defined by
\[
d_r(C) = \min \{ w(D) : D \text{ is an } r\text{-dimensional subcode of } C \}. \]

If there exists an \( r \)-dimensional subcode in which all non-zero words have minimum weight, which we shall call a minimum weight subcode, then
\[
d_r(C) = (2^r - 1) d_1(C)/2^{r-1}. \tag{4} \]

In our case where \( C = C_h \) we recall the following proposition from [G-V2].

(1.1) Proposition. The weight \( w(D) \) of the \( r \)-dimensional subcode \( D \subseteq C_h \) satisfies
\[
w(D) = q - (C(D)(F_q) - 1)/2^r. \]

We immediately see that subcodes of low weight correspond to fibre products with many points.

2. Quadratic Forms and Codes

To a non-zero word \( c \neq 0 \) of \( C_h \) (or to \( R \in R_h - \{0\} \)) we can associate not only a curve but also a quadratic form. Indeed, the expression
\[
Q_{a,h}(x) = \text{Tr}(xR(x)),
\]
defines on the \( F_2 \)-vector space \( F_q, 2^m \) a quadratic form over \( F_2 \) in \( m \) variables. In the sequel we denote \( Q_{a,h}(x) \) simply by \( Q(x) \). The associated symmetric bilinear form
\[
B(x, y) = \text{Tr}(xR(y) + yR(x)),
\]
which is also symplectic, has radical
\[
W = \{ x \in F_q : B(x, y) = 0 \text{ for all } y \in F_q \}. \]

This \( F_2 \)-vector space \( W \) has dimension \( w \) with \( m - w \equiv 0 \pmod{2} \) since \( B \) is symplectic. If \( a \neq 0 \) then \( 0 \leq w \leq 2h \) (see [G-V1]). Quadratic forms \( Q \) and \( Q' \) in \( F_2[x_1, \ldots, x_m] \) which can be obtained from each other by a coordinate transformation are called equivalent quadratic forms, denoted by \( Q \sim Q' \). We define the rank of \( Q \) by
\[
\text{rk}(Q) = \min_{Q' \sim Q} \{ \text{number of variables actually occurring in } Q' \}. \]
In characteristic 2 the quadratic form $Q(x)$ is not necessarily zero on the radical $W$. Therefore we introduce

$$W_0 = \{ x \in W : Q(x) = 0 \}.$$ 

This is a vector space of dimension $w_0$ with $w_0 = w$ or $w_0 = w - 1$, see [G-V1]. The theory of quadratic forms in characteristic 2 (see [A]) yields

(2.1) **Proposition.** The quadratic form $Q(x) = \text{Tr}(xR(x))$, with $R \in R_k - \{0\}$, has rank $m - w_0$.

Over $\mathbb{F}_2$ the classification of quadratic forms $Q(x)$ is strongly related to the number of zeros $N(Q) = \# \{ x \in \mathbb{F}_q : Q(x) = 0 \}$ (see [L-N, Chap. 6]). Indeed, we have:

(2.2) **Proposition.** If $Q(x)$ has rank $m - w - 1$ (odd) then

$$Q(x) \sim X_1 X_2 + \cdots + X_{m-w-1} X_{m-w} + X_{m-w+1}^2$$

and $N(Q) = 2^{w-1} \cdot 2^{m-w} = q/2$.

If $Q(x)$ has rank $m - w$ (even) then either

$$Q(x) \sim X_1 X_2 + \cdots + X_{m-w-1} X_{m-w}$$

and $N(Q) = 2^w (2^{m-w-1} + 2^{(m-w-2)/2}) = (q + \sqrt{q^2w})/2$,

or

$$Q(x) \sim X_1 X_2 + \cdots + X_{m-w-1} X_{m-w} + X_{m-w-1}^2 + X_{m-w}^2$$

and $N(Q) = 2^w (2^{m-w-1} - 2^{(m-w-2)/2}) = (q - \sqrt{q^2w})/2$.

We have simple relations for the weight of a word $c_R \in \mathscr{C}_h$ and the number of rational points on the corresponding curve:

$$w(c_R) = q - N(Q_R) \quad \text{and} \quad \# C_R(\mathbb{F}_q) = 2N(Q_R) + 1. \quad (5)$$

(2.3) **Remark.** Words in $\mathscr{C}_h$ with $w = 2h$ or $w = 2h - 1$ correspond to $R$ of degree $2^h$, which means that the corresponding curves have genus $2^h - 1$.

3. **THE CONSTRUCTION OF LOW WEIGHT SUBCODES OF $\mathscr{C}_h$ FOR ODD $m$**

In this section we take $q = 2^m$ with $m$ odd, $m \geq 3$. To find minimum weight subcodes in $\mathscr{C}_h$ with $0 < h \leq (m-1)/2$ we exploit the following simple observation.
(3.1) **Proposition.** For \(0 \leq w \leq 2h-1\) with \(m-w \equiv 0 \pmod{2}\) the expression

\[
\sum_{i=1}^{(m-w)/2} \text{Tr}(a_i x) \text{Tr}(b_i x)
\]  

(6)

with \(\{a_i, b_i : i = 1, \ldots, (m-w)/2\} \subset \mathbb{F}_q\) independent over \(\mathbb{F}_2\), is a quadratic form which is equivalent to \(X_1 X_2 + \cdots + X_{m-w-1} X_{m-w}\) and which has \((q + \sqrt{q^2})/2\) zeros in \(\mathbb{F}_q\).

**Proof.** Since \(\text{Tr}(ax) \in \mathbb{F}_q[x_1, \ldots, x_m]\) is a linear form, (6) is indeed a quadratic form. The \(\mathbb{F}_2\)-independence of \(\{a_i, b_i\}\) implies that the matrix of the transformation \(X_{m-w-1} = \text{Tr}(a_i x)\) and \(X_{m-w} = \text{Tr}(b_i x)\) for \(i = 1, \ldots, (m-w)/2\) has rank \(m-w\). This matrix can be completed to a coordinate transformation in the \(m\)-dimensional \(\mathbb{F}_2\)-vector space \(\mathbb{F}_q^m\) which yields the equivalence. The number of zeros follows from Proposition (2.2).

We shall denote the quadratic form (6) by \(Q(a_1, \ldots, a_{(m-w)/2}, b_1, \ldots, b_{(m-w)/2})\) or simply by \(Q(a, b)\). The form \(Q(a, b)\) evaluated on \(\mathbb{F}_q^*\) induces a word of length \(q-1\) and often we shall identify the word and the form. If the \(\mathbb{F}_2\)-rank of the set

\[\{a_1, \ldots, a_{(m-w)/2}, b_1, \ldots, b_{(m-w)/2}\},\]

which we shall also shortly denote by \(\{a, b\}\), is \(m-w\) then \(Q(a, b)\) has weight \((q - \sqrt{q^2})/2\).

First we derive a criterion for words induced by \(Q(a, b)\) to be in \(\mathcal{C}_q\).

(3.2) **Proposition.** If the elements \(a_i, b_i \in \mathbb{F}_q\) with \(1 \leq i \leq (m-w)/2\) satisfy the system of equations

\[
\sum_{i=1}^{(m-w)/2} (a_{2^h i}^2 b_i + a_i b_{2^h i}^2) = 0
\]

(7)

for \(j = h+1, \ldots, (m-1)/2\), then the word of length \(q-1\) induced by evaluating \(\sum_{i=1}^{(m-w)/2} \text{Tr}(a_i x) \text{Tr}(b_i x)\) on \(\mathbb{F}_q^m\) is a codeword in \(\mathcal{C}_q\).

**Proof.** Observe that for \(a, b \in \mathbb{F}_q\) we have

\[
\text{Tr}(ax) \text{Tr}(bx) = \text{Tr}(\text{Tr}(ax) bx) = \text{Tr} \left( \sum_{j=0}^{m-1} a_{2^j} bx^{2^j+1} \right).
\]

The trace map on \(\mathbb{F}_q\) satisfies

\[
\text{Tr}(a_{2^j} bx^{2^j+1}) = \text{Tr}(abx^{2^j}x^{2^j+1})
\]
which implies that
\[
\text{Tr}(ax) \text{Tr}(bx) = \text{Tr} \left( \sum_{j=0}^{(m-1)/2} (a^{2^j} b + a^{2^j} x^{2^j+1}) \right).
\]
The argument of \(\text{Tr}\) must be of the form \(xR(x)\) with \(R \in R_a\), hence we do not want terms \(x^{2^j+1}\) with \(j > h\) to appear in the argument of \(\text{Tr}\). This condition yields the system of equations (7).

(3.3) Corollary. For fixed \(a_1, ..., a_{(m-w)/2}\) the words \(Q(a,b)\) induced by the solutions \((b_1, ..., b_{(m-w)/2})\) of (7) form a subcode of \(\mathcal{G}_h\).

Proof. From the linearized character of (7) in the \(b_i\) we immediately see that the set of solutions \((b_1, ..., b_{(m-w)/2}) \in F_q^{(m-w)/2}\) is a \(F_2\)-subspace of \(F_q^{(m-w)/2}\).

For the sequel we fix a \(F_2\)-independent subset \(\{a_i : 1 \leq i \leq (m-w)/2\}\) of \(F_q\) and we may assume \(a_1 = 1\), which is not a restriction because the equations in (7) are homogeneous. Furthermore, we assume that the system of equations is not empty. Since we are especially interested in subcodes consisting of minimum weight words we take \(w = 2h - 1\) and we put \((m-w)/2 = (m-2h+1)/2 = M\). In this situation we have \(M-1\) equations with \(M\) unknowns \(b_i\).

(3.4) Proposition. For fixed \(a_1, a_2, ..., a_M\) in \(F_q\) the system of equations (7) has at least \(q\) solutions \((b_1, b_2, ..., b_M)\) in \(F_q^M\).

Proof. If we take an equation \(b_1^{2^j} + b_1 = \sum_{i=2}^{M} a_i b_i^{2^j} + a^{2^j} b_i\) from (7) and if we apply the transformation \(b_1 = b_i^{2^j} + \sum_{i=2}^{M} a_i b_i\), then this equation becomes
\[
b_i^{2^j} + b_i = \sum_{i=2}^{M} a_i^{2^j} + 2^j \sqrt{a_i} b_i. \tag{8}
\]
Note that (8) has \(q^{M-1}\) solutions if at least one of the coefficients of the \(b_i\) on the right side is non-zero. Otherwise the number of solutions depends on the number of solutions of \(X^{2^j} + X = 0\) in \(F_q\) and is at least \(2q^{M-1}\). So the dimension of the \(F_2\)-vector space of solutions of an equation from (7), with \(mM\) unknowns over \(F_2\), is at least \(m(M-1)\). Since we have \(M-1\) equations our assertion follows.

The actual number of solutions of our system depends on the choice of the \(a_i\) as we shall see in examples in the next section. To solve the system (7) we repeatedly carry out transformations as described in the proof of Proposition (3.4) and then finally we arrive at an equation of the form
\[
S(y) = R(x),
\]
where $S$ and $R$ are $F_2$-linearized polynomials with coefficients in $F_q$. This equation can be solved by lowering the degree step by step. If degree $(S) \leq (R)$ we start with the substitution $y = y' + f(x)$ with a suitable $2$-linearized polynomial $f$. As long as the resulting polynomials are not zero we continue the descent until we arrive at a linear equation in two unknowns which has $q$ solutions.

To obtain words of minimum weight in $C_h$ we must have solutions $(b_1, \ldots, b_M)$ of (7) where $\text{rk}_{F_2}(\{a, b\}) = 2M$.

(3.5) \textbf{Proposition.} Let $(b_1, \ldots, b_M)$ be a solution of (7). Then we have:

\[
\text{rk}_{F_2}(\{a_1, \ldots, a_M, b_1\}) = M + 1
\]

implies

\[
\text{rk}_{F_2}(\{a_1, \ldots, a_M, b_1, \ldots, b_M\}) = 2M.
\]

\textbf{Proof.} Denote $\sum_{i=1}^M \text{Tr}(a_i x) \text{Tr}(b_i x) \in C_h$ by $\sum_{i=1}^M X_{2i-1}X_2$. Assume that $\{a, b\}$ has $F_2$-rank $< 2M$ with $b_M \in \langle a_1, \ldots, a_M, b_1, \ldots, b_{M-1} \rangle$, the $F_2$-subspace of $F_q$ generated by the elements between the pointed brackets. This implies $X_{2M} = \sum_{i=1}^M X_i$ with $X_i \in F_2$ not all zero. Then

\[
\sum_{i=1}^M X_{2i-1}X_2 = \sum_{i=1}^{M-1} (X_{2i-1} + \sigma_2 X_{2M-1})(X_2 + \sigma_{2i-1}X_{2M-1})
\]

\[
+ (\sigma_{2M-1} + \sum_{i=1}^{M-1} \sigma_{2i-1} \sigma_2) X_{2M-1}^2.
\]

Note that $X_{2M-1} = (\text{Tr}(a_M x))^2 = (\text{Tr}(a_M x))^2 \in C_h$ and so

\[
\sum_{i=1}^M (X_{2i-1} + \sigma_2 X_{2M-1})(X_2 + \sigma_{2i-1}X_{2M-1})
\]

\[
= \sum_{i=1}^{M-1} \text{Tr}(a_i x) \text{Tr}(b_i x) \in C_h.
\]

Here $a_i^{(1)} = a_i + \sigma_2 a_M$ and $b_i^{(1)} = b_i + \sigma_{2i-1} a_M$ for $1 \leq i \leq M - 1$ while

\[
\text{rk}_{F_2}(\{a_i^{(1)}, b_i^{(1)}; 1 \leq i \leq M - 1\}) \leq 2M - 2.
\]

If $\text{rk}_{F_2}(\{a_i^{(1)}, b_i^{(1)}; 1 \leq i \leq M - 1\}) = 2M - 2$ we have found a word in $C_h$ of weight $(q - \sqrt{q^{2M+1}})/2$ which is smaller than the minimum weight. On the other hand if the $F_2$-rank of $\{a_i^{(1)}, b_i^{(1)}; 1 \leq i \leq M - 1\} < 2M - 2$ we repeat the foregoing procedure. In the end we arrive at a non-zero word

\[
\text{Tr}(a_i + L_1(a_2, \ldots, a_m) x) \text{Tr}(b_i + L_2(a_2, \ldots, a_m) x) \in C_h,
\]

with $F_2$-linear combinations $L_1$ and $L_2$ of $a_2, \ldots, a_m$, which also violates the minimum weight.  

\[\square\]
For fixed $a_i$ we let $S$ be the $F_2$-vector space of solutions $(b_1, ..., b_M)$ of (7) and $V$ the image in $F_q$ of the projection of $S$ on the first coordinate $b_1$.

(3.6) **Theorem.** If $r = \dim_{F_2}(V) - M > 0$ then there exists a minimum weight subcode of $G_0$ of dimension $r$.

**Proof.** Since $r = \dim_{F_2}(V) - M > 0$ we can find $b_1^{(1)}, ..., b_r^{(r)}$ in $V$ such that

$$\rk_{F_2}(\{a_1, ..., a_M, b_1^{(1)}, ..., b_r^{(r)}\}) = M + r.$$ 

By Proposition (3.5) the corresponding solutions $(b_1^{(1)}, b_2^{(1)}, ..., b_M^{(1)})$ for $i = 1, ..., r$ induce words of minimum weight in $G_0$. These words generate an $r$-dimensional subcode of minimum weight.

From Proposition (3.4) we derive that $\dim_{F_2}(S) \geq m$. The following proposition provides us with a lower bound for $\dim_{F_2}(V)$.

(3.7) **Proposition.** Let $(b_1, b_2, ..., b_M) \in S$ then $(b_1, b_2, ..., b_M) \in S$ if and only if $(b_2, ..., b_M) = (b_2, ..., b_M) + (a_2, ..., a_M) A$, where $A$ is a symmetric $(M-1) \times (M-1)$ matrix over $F_2$.

**Proof.** The “if-part” follows immediately by substitution in (7). For the “only if-part” we observe that

$$\sum_{i=2}^{M} \Tr(a_i x) \Tr(b_i^{(i)} x)$$

represents a word in $G_0$. Just as in the proof of Proposition (3.5) this implies that for $i = 2, ..., M$ the $(b_i + b_i)$ are $F_2$-dependent of $\{a_2, ..., a_M\}$ or $b_i + b_i = \sum_{j=2}^{M} \pi_i a_j$.

By substitution in (9) we obtain

$$\sum_{i=2}^{M} \pi_i (\Tr(a_i x))^2 + \sum_{i=2}^{M-1} \Tr(a_i x) \Tr(\sum_{i<j}^{M} \beta_{i,j} a_j) x$$

with $\beta_{i,j} = \pi_i + \pi_j$. The second term in (10) also represents a word in $G_0$. Again to avoid contradiction with the minimum weight in $G_0$ we must have that $\sum_{i<j}^{M} \beta_{i,j} a_j$ is dependent of $a_2, ..., a_{M-1}$ for $i = 2, ..., M$. We then find successively: $\beta_{i,M} = 0$ for $2 \leq i \leq M - 1$, $\beta_{i,M-1} = 0$ for $2 \leq i \leq M - 2$ until $\beta_{23} = 0$. This means that the $(M-1) \times (M-1)$ matrix $A = (\pi_{i,j})$ is symmetric.
Since we can choose $M(M - 1)/2$ entries in $A$ independently we have

(3.8) **Corollary.** The dimension of $V$ over $\mathbb{F}_2$ satisfies

$$\dim_{\mathbb{F}_2}(V) = \dim_{\mathbb{F}_2}(S) - \left(\frac{M}{2}\right) \geq m - \left(\frac{M}{2}\right).$$

In the following section we shall illustrate the ideas and results of this section by some examples.

4. **Determination of Generalized Hamming Weights and Curves with Many Points**

We have

$$q = 2^m \quad \text{with } m \geq 3 \text{ odd},$$

$$0 < h \leq (m - 1)/2,$$

$$w = 2h - 1 \quad \text{and} \quad M = (m - 2h + 1)/2.$$

I. **The Case $h = (m - 1)/2$**

Since the system (7) is empty, Propositions (3.1) and (3.2) imply that Tr($x$) Tr($bx$) $\in C_h$ has minimum weight for $b \in \mathbb{F}_q - \mathbb{F}_2$. By choosing $\mathbb{F}_2$-independent elements $b(1), \ldots, b(m - 1)$ in $\mathbb{F}_q$ we find words which generate a $(m - 1)$-dimensional subcode of $C_h$ of minimum weight. Combining (4) and Proposition (2.2) we find:

(4.1) **Proposition.** The generalized Hamming weights of the binary code $C_{(m - 1)/2}$ of length $2^m - 1$ satisfy

$$d_r(C_{(m - 1)/2}) = (2^r - 1) \cdot 2^{m - r - 1} \quad \text{for } 1 \leq r \leq m - 1.$$  

Applying the fibre product construction from Section 1 to these minimum weight subcodes we get:

(4.2) **Corollary.** For $1 \leq r \leq m - 1$ there exist curves $C_r$ defined over $\mathbb{F}_q = \mathbb{F}_{2^m}$ of genus $g(C_r) = (2^r - 1) \sqrt{q/8}$ and # $C_r(\mathbb{F}_q) = q + 1 + (2^r - 1) q/2$.

**Proof.** The curves corresponding to the non-zero words in the $r$-dimensional minimum weight subcode $D_r$ have genus $g = 2^{h-1} = 2^{(m - 3)/2}$ and according to (5) $f_{\text{Frob}} = q - 2N(\text{Tr}(xR(x))) = -\sqrt{q2^{2m-1}}$. From (2) and (3) we obtain $g(C_r^{(2r)}) = (2^r - 1) 2^{(m - 3)/2}$ and

$$# C_r^{(2r)}(\mathbb{F}_q) = q + 1 + (2^r - 1) \sqrt{q2^{2m-1}} = 2^m + 1 + (2^r - 1) 2^{m - 1}.$$
This result confirms our earlier result [G-V3, Theorem 5].

II. The Case $h=(m-3)/2$

We take $m \geq 5$. In this case $M=2$; fix $\{1, a_2\}$ with $\mathbb{F}_2$-rank 2 in $\mathbb{F}_q$. The system (7) consists of the equation

\[ b_1^{2^{m-1}/2} + b_1 = a_2 b_2^{2^{m-1}/2} + a_2^{2^{m-1}/2} b_2. \]  

(11)

The transformation $b_1 = b_1' + 2^{m-1}/2 \sqrt{a_2} b_2$ changes (11) into

\[ b_1' = b_1^{2^{m-1}/2} + b_1' = (a_2^{2^{m-1}/2} + 2^{m-1}/2 \sqrt{a_2}) b_2. \]  

(12)

The coefficient of $b_2$ in (12) is non-zero because $\{1, a_2\}$ has $\mathbb{F}_2$-rank 2. Hence (11) has $q$ solutions $(b_1, b_2)$ or $\dim_{F_2}(S) = m$. From Corollary (3.8) we deduce $\dim_{F_2}(V) = m-1$ and we find by Theorem (3.6): 

(4.3) Proposition. (i) The generalized Hamming weights of $\mathcal{C}_{(m-3)/2}$ of length $2^m-1$ satisfy

\[ d_r(\mathcal{C}_{(m-3)/2}) = (2^r-1)(2^{m-1} - 2^{m-3})/2^{r-1} \quad \text{for} \quad 1 \leq r \leq m-3. \]

(ii) For $1 \leq r \leq m-3$ there exist curves $C_r$ defined over $\mathbb{F}_{2^r}$, with $m \geq 5$ odd, of genus $g(C_r) = (2^r-1)2^{(m-5)/2}$ and with $\# C_r(\mathbb{F}_{2^r}) = 2^m + 1 + (2^r-1)2^{m-2}$.

Proof. The proof is similar to the proof of Proposition (4.1) and Corollary (4.2).

(4.4) Example. For $m=7$ we have $h=2$ and $\mathcal{C}_2 = BCH(3)^\perp$, the dual of the 3-error correcting BCH code of length 127. The former proposition yields:

- The generalized Hamming weights $d_r$, $1 \leq r \leq 4$, of $BCH(3)^\perp$ of length 127 are
  \[ d_1 = 48, \quad d_2 = 72, \quad d_3 = 84, \quad d_4 = 90. \]

- There exist curves $C_r$ defined over $\mathbb{F}_{128}$ with
  \[
  \begin{array}{c|c|c|c}
  r & g(C_r) & \# C_r(\mathbb{F}_{128}) & \text{Wirtz} \\
  
  \hline
  1 & 225 & 161 & (172) \\
  2 & 353 & 225 \ (225-261) \\
  3 & 609 & 353 \ (289-437) \\
  4 & 1219 & 609 \ (369-789) \\
  \end{array}
  \]
III. The Case $h = (m - 5)/2$

We take $m \geq 7$. In this case $M = 3$; fix $\{1, a_2, a_3\}$ $F_2$-independent in $F_q$.

The system of equations is:

\[
\begin{align*}
(13) & \quad b_1^{2^m - 3^2} + b_1 = \sum_{i=2}^{3} (a_i b_i^{2^m - 3^2} + a_i^{2^m - 3^2} b_i), \\
(14) & \quad b_1^{2^m - 1^2} + b_1 = \sum_{i=2}^{3} (a_i b_i^{2^m - 1^2} + a_i^{2^m - 1^2} b_i).
\end{align*}
\]

Apply the linear transformation $b_1 = b_1 + \sqrt[2]{a_2} b_2 + \sqrt[3]{a_3} b_3$ with $s = 2^m - 1^2$.

Equation (14) becomes linear in $b_2$ and $b_3$ with non-zero coefficients. We express $b_3$ in $b_2$ and $b_3$. Then (13) becomes an equation in $b_1$ and $b_2$ of the form $S(b_1) = R(b_2)$ with $R$ and $S$ $2$-linearized. In general $S(b_1)$ has degree $2^m - 2$ and $R(b_2)$ has degree $2^m - 3^2$ and we find $q$ solutions $(b_1, b_2, b_3)$ (see the remarks after Proposition (3.4)). Just as in the cases I and II we find a proposition on generalized Hamming weights and on curves.

The existence of $q$ solutions is guaranteed by Proposition (3.4). However for special choices of $\{1, a_2, a_3\}$ the number of solutions is a multiple of $q$.

To illustrate this remark we consider the following example.

(4.5) Example. For $m = 9$ and $h = 2$ our system is:

\[
\begin{align*}
(15) & \quad b_1^8 + b_1 = a_2 b_2^8 + a_2^3 b_2 + a_3 b_3^8 + a_3^3 b_3, \\
(16) & \quad b_1^{16} + b_1 = a_2 b_2^{16} + a_2^{16} b_2 + a_3 b_3^{16} + a_3^{16} b_3.
\end{align*}
\]

Now we choose a special $F_2$-independent triple: $\{1, a_2, a_3\} \subset F_8$. Then we can write (15) as

\[
(17) \quad b_1^8 + b_1 = a_2^2 b_2^8 + a_2 b_2 + a_3^2 b_3^8 + a_3 b_3.
\]

The solutions of (17) are $\{ (b_1, b_2, b_3) \in F_8^3 : b_1 + a_2 b_2 + a_3 b_3 = \mu \in F_8 \}$. Take a value $\mu \in F_8$ and substitute $b_1 = \mu + a_2 b_2 + a_3 b_3$ in (16). We find an equation in $b_2$ and $b_3$ which has $q$ solutions $(b_2, b_3)$, so the original system has $8q$ solutions and $\dim F_2(S) = 12$. Then Corollary (3.8) and Theorem (3.6) yield:

- The generalized Hamming weights $d_r$ of $BCH(3)$ of length 511 satisfy

\[
d_r = (2^r - 1) d_1 / 2^{r-1} = 224(2^r - 1) / 2^{r-1} \quad \text{for} \quad 1 \leq r \leq 6.
\]
There exist curves $C_r$ defined over $F_{512}$ with

<table>
<thead>
<tr>
<th>$r$</th>
<th>$g(C_r)$</th>
<th>$# C_r(F_{512})$</th>
<th>Upper bound (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>577</td>
<td>603</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>705</td>
<td>783</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>961</td>
<td>1143</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>1473</td>
<td>1863</td>
</tr>
</tbody>
</table>

Combination of the cases also produce good curves as we shall show by an example.

(4.6) Example. Take $m = 7$ and fix $F_2$-independent $\{1, a_2\}$ in $F_{128}$. By Proposition (4.3) in case II (i.e. with $h = (m - 3)/2$) there exists a word of minimum weight (viz., 48) in $G_3$ of the form $\text{Tr}(x) \text{Tr}(b_1 x) + \text{Tr}(a_2 x) \text{Tr}(b_2 x)$. Now $c_1 = \text{Tr}(x) \text{Tr}(b_1 x)$ and $c_2 = \text{Tr}(a_2 x) \text{Tr}(b_2 x)$ are words in $G_3$ since in the corresponding $\text{Tr}(x R(x))$ the degree of $x R(x)$ is equal to $2^{(m - 1)/2} + 1 = 9$. From Proposition (2.2) it follows that $c_1$ and $c_2$ have weight 32. The curves corresponding to $c_1$, $c_2$ and $c_1 + c_2$ have genus 4, 4, 2 respectively and trace of Frobenius $-64$, $-64$, $-32$ respectively. So the fibre product curve corresponding to the subcode of $G_3$ generated by $c_1$ and $c_2$ is a curve over $F_{128}$ of genus $g = 10$ and $129 + 160 = 289$ rational points. The upper bound (1) in this case is 349.

Applying the same approach to a word

$$\text{Tr}(x) \text{Tr}(b_1 x) + \text{Tr}(a_2 x) \text{Tr}(b_2 x) + \text{Tr}(a_3 x) \text{Tr}(b_3 x)$$

of minimum weight in $G_4$ over $F_{128}$ one finds a curve of genus $g = 9$ with 241 rational points (Wirtz: 209–327).

5. The Construction of Low Weight Subcodes of $G_h$ for Even $m$

For even $m$ the situation is somewhat more complicated as we shall explain in this section.

We take $q = 2^m$ with $m$ even, $m \geq 4$ and we consider the code $G_h$ with $0 < h < m/2$. Words in $G_h$ correspond to quadratic forms with $0 \leq w \leq 2h$ and $m - w \equiv 0 \pmod{2}$.

In the same way as in Proposition (3.2) we find a system of equations as condition for $\sum_{j=1}^{(m-3)/2} \text{Tr}(a, x) \text{Tr}(b, x)$ to induce a word in $G_h$. For $j = h + 1, \ldots, (m - 2)/2$ the equations are of the same form as in (7) but for
If \( j = m/2 \) we find a term \( a^{2m^2}b^{x^{2m^2}+1} \) in the expansion of \( \text{Tr}(ax) \text{Tr}(bx) \) which we want to neglect. To achieve that we could require

\[
\sum_{i=1}^{(m-w)/2} a_i^{2m^2} b_i = 0.
\]

However, note that \( \text{Tr}(x^{2m^2}+1) = 0 \) for \( x \in \mathbb{F}_q \) since \( x^{2m^2}+1 \in \mathbb{F}_q \). So we can also disregard the term with \( x^{2m^2}+1 \) if we require \( \sum_{i=1}^{(m-w)/2} a_i^{2m^2} b_i \in \mathbb{F}_q \) or if we require \( \sum_{i=1}^{(m-w)/2} (a_i^{2m^2} b_i + a_i b_i^{2m^2}) = 0 \) which is less restrictive than \( \sum_{i=1}^{(m-w)/2} a_i^{2m^2} b_i = 0 \) and of the same form as the other equations. For the sake of completeness we formulate the counterpart of Proposition (3.2).

\[(5.1) \ \text{Proposition.} \ \text{If the elements } a_i, b_i \in \mathbb{F}_q \text{ with } 1 \leq i \leq (m-w)/2 \text{ satisfy the system of equations}
\[
\sum_{i=1}^{(m-w)/2} (a_i^{2m^2} b_i + a_i b_i^{2m^2}) = 0
\]

for \( j = h + 1, \ldots, m/2 \), then \( \sum_{i=1}^{(m-w)/2} \text{Tr}(a,x) \text{Tr}(b,x) \) induces a codeword in \( \mathcal{C}_h \).

Words of minimum weight in \( \mathcal{C}_h \) correspond to quadratic forms with \( w = 2h \) and we set \( M = (m-2h)/2 \). Furthermore, we fix a \( \mathbb{F}_2 \)-independent subset \( \{a_i; 1 \leq i \leq M\} \subset \mathbb{F}_q \), and if we wish we may assume that \( a_1 = 1 \). The situation is complicated by the fact that (18) has \( M \) equations in \( M \) unknowns \( b_i \) hence we cannot guarantee the existence of \( q \) solutions \( (b_1, \ldots, b_M) \) as in the odd case. By successively applying linear transformations to (18) we finally arrive at an equation of the form \( S(x) = 0 \) with \( S \) a \( 2 \)-linearized polynomial. The actual number of solutions of (18) depends on the choice of the subset \( \{a_i; 1 \leq i \leq M\} \).

To find generalized Hamming weights of \( \mathcal{C}_h \) and curves with many points we employ the same strategy as indicated in Section 3. The reader can easily convince himself that for even \( m \) we have the analogues of (3.5), (3.6), (3.7) and (3.8). We shall illustrate the strategy by working out some examples.

1. The Case \( h = (m-2)/2 \)

The system (18) consists of only one equation:

\[
b_1^{2m^2} + b_1 = 0
\]

which has \( 2^{m-2} \) solutions or \( \dim_f(S) = m/2 \). We find by Theorem (3.6): 

\[(5.2) \ \text{Proposition.} \ \text{(i)} \ The \ generalized \ Hamming \ weights \ of \ \mathcal{C}_{(m-2)/2}
\]

of length \( 2^{m-1} \) satisfy \( d_r(\mathcal{C}_{(m-2)/2}) = (2^r - 1)(2^{m-1} - 2^{m-2})/2^{r-1} \) for \( 1 \leq r \leq (m-2)/2 \).
(ii) For \(1 \leq r \leq (m - 2)/2\) there exist curves \(C_r\) defined over \(\mathbb{F}_{2^m}\) of genus \(g(C_r) = (2^r - 1) (2^{(m-4)/2} + 1)\) and \(\# C_r(\mathbb{F}_q) = 2^m + 1 + (2^r - 1) 2^{m-1}\); these curves attain the Hasse–Weil upper bound.

(5.3) Example. For \(m = 6\) we have \(h = 2\) and for \(\mathfrak{C}_5 = \text{BCH}(3)^{-}\) of length 63 holds \(d(BCH(3)^{-}) = 24\). Furthermore there exist curves \(C_r\) defined over \(\mathbb{F}_{64}\) with

<table>
<thead>
<tr>
<th>(r)</th>
<th>(g(C_r))</th>
<th>(# C_r(\mathbb{F}_{64}))</th>
<th>Wirtz</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>97</td>
<td>97</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>161</td>
<td>155–161</td>
</tr>
</tbody>
</table>

For \(m = 8\) we have \(h = 3\) and from Proposition (5.2.ii) we obtain curves \(C_r\) defined over \(\mathbb{F}_{256}\) with

<table>
<thead>
<tr>
<th>(r)</th>
<th>(g(C_r))</th>
<th>(# C_r(\mathbb{F}_{256}))</th>
<th>Upper bound (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>385</td>
<td>385</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>641</td>
<td>641</td>
</tr>
<tr>
<td>3</td>
<td>28</td>
<td>1153</td>
<td>1153</td>
</tr>
</tbody>
</table>

II. The Case \(h = (m - 4)/2\)

We take \(m \geq 6\). The system of equations is

\[
\begin{align*}
&b_1^{2^{m-1} + 1} + b_1 = a_2 b_2^{2^{m-1} + 1} + a_2^{2^{m-1} + 1} b_2, \\
&b_1^{2^{2^{m}} + 1} + b_1 = a_2 b_2^{2^{2^{m}} + 1} + a_2^{2^{2^{m}} + 1} b_2.
\end{align*}
\]

If we employ the transformation \(b_1 = b'_1 + \sqrt{a_2} b_2\) we get

\[
\begin{align*}
&b_1^{2^{2^{m}} + 1} + b'_1 = (a_2 + \sqrt{a_2}) b_2^{2^{2^{m}} + 1} + (a_2^{2^{2^{m}} + 1} + 2^{2^{m}} \sqrt{a_2}) b_2, \\
&b_1^{2^{2^{m}} + 1} + b'_1 = (a_2^{2^{2^{m}} + 1} + 2^{2^{m}} \sqrt{a_2}) b_2 = 0.
\end{align*}
\]

The second equation in (19) implies \(b'_1 \in \mathbb{F}_{/\sqrt{a_2}}\).

Now we choose \(a_2 \in \mathbb{F}_4 - \mathbb{F}_2\) then the first equation in (19) becomes

\[
\begin{align*}
&b_1^{2^{2^{m}} + 1} + b'_1 = b_2^{2^{2^{m}} + 1} + b_2.
\end{align*}
\]

This implies \(b_2 = b'_1 + \sqrt{a_2}\) if g.c.d. \((m, (m - 2)/2) = 1\) or \(b_2 = b'_1 + \sqrt{a_2}\) if g.c.d. \((m, (m - 2)/2) = 2\). We have proved:

(5.4) Proposition. If we take \(\{1, a_2\}\) with \(a_2 \in \mathbb{F}_4 - \mathbb{F}_2\) then the system of equations (19) has \(2\sqrt{q}\) solutions \((b_1, b_2)\) for \(m \equiv 0 \pmod{4}\) and \(4\sqrt{q}\) solutions for \(m \equiv 2 \pmod{4}\).
We shall leave the formulation of the analogue of (5.2) to the reader. It implies that there exist curves over $\mathbb{F}_{256}$ of genus 2 with 321 points and of genus 6 with 449 points; both values are maximal. Moreover $d_2(BCH(3)^+) = \frac{1}{2}d_2(BCH(3)^-)$, $d_1(BCH(3)^+) = 144$ for $BCH(3)^+$ of length 255.

Finally, it is also possible to obtain good curves by applying the method to a combination of the cases. We shall illustrate this with an example.

(5.5) Example. Take $m = 6$ and let $b \in \mathbb{F}_4 - \mathbb{F}_2$. As we saw in case I of Section 4, the word $\text{Tr}(x)\text{Tr}(bx) \in \mathbb{G}_2$ if $b^8 + b = 0$, i.e., $b \in \mathbb{F}_8$. On the other hand case II implies that $\text{Tr}(x)\text{Tr}(bx) + \text{Tr}(px)\text{Tr}(b_2x) \in \mathbb{G}_1$ if

$$b_1^4 + b_1 = \rho b_4^4 + \rho b_2, \quad b_2^4 + b_2 = \rho b_4^5 + \rho^2 b_2.$$ (20)

One checks that the solutions $(b_1, b_2)$ of (20) can be written as $(b_1, b_2) = s(\rho, 1) + t(1, \rho)$ with $s \in \mathbb{F}_4$, $t \in \mathbb{F}_4$. Now choose $b \in \mathbb{F}_8 - \mathbb{F}_2$ and $(b_1, b_2) = b(\rho, 1)$ then \{1, $\rho$, $bp$, $b$\} has $\mathbb{F}_2$-rank 4. The corresponding words are

$$c_1 = \text{Tr}(x)\text{Tr}(bx) = \text{Tr}(bx^2 + (b^2 + b)x^3 + (b^4 + b)x^5) \text{ of weight 16},$$

$$c_2 = \text{Tr}(x)\text{Tr}(bp x) + \text{Tr}(px)\text{Tr}(bx) = \text{Tr}(b^2 + b)x^3 \text{ of weight 24}.$$ Then $c_1 + c_2 = \text{Tr}(x)\text{Tr}(bp^2 x) + \text{Tr}(px)\text{Tr}(bx^2) = \text{Tr}(bx^2 + (b^4 + b)x^3) \in \mathbb{G}_2$ has weight 24 as follows from Proposition (2.2) since \{1, $\rho$, $bp$, $b$\} also has $\mathbb{F}_2$-rank 4. The curves have genus 2, 1, 2 respectively and trace of Frobenius $-32$, $-16$, $-16$ respectively. So we find a fibre product curve over $\mathbb{F}_{64}$ of genus 5 with 65+64=129 rational points (Wirtz: 101–145).

(5.6) Remark. For $b \in \mathbb{F}_8 - \mathbb{F}_2$ the quadratic form $\text{Tr}(bx^9)$ vanishes on $\mathbb{F}_{64}$. This implies that the curve $y^2 + y = bx^9$ with genus 4 has trace of Frobenius $-64$. The fibre product of this curve and

$$y^2 + y = bx^9 + (b^4 + b)x^5 + (b^2 + b)x^3 + bx^2$$

has genus $g = 10$ and 193 points over $\mathbb{F}_{64}$ (Wirtz: 139–225).

References


