Roots of an orthogonal matrix - solution.

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form $f(y_i;\mu,\alpha) = (1/\alpha)g((y_i - \mu)/\alpha), -\infty < y_i < \infty$ (Lehmann, 1983, chaps. 1 and 3). The estimation problem remains invariant under the transformations $y_i^* = dy_i + k$, $\mu^* = d\mu + k$, and $\alpha^* = d\alpha$, provided the MLE exists. Other members of the location-scale family that are useful for the analysis of duration data include the log-normal and the two-parameter exponential distributions.

NOTE

1. Excellent solutions also have been proposed independently by G. Dhaene and by S.K. Sapra, the poser of the problem.

REFERENCE

Lehmann, E.L. (1983) *Theory of Point Estimation.* New York: Wiley.

96.5.3. *Roots of an Orthogonal Matrix*—Solution.[1] Two solutions have been proposed independently by H. Peter Boswijk and by Maozo Lu. Each of these solutions, which are published below, contains an interesting derivation.

1. Solution proposed by H. Peter Boswijk. Let A be an $n \times n$ orthogonal matrix, so that $A'A = AA' = I_n$. We wish to characterize the eigenvalues λ of A, i.e., the solutions to

$$Az = \lambda z, \tag{1}$$

with z a complex n-vector $z = x + iy \in \mathbb{C}^n$, where $x, y \in \mathbb{R}^n$, and $i := \sqrt{-1}$. The conjugate of z is denoted by $\bar{z} := x' + iy'$, and the modulus of z is $|z| := \sqrt{\bar{z}z} = \sqrt{x'x + y'y}$. We shall only be interested in the nontrivial solution $z \neq 0$, so that $|z| \neq 0$. Because rank $A = n$, we also have $\lambda \neq 0$.

Premultiplication by $\bar{z}$ in (1) yields

$$\bar{z}Az = \lambda\bar{z}z \Rightarrow \lambda = \frac{\bar{z}Az}{|z|^2}. \tag{2}$$

However, premultiplication by $\bar{z}A'$ gives

$$\bar{z}A'Az = \bar{z}z = \lambda\bar{z}A'z \Rightarrow \lambda^{-1} = \frac{\bar{z}A'z}{|z|^2}. \tag{3}$$

Because $\bar{z}A'z = \overline{\bar{z}Az}$, (2) and (3) together lead to

$$\lambda^{-1} = \bar{\lambda} \Rightarrow |\lambda| = 1 \Leftrightarrow \lambda = e^{i\phi}, \qquad \phi \in [0,2\pi],$$

that is, λ may be any point on the unit circle, not just 1 or -1.

Remark. This derivation is an adaptation of the proof of Theorem 1.7 by Magnus and Neudecker (1988). They replaced $\bar{z}$ by z' and suggested that $\lambda = (z'Az)/(z'z) = (z'A'z)/(z'z) = \lambda^{-1} \Rightarrow \lambda = \pm 1$. However, this argument is incorrect because $z'z$ and $z'Az$ may be zero for complex z. The simplest example is the orthogonal matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$ and corresponding eigenvectors $z_1 = (1, i)'$ and $z_2 = (i, 1)'$; thus, $z_1'z_1 = z_2'z_2 = z_1'Az_1 = z_2'Az_2 = 0$.

REFERENCE

Magnus, J.R. & H. Neudecker (1988) *Matrix Differential Calculus with Applications in Statistics and Econometrics*. Chichester: John Wiley & Sons.

2. Solution proposed by Maozo Lu. Let A be an $n \times n$ real orthogonal matrix, that is, $A^T = A^{-1}$; then there exists another $n \times n$ real orthogonal matrix Q such that

$$D = Q^TAQ = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_p & & & \\ & & & A_1 & & \\ & & & & \ddots & \\ & & & & & A_k \end{bmatrix},$$

where each $\lambda_i = \pm 1, i = 1, \dots, p$; each A_j is a 2×2 real matrix and has the form

$$A_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix},$$

in which $\theta_j \in \mathbf{R}^1, j = 1, \dots, k$; and $p + k = n$ (see Horn and Johnson,1985, Corollary 2.5.14).

The eigenvalues of orthogonal matrix A are solutions to the characteristic equation $|A - \lambda I_n| = 0$, which takes the following equivalent form due to the nonsingularity and orthogonality of matrix Q:

$$\begin{aligned} |A - \lambda I_n| &= |Q^T||A - \lambda I_n||Q| \\ &= |Q^TAQ - \lambda Q^TQ| = |D - \lambda I_n| = 0; \end{aligned}$$

therefore,

$$|A - \lambda I_n| = \left[\prod_{i=1}^{p} (\lambda_i - \lambda)\right]\left[\prod_{j=1}^{k} (\lambda^2 - 2\cos\theta_j + 1)\right] = 0.$$

The solutions to the above equation, i.e., the eigenvalues of orthogonal matrix A, take one of the following forms:

$$\begin{aligned} &\lambda_i = \pm 1, && i = 1, \dots, p \\ &\lambda_j = \cos\theta_j \pm i\sin\theta_j, && j = 1, \dots, k. \end{aligned}$$

Hence, it is clear that not all λ_i's equal ± 1, but for all the eigenvalues, we have $||\lambda_i|| = 1$, $i = 1,\ldots,n$. It is also clear that orthogonal matrix A only has eigenvalues ± 1 if and only if it is symmetric.

NOTE

1. Excellent solutions also have been proposed by R.W. Farebrother, G. Trenkler, and by K.M. Abadir and K. Hadri, the posers of the problem.

REFERENCE

Horn, R.A. & C.R. Johnson (1985) *Matrix Analysis*. New York: Cambridge University Press.

96.5.4. *On the Bias of Standard Errors of the LS Residual under Nonnormal Errors*—Solution,[1] proposed by Offer Lieberman, Aman Ullah, and Robert Breunig. Let us write

$$s_u = (s_u^2)^{1/2} = (s_u^2 - \sigma^2 + \sigma^2)^{1/2} = \sigma\left(1 + \frac{s_u^2 - \sigma^2}{\sigma^2}\right)^{1/2}$$

$$\simeq \sigma\left[1 + \frac{1}{2}\left(\frac{s_u^2 - \sigma^2}{\sigma^2}\right) - \frac{1}{8}\left(\frac{s_u^2 - \sigma^2}{\sigma^2}\right)^2\right] \tag{8}$$

up to $O_p(n^{-1})$, where $s_u^2 - \sigma^2 = O_p(n^{-1/2})$. Taking expectations on both sides of (8) and using $Es_u^2 = \sigma^2$ (see (11) below) we get $O(n^{-1})$,

$$E(s_u - \sigma) = -\frac{1}{8\sigma^3} E(s_u^2 - \sigma^2)^2. \tag{9}$$

Further, up to $O(n^{-3/2})$, we get

$$E\{(\hat{V}(b_j))^{1/2}\} - (V(b_j))^{1/2} = (E\{s_u - \sigma\})a_{jj}^{1/2}. \tag{10}$$

To obtain $E(s_u^2 - \sigma^2)^2$ under nonnormal errors (2), we note that (see Ullah and Srivastava, 1994, eq. 2.18)

$$\begin{aligned} E(u'Mu) &= \sigma^2 \operatorname{tr}(M) = (n-k)\sigma^2 \\ E(u'Mu)^2 &= \sigma^4[\gamma_2 \operatorname{tr}(\dot{M}) + (n-k)(n-k+2)]. \end{aligned} \tag{11}$$

Thus, $Es_u^2 = \sigma^2$ and

$$\begin{aligned} V(s_u^2) = E(s_u^2 - \sigma^2)^2 = Es_u^4 - \sigma^4 &= \frac{1}{(n-k)^2} E(u'Mu)^2 - \sigma^4 \\ &= \frac{2\sigma^4}{(n-k)}\left[1 + \gamma_2 \frac{\operatorname{tr}(\dot{M})}{2(n-k)}\right], \end{aligned} \tag{12}$$

which is the result in (6). Next, substituting (12) in (9) provides the result in (4). Further, substituting (9) and (12) in (10) provides the result in (5). Note that σ^4 in (4) and (5) should be read as σ^3.

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