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Approximation on a disk IV

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ABSTRACT

It is shown that on closed disks $D$ around the origin in the complex plane and for every integer $m \geq 2$, not every continuous function on $D$ can be approximated uniformly on $D$ by polynomials in $z^m$ and $(z/(1 + z))^m$.

In a series of papers [1], [2], [3] the following situation is investigated. Let the function $g$ be defined in a neighborhood of the origin in the complex plane, of class $C^1$, with $g(0) = 0$, $g_z(0) = 0$, $g_{zz}(0) = 1$, and such that $z^m$ and $g^m$ separate points near 0 for some $m \geq 2$. Let $D$ be a small closed disk in the plane, centered at the origin. Is it possible to approximate every continuous function on $D$ uniformly on $D$ by polynomials in $z^m$ and $g^m$? In [2] we have shown that the answer no is possible for $m = 2$. For a large class of functions $g$ the answer is yes ($m = 2$ in [1], generalized in [3] for arbitrary $m$). In this paper we show that for arbitrary $m$ the answer no also is possible.

Theorem. Let $D$ be a closed disk in the plane, centered at the origin with radius less than 1. Let $[z^m, (z/(1 + z))^m : D]$ be the uniform closure in $C(D)$ of the polynomials in $z^m$ and $(z/(1 + z))^m$.

Then $[z^m, (z/(1 + z))^m : D] \neq C(D)$ for $m = 2, 3, \ldots$
which case the statement of the theorem is trivial) but if \( D \) is small enough (radius of \( D \) less than \( \sin \pi/m \)) points are separated.

The case \( m = 2 \) is the main result of [2]. The essential ingredient, writing

\[
g_a(z) = \frac{\bar{z}}{1 + az}
\]

and

\[
X_a = \{(z, g_a(z)) : z \in D \}
\]
is that \( X_1 \cup X_{-1} \) is not polynomially convex if \( D \) is a (small) disk centered at 0. This is shown by looking at the intersection of \( X_1 \cup X_{-1} \) and the analytic variety \( V \) in \( \mathbb{C}^2 \) defined by the equation

\[
\zeta_1 \zeta_2 - t\zeta_1 - t\zeta_2 = 0
\]

(\( t \) a small nonzero real number).

We modify the definition of \( V \) to show:

**Proposition.** If \( p \) and \( q \) are two different complex numbers then \( X_p \cup X_q \) is not polynomially convex if \( D \) is any (small) disk centered at 0.

**Proof.** Let \( D \) be so small that this disk misses the singular points of \( g_p \) and \( g_q \). Let \( \alpha \in \mathbb{C}, t \in \mathbb{C}, \) \( t \) nonzero, and define the variety \( V \) in \( \mathbb{C}^2 \) by the equation

\[
\zeta_1 \zeta_2 (1 + i\alpha) - t\zeta_2 - i\zeta_1 = 0.
\]

Note that if \( \zeta_1 \neq t/(1 + i\alpha) \) the value of \( \zeta_2 \) is uniquely determined by \( \zeta_1 \).

The fact that the point \( (z, \frac{\bar{z}}{1 + p\bar{z}}) \) belongs to \( V \) means that

\[
z\bar{z}(1 - \frac{t}{1 + p\bar{z}}) - tz - i\bar{z} = 0.
\]

Now choose \( \alpha = (p + q)/2 \), then \( p - \alpha = -(q - \alpha) \), and choose \( t \) a small nonzero complex number such that \( i(p - \alpha) \) and \( i(q - \alpha) \) are both real-valued, of opposite sign.

So if \( t \) is sufficiently small, then:

\[
X_p \cap V = \left\{(z, g_p(z)) : \left| z - \frac{t}{1 - i(p - \alpha)} \right| = \frac{|t|}{1 - i(p - \alpha)} \right\}
\]

and

\[
X_q \cap V = \left\{(z, g_q(z)) : \left| z - \frac{t}{1 - i(q - \alpha)} \right| = \frac{|t|}{1 - i(q - \alpha)} \right\}.
\]

Note that the exceptional point \( z = \frac{t}{1 + i\alpha} \) belongs (for small \( t \)) to the interior of both circles \( |z - \frac{t}{1 + i\alpha}| = |t|/(1 + i\alpha) \).

The polynomially convex hull of \( X_p \cup X_q \) contains the open subset \( O \) of \( V \) bounded by the two closed curves \( X_p \cap V \) and \( X_q \cap V \), so \( X_p \cup X_q \) is not polynomially convex. \( \square \)

**Remark.** Note that for all complex numbers \( a \) and \( b \) the intersection of the disk

\[
X = \{(z, bg_a(z)) : z \in D \}
\]

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where $D$ does not contain the singular point of $g_2$, and the variety $V$ ($t$ small and $\neq 0$) is either a simple closed curve, which projects onto a circle in the $\zeta_1$-plane, or else consists of at most two points.

**Proof of the theorem.** We give the proof for $m = 3$. The modifications for the case of an arbitrary value of $m$ will be obvious.

We assume that the radius of $D$ is less than $\frac{1}{2}\sqrt{3}$, so $z^3$ and $(\frac{\bar{z}}{(1 + \bar{z})})^3$ separate the points of $D$. Let $X = \{(z^3, (\frac{\bar{z}}{(1 + \bar{z})})^3) : z \in D\}$, let $\rho = e^{2\pi i/3}$, $\rho^2 = \bar{\rho}$ be the cubic roots of unity and let $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Pi(\zeta_1, \zeta_2) = (\zeta_1^3, \zeta_2^3)$.

Then $\Pi^{-1}(X) = X_1 \cup \cdots \cup X_9$ with

$$X_1 = \left\{ \left( z, \frac{\bar{z}}{1 + \bar{z}} \right) : z \in D \right\} = \left\{ \left( w, \frac{\bar{w}}{1 + \bar{w}} \right) : w \in D \right\}$$

$$X_2 = \left\{ \left( \rho z, \frac{\bar{\rho} \bar{z}}{1 + \bar{\rho} \bar{z}} \right) : z \in D \right\} = \left\{ \left( \frac{\bar{w}}{1 + \bar{w}} \right) : w \in D \right\}$$

$$X_3 = \left\{ \left( \rho \bar{z}, \frac{\rho \bar{z}}{1 + \bar{\rho} \bar{z}} \right) : z \in D \right\} = \left\{ \left( \frac{\bar{w}}{1 + \bar{w}} \right) : w \in D \right\}.$$  

The disks $X_4, X_5, X_6$ are obtained by multiplying the second coordinates in $X_1, X_2, X_3$ by $\rho$ while $X_7, X_8, X_9$ are gotten similarly by multiplication by $\bar{\rho}$.

From the proof of the proposition with $p = 1$ and $q = \rho$, it follows that $X_1 \cup X_2$ is not polynomially convex because its hull contains an open subset $O$ of the appropriate variety $V$. By the remark above $O$ is not contained in $X_3 \cup \cdots \cup X_9$, so $\Pi^{-1}(X) = X_1 \cup \cdots \cup X_9$ is not polynomially convex. It follows that $X$ is not polynomially convex, hence $P(X) \neq C(X)$ which is equivalent to $[z^3, (\frac{\bar{z}}{(1 + \bar{z})})^3 : D] \neq C(D)$. □

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**REFERENCES**

