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Approximation on a disk IV

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ABSTRACT

It is shown that on closed disks $D$ around the origin in the complex plane and for every integer $m \geq 2$, not every continuous function on $D$ can be approximated uniformly on $D$ by polynomials in $z^m$ and $(z/(1 + z))^m$.

In a series of papers [1], [2], [3] the following situation is investigated. Let the function $g$ be defined in a neighborhood of the origin in the complex plane, of class $C^1$, with $g(0) = 0$, $g_z(0) = 0$, $g_z(0) = 1$, and such that $z^m$ and $g^m$ separate points near 0 for some $m \geq 2$. Let $D$ be a small closed disk in the plane, centered at the origin. Is it possible to approximate every continuous function on $D$ uniformly on $D$ by polynomials in $z^m$ and $g^m$? In [2] we have shown that the answer no is possible for $m = 2$. For a large class of functions $g$ the answer is yes ($m = 2$ in [1], generalized in [3] for arbitrary $m$). In this paper we show that for arbitrary $m$ the answer no also is possible.

Theorem. Let $D$ be a closed disk in the plane, centered at the origin with radius less than 1. Let $[z^m, (z/(1 + z))^m : D]$ be the uniform closure in $C(D)$ of the polynomials in $z^m$ and $(z/(1 + z))^m$.

Then $[z^m, (z/(1 + z))^m : D] \neq C(D)$ for $m = 2, 3, \ldots$

Note that it is enough to give a proof for sufficiently small disks $D$. Also note that it may happen that points of $D$ are not separated by the two generators (in
which case the statement of the theorem is trivial) but if $D$ is small enough (radius of $D$ less than $\sin \pi/m$) points are separated.

The case $m=2$ is the main result of [2]. The essential ingredient, writing
\[
g_a(z) = \frac{\bar{z}}{1 + a\bar{z}}
\]
and
\[
X_a = \{(z, g_a(z)) : z \in D\}
\]
is that $X_1 \cup X_{-1}$ is not polynomially convex if $D$ is a (small) disk centered at 0. This is shown by looking at the intersection of $X_1 \cup X_{-1}$ and the analytic variety $V$ in $\mathbb{C}^2$ defined by the equation
\[
\zeta_1 \zeta_2 - t\zeta_1 - t\zeta_2 = 0
\]
($t$ a small nonzero real number).

We modify the definition of $V$ to show:

**Proposition.** If $p$ and $q$ are two different complex numbers then $X_p \cup X_q$ is not polynomially convex if $D$ is any (small) disk centered at 0.

**Proof.** Let $D$ be so small that this disk misses the singular points of $g_p$ and $g_q$. Let $\alpha \in \mathbb{C}$, $t \in \mathbb{C}$, $t$ nonzero, and define the variety $V$ in $\mathbb{C}^2$ by the equation
\[
\zeta_1 \zeta_2 (1 + i\alpha) - t\zeta_2 - i\zeta_1 = 0.
\]
Note that if $\zeta_1 \neq t/(1 + i\alpha)$ the value of $\zeta_2$ is uniquely determined by $\zeta_1$.

The fact that the point $(z, z/(1+p\bar{z}))$ belongs to $V$ means that
\[
z\bar{z}(1 - i(p - \alpha)) - t\bar{z} - iz = 0.
\]
Now choose $\alpha = (p + q)/2$, then $p - \alpha = -(q - \alpha)$, and choose $t$ a small nonzero complex number such that $i(p - \alpha)$ and $i(q - \alpha)$ are both real-valued, of opposite sign.

So if $t$ is sufficiently small, then:

\[
X_p \cap V = \left\{(z, g_p(z)) : \left|z - \frac{t}{1 - i(p - \alpha)}\right| = \frac{|t|}{1 - i(p - \alpha)}\right\}
\]
and
\[
X_q \cap V = \left\{(z, g_q(z)) : \left|z - \frac{t}{1 - i(q - \alpha)}\right| = \frac{|t|}{1 - i(q - \alpha)}\right\}.
\]

Note that the exceptional point $z = t/(1 + i\alpha)$ belongs (for small $t$) to the interior of both circles $|z - t/(1 \pm i(p - \alpha))| = |t|/(1 \pm i(p - \alpha))$.

The polynomially convex hull of $X_p \cup X_q$ contains the open subset $O$ of $V$ bounded by the two closed curves $X_p \cap V$ and $X_q \cap V$, so $X_p \cup X_q$ is not polynomially convex.

**Remark.** Note that for all complex numbers $a$ and $b$ the intersection of the disk
\[
X = \{(z, bg_a(z)) : z \in D\}
\]
where $D$ does not contain the singular point of $g_z$, and the variety $V$ ($t$ small and $\neq 0$) is either a simple closed curve, which projects onto a circle in the $\zeta_1$-plane, or else consists of at most two points.

**Proof of the theorem.** We give the proof for $m = 3$. The modifications for the case of an arbitrary value of $m$ will be obvious.

We assume that the radius of $D$ is less than $\frac{1}{2}\sqrt{3}$, so $z^3$ and $(\bar{z}/(1 + \bar{z}))^3$ separate the points of $D$. Let $X = \{(z^3, (\bar{z}/(1 + \bar{z}))^3) : z \in D\}$, let $1, \rho = e^{2\pi i/3}, \rho^2 = \bar{\rho}$ be the cubic roots of unity and let $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Pi(\zeta_1, \zeta_2) = (\zeta^3_1, \zeta^3_2)$.

Then $\Pi^{-1}(X) = X_1 \cup \ldots \cup X_9$ with

\[
X_1 = \left\{(z, \frac{\bar{z}}{1 + \bar{z}}) : z \in D\right\} = \left\{(w, \frac{\bar{w}}{1 + \bar{w}}) : w \in D\right\}
\]

\[
X_2 = \left\{(\rho z, \frac{\bar{\rho} \bar{z}}{1 + \bar{z}}) : z \in D\right\} = \left\{(w, \frac{\bar{w}}{1 + \bar{w}}) : w \in D\right\}
\]

\[
X_3 = \left\{(\bar{\rho} z, \frac{\rho \bar{z}}{1 + \bar{z}}) : z \in D\right\} = \left\{(w, \frac{\bar{w}}{1 + \bar{w}}) : w \in D\right\}.
\]

The disks $X_4, X_5, X_6$ are obtained by multiplying the second coordinates in $X_1, X_2, X_3$ by $\rho$ while $X_7, X_8, X_9$ are gotten similarly by multiplication by $\bar{\rho}$.

From the proof of the proposition with $p = 1$ and $q = \rho$, it follows that $X_1 \cup X_2$ is not polynomially convex because its hull contains an open subset $O$ of the appropriate variety $V$. By the remark above $O$ is not contained in $X_3 \cup \ldots \cup X_9$, so $\Pi^{-1}(X) = X_1 \cup \ldots \cup X_9$ is not polynomially convex. It follows that $X$ is not polynomially convex, hence $P(X) \neq C(X)$ which is equivalent to $[z^3, (\bar{z}/(1 + \bar{z}))^3] : D] \neq C(D)$. □

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**REFERENCES**

