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Explicit Complete Curves in the Moduli Space of Curves of Genus Three

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Abstract. We explicitly describe complete, one-dimensional subvarieties of the moduli space of smooth complex curves of genus 3.

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Introduction

Consider \( \mathcal{M}_g \), the moduli space of smooth curves of genus \( g \) over the field of complex numbers \( \mathbb{C} \). For \( g \geq 2 \), \( \mathcal{M}_g \) is a quasi-projective variety of dimension \( 3g - 3 \). Note that \( \mathcal{M}_g \) is not complete as smooth curves can degenerate to singular ones. In fact, \( \mathcal{M}_2 \) is affine. However, \( \mathcal{M}_g \) contains complete curves if \( g \geq 3 \). This follows from the existence of a projective compactification \( \bar{\mathcal{M}}_g \) of \( \mathcal{M}_g \) in which the boundary \( \bar{\mathcal{M}}_g \setminus \mathcal{M}_g \) has codimension \( \geq 2 \) (take the closure of the image of \( \mathcal{M}_g \) in the Satake compactification of \( \mathcal{A}_g \), the moduli space of principally polarized abelian varieties). The complete curves are obtained by cutting \( \bar{\mathcal{M}}_g \) with sufficiently many hypersurfaces in general position. An upper bound for the dimension of a complete subvariety of \( \bar{\mathcal{M}}_g \) is \( g - 2 \) if \( g \geq 2 \) ([2]). So these complete curves achieve this bound if \( g = 3 \).

Harris (in [6]) notes that these curves are not very explicit, although constructions of explicit complete families of smooth curves are known. In [4] an explicit one-dimensional family is given for every genus \( g \geq 4 \), but for \( g = 3 \) a less explicit family is exhibited. The aim of this note is to produce explicit examples of complete families of smooth curves of genus 3 having a moduli theoretic interpretation.

Briefly, the construction is as follows: Fix a smooth base curve \( C_3 \) of genus 3 and fix a complete curve \( F \subset C_3 \times C_3 \setminus \Delta \). Construct a complete family of smooth double curves \((C_f \xrightarrow{\pi_f} C_3)_{f \in F}\), where \( \pi_f \) is branched over the two points determined by \( f \in F \) (in fact such a family may exist only over a finite cover of \( F \) due to monodromy obstructions). To the covers \( C_f \xrightarrow{\pi_f} C_3 \) we can associate their Prym varieties. We obtain a complete family of three-dimensional principally

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polarized abelian varieties $\text{Prym}(C_f/C_3)_{f \in F}$, which turn out to be Jacobians of smooth curves if $C_3$ is not hyperelliptic. So we get a complete family of Jacobians of smooth genus 3 curves.

In the first section we present a specific construction of complete families of three-dimensional principally polarized abelian varieties, depending on five parameters. The base is one-dimensional and the fibres are Prym varieties of branched double covers of curves. We will see in Section 2 that these fibres are actually Jacobians of smooth curves. The corresponding families of curves are then constructed via the so-called ‘trigonal construction’. Section 3 contains a calculation of invariants of these families, in particular the number of hyperelliptic fibres. Finally, we show that these families are general in the following sense: the generic smooth curve of genus 3 occurs as a fibre of one of the families.

1. The Construction

Fix an elliptic curve $E = (E, 0)$. Let $\phi: C_3 \to E$ be a double cover ramified at four points of $E$, so that the genus of $C_3$ is 3. We want $C_3$ to be non-hyperelliptic. This turns out to be an open condition on the four branch points: let $B$ be the branch divisor, $\tilde{B}$ the ramification divisor of $\phi$, and let $L$ denote the unique line bundle on $E$ satisfying $\phi^*L \cong \mathcal{O}_{C_3}(\tilde{B})$. Then $C_3$ is hyperelliptic if and only if $B = B_1 + B_2$ for some $B_1, B_2 \subset |L|$ (if $C_3$ is hyperelliptic, then $\phi$ acts freely on $\text{Supp}(\tilde{B})$ and $L = \phi_*(x + \sigma x)$ for an $x \in \text{Supp}(\tilde{B})$).

Fix a point $t \neq 0$ in $E$. Set $\Delta_t = \{(x, x+t) \mid x \in E\} \subset E \times E$. Let $F = F(t) \subset C_3 \times C_3$ be the inverse image of $\Delta_t$ under the natural map $\phi \times \phi: C_3 \times C_3 \to E \times E$. Then in $C_3 \times C_3$ we have that $F \cap \Delta = \emptyset$, thus $F$ parametrizes pairs of distinct points on $C_3$. If we suppose -- as we shall do in the following -- that $t \notin \{\phi(p) - \phi(q) \mid p, q$ ramification points of $\phi\}$, then one easily verifies that $F$ is smooth of genus 9.

Denote by $\pi_1, \pi_2: F \to C_3$ the maps induced by the projections of $C_3 \times C_3$ onto the first, respectively the second coordinate, and denote by $\Gamma_{\pi_i} \subset F \times C_3$ the graph of $\pi_i$, for $i = 1, 2$. We want to have a double cover of $F \times C_3$ ramified precisely over $\Gamma_{\pi_1} + \Gamma_{\pi_2}$. Such a cover may not exist due to monodromy obstructions. To overcome these, we consider the natural map $F \to \text{Pic}^2(C_3), x \mapsto [\pi_1(x) + \pi_2(x)]$, and the squaring map $\text{sq}: \text{Pic}^1(C_3) \to \text{Pic}^2(C_3), L \mapsto L \otimes L$. Define the curve $B$ as the fibred product $B = F \times_{\text{Pic}^2(C_3)} \text{Pic}^1(C_3)$.

$$
\begin{array}{ccc}
B & \longrightarrow & \text{Pic}^1(C_3) \\
\downarrow f & & \downarrow \text{sq} \\
F & \longrightarrow & \text{Pic}^2(C_3)
\end{array}
$$
Denote the map \( B \to F \) by \( f \), and set \( \Gamma = \Gamma_{\pi_1 f} + \Gamma_{\pi_2 f} \subset B \times C_3 \). The pull-back to \( B \times C_3 \) of the universal line bundle over \( \text{Pic}^1(C_3) \times C_3 \) gives a line bundle \( \mathcal{L} \) on \( B \times C_3 \) satisfying \( \mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma) \).

Using this \( \mathcal{L} \) we make a double cover \( C \to B \times C_3 \), ramified precisely along \( \Gamma \). This is a well-known construction: embed \( B \times C_3 \) in the total space of the line bundle corresponding to \( \mathcal{O}_{B \times C_3}(\Gamma) \) with the section \( s \) of \( \mathcal{O}_{B \times C_3}(\Gamma) \) which vanishes precisely once along \( \Gamma \). Define \( C \) as the inverse image of \( s(B \times C_3) \) under the natural squaring map on local sections \( \mathcal{L} \xrightarrow{\text{sq}} \mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma) \). Here \( \mathcal{L} \) denotes the total space of the line bundle \( \mathcal{L} \), by abuse of notation.

\[
\begin{array}{ccc}
C & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
B \times C_3 & \longrightarrow & B \times C_3 \\
\end{array}
\]

The composition with the projection \( B \times C_3 \to B \) yields a complete family of curves \( \pi_C: C \to B \). Let \( C_b \) denote the fibre of \( \pi_C: C \to B \) over \( b \in B \). Each \( C_b \) is smooth of genus 6 and maps \( 2:1 \) to \( C_3 \) with two ramification points, via the composition \( C \to B \times C_3 \to C_3 \). Note that the map \( f: B \to F \) is étale of degree 64, so that \( B \) is a smooth curve, but possibly reducible. Thus \( \pi_C: C \to B \) is a complete, possibly reducible family of smooth curves.

We can associate to every double covering \( C_b \to C_3 \) its Prym variety (see [7]). This is a three-dimensional, principally polarized abelian variety \( \text{Prym}(C_b/C_3) = (P, \Xi) \). \( P \) is defined as the kernel of the norm map \( \text{Nm}: \text{Jac}(C_b) \to \text{Jac}(C_3) \) (which is connected since \( C_b \to C_3 \) is ramified). The Jacobian of \( C_3 \) injects in the Jacobian of \( C_b \), so up to isogeny \( \text{Jac}(C_b) \) splits as the product of \( \text{Jac}(C_3) \) and another factor, which is \( P \). The polarization on \( \text{Jac}(C_b) \) gives rise to a polarization of \( P \times \text{Jac}(C_3) \): \( P \times \text{Jac}(C_3) \to \text{Jac}(C_b) \to \text{Jac}(C_3) \to P \times \text{Jac}(C_3) \). This polarization splits and induces on \( P \) twice a principal polarization \( \Xi \).

It is straightforward to globalize this construction. The family \( \pi_C: C \to B \) has sections. Thus we have the associated family \( \pi_C: C \to B \) of Jacobians \( JC \to B \). Consider the norm map \( \text{Nm}: JC \to B \times \text{Jac}(C_3) \), induced by the usual norm map \( \text{Nm}: JC_b \to b \times \text{Jac}(C_3) \). Define \( \mathcal{P} \) as \( \text{Nm}^{-1}(B \times 0) \subset JC \). The projection \( JC \to B \) yields a family \( \pi_\mathcal{P}: \mathcal{P} \to B \) of 3-dimensional principally polarized abelian varieties.

We now need Theorem 2.1. The theorem says that the 1-dimensional family of Pryms \( \mathcal{P} \to B \) is in fact a family of Jacobians of smooth curves. Note that here we use that \( C_3 \) is non-hyperelliptic. For readability the statement and proof of this theorem is postponed to the next section. Summarizing, we have the following result:
**THEOREM 1.1.** The above construction yields a complete, one-dimensional family \( \pi_C : P \to B \) of three-dimensional Jacobians of smooth curves.

Note that the construction depends on five parameters: namely on \( E \), on the double cover \( \phi : C_3 \to E \) and on the point \( t \in E \). Thus we obtain in fact a five-dimensional family of complete, one-dimensional families of three-dimensional Jacobians.

**APPENDIX: THE CORRESPONDING FAMILY OF CURVES**

We will exhibit in this appendix the family of smooth curves corresponding to the family of Jacobians \( \pi_P : P \to B \), using the so-called trigonal construction. We explain this first for a single Prym variety \( \text{Prym}(C_b/C_3) \). For the details and proofs we refer to [3].

The map \( C_b \to C_3 \) has two branch points, which we denote by \( p, p' \in C_3 \). Consider \( C_3 \) canonically embedded in \( |K_{C_3}|^* \cong \mathbb{P}^2 \). The line through \( p \) and \( p' \) cuts out on \( C_3 \) two other points. Choose one of them and call it \( p'' \). Let \( P \) denote the linear system \( |K_{C_3} - p''| \). Projecting from \( p'' \) determines a three-to-one map \( C_3 \to P \cong \mathbb{P}^1 \).

Assume that this map does not ramify at \( p \) or \( p' \). Let \( C_b^{(3)} \) (respectively \( C_3^{(3)} \)) denote the threefold symmetric product of \( C_b \) (respectively \( C_3 \)). The line \( P \) embeds naturally in \( C_3^{(3)} \). Consider the inverse image of \( P \subset C_3^{(3)} \) under the natural map \( C_b^{(3)} \to C_3^{(3)} \). This is a curve with two smooth, isomorphic components \( T_0, T_1 \). Their intersection consists of two distinct points \( q, q' \), in which they meet transversely. Both curves map with degree four to \( P \) and \( q, q' \) are simple ramification points on each curve mapping to the same point on \( P \). Apart from \( q \) and \( q' \) the ramification on the tetragonal curves \( T_0 \) and \( T_1 \) arises from the ramification of the \( g_3^1 \) on \( C_3 \). Namely if \( C_3 \to P \) is simply branched over \( P \in \mathbb{P} \), then also the maps \( T_i \to P \) are simply branched over \( P \). If \( C_3 \to P \) is completely branched over \( P \in \mathbb{P} \) then the maps \( T_i \to P \) have a point of ramification index 3 over \( P \). In particular the genera of the \( T_i \) are 3.

If, on the other hand, the triple cover \( C_3 \to P \) is ramified at \( p \) or \( p' \), then the situation is different. The points \( q \) and \( q' \) coincide, i.e. the inverse image still has two smooth isomorphic components \( T_0 \) and \( T_1 \) which now meet in one point \( q \), and the fourfold cover \( T_i \to P \) is totally ramified in \( q \).

The curves \( T_0 \) and \( T_1 \) map naturally to \( \text{Pic}^3(C_b) \), since they live in \( C_b^{(3)} \). The norm map \( \text{Nm} : \text{Pic}^3(C_b) \to \text{Pic}^3(C_3) \) maps their images to the image of \( P \) in \( \text{Pic}^3(C_3) \), which is a point. The images of \( T_0 \) and \( T_1 \) in \( \text{Pic}^3(C_b) \) lie thus in a suitable translate of \( P = \ker(\text{Nm} : \text{Jac}(C_b) \to \text{Jac}(C_3)) \). Moreover, the maps \( T_i \to \text{Pic}^3(C_b) \), \( i = 0, 1 \), induce isomorphisms of principal polarized abelian varieties \( \text{Jac}(T_i) \to \text{Prym}(C_b/C_3) \). The trigonal construction goes back to Recillas and was generalized by Donagi.

To globalize this construction reconsider the curve \( F \subset C_3 \times C_3 \). Every point \( x \in F = F(t) \) determines two distinct points \( \pi_1(x), \pi_2(x) \in C_3 \), and the line
spanned by these points in the canonical embedding of $C_3$ we denote by $\ell(x)$. Let $K(x)$ denote the hyperplane section $C_3 \cdot \ell(x)$. Consider the curve

$$\tilde{F} = \tilde{F}(t) = \{(x, c) \in F \times C_3 | c \in \text{Supp}(K(x) - \pi_1(x) - \pi_2(x))\}.$$ 

Then $\tilde{F}$ is smooth since $F$ and $C_3$ are. Clearly $\tilde{F}$ maps with degree two onto $F$. This map is ramified in 64 points (see Lemma 3.3 for the computation), hence $\tilde{F}$ is irreducible of genus 49. A point $(x, p'')$ of $\tilde{F}$ determines on $C_3$ a $g^1_3$, namely $|K_{C_3} - p''|$, plus two unordered, distinct points contained in a divisor of this $g^1_3$, namely $\pi_1(x)$, $\pi_2(x)$. $\tilde{F}$ maps to $\text{Pic}^2(C_3)$ via the composition $\tilde{F} \to F \to \text{Pic}^2(C_3)$, and also $\text{Pic}^1(C_3)$ does via the squaring map $\text{Pic}^1(C_3) \to \text{Pic}^2(C_3)$. Let $\tilde{B} = \tilde{F} \times_{\text{Pic}^2(C_3)} \text{Pic}^1(C_3)$. Note that $\tilde{B}$ maps two-to-one to $B$.

The pull-back of $C \to B$ to $\tilde{B}$ yields a double cover $\tilde{C} \to \tilde{B} \times C_3$. The composition $\tilde{C} \to \tilde{B} \times C_3 \to \tilde{B}$ is a one-dimensional family of double covers. The extra structure now is a three-to-one map $\tilde{C} \to \tilde{B} \times C_3$ which lifts the involution of $C \to S$ setwise, and its quotient by this involution we denote by $\tilde{T}$. The (singular) surface $\tilde{T}$ has an involution which preserves the fibres of $\tilde{T} \to S$ setwise, and its quotient by this involution we denote by $\tilde{T}$. The surface $\tilde{T}$ is smooth. The natural map $\tilde{T} \to S$ composed with the projection $S \to \tilde{B}$ yields a family $\tilde{T} \to \tilde{B}$ of smooth tetragonal curves.

COROLLARY 1.2. The above construction gives a complete, one-dimensional family $\tilde{T} \to S \to \tilde{B}$ of smooth, tetragonal curves of genus 3.

2. Pryms of Double Coverings of Genus 3 Curves Ramified in Two Points

This section is devoted to the proof of the theorem below. It shows that the Prym variety associated to a ramified double covering $\tilde{Z}/Z$ ramified in two points, where the genus of $Z$ is 3, is a Jacobian of a smooth curve as long as the base curve is not hyperelliptic.

In the following let $Z$ be a smooth genus 3 curve, and let $\pi_Z: \tilde{Z} \to Z$ be a double cover of smooth curves, with two branch points $p, p' \in Z$ and ramification points $\tilde{p}, \tilde{p}' \in \tilde{Z}$. Associated to this cover is a (unique) line bundle $L_{\tilde{Z}/Z}$ satisfying $(\pi_Z)^*(L_{\tilde{Z}/Z}) \cong O_{\tilde{Z}}(\tilde{p} + \tilde{p}')$. The data $(Z, p + p', L_{\tilde{Z}/Z})$ determine the cover $\pi_Z: \tilde{Z} \to Z$ up to isomorphism. The Prym variety $P = \text{Prym}(\tilde{Z}/Z)$ is a threedimensional abelian variety which comes with a natural principal polarization, denoted by $\Xi$.

THEOREM 2.1. (1) If $Z$ is hyperelliptic, and if $p, p'$ are interchanged by the hyperelliptic involution of $Z$, then $(P, \Xi)$ is the Jacobian of a smooth irreducible hyperelliptic curve, or a product of two such.
(2) If $Z$ is hyperelliptic and $p$, $p'$ are not interchanged by the hyperelliptic involution of $Z$, then $(P, \Xi)$ is a Jacobian of a smooth irreducible hyperelliptic curve.

(3) If $Z$ is not hyperelliptic, then $(P, \Xi)$ is a Jacobian of a smooth irreducible curve. This curve is hyperelliptic if and only if for some $p'' \in Z$, $p + p' + 2p''$ is a canonical divisor and $h^0(\mathcal{L}_{\bar{Z}/Z}(p + p' + p'')) = 0$.

Proof. Let $W = Z/(p \sim p')$ be the curve obtained from $Z$ by identifying $p$ and $p'$, and let $\bar{W} = \tilde{Z}/(\tilde{p} \sim \tilde{p}')$. Denote by $f_N: Z \to W$ the normalisation mapping. Note that $p_a(W) = 4$, $p_a(\bar{W}) = 7$.

The idea is to use the theory from [1]: the induced map $\pi_W: \bar{W} \to W$ is an admissible covering in the sense of Beauville, and $\text{Prym}(\bar{Z}/Z) = \text{Prym}(\bar{W}/W)$. Here the right-hand side is the Prym variety associated by Beauville to the covering $\bar{W}/W$. It lives canonically in $JW^* = \{\text{line bundles } L \text{ on } W \mid 2 \deg L = \deg \omega_{\bar{W}}\}$ as $P = \text{Prym}(\bar{W}/W) = \{L \mid Nm L \cong \omega_{\bar{W}}, h^0(L) \text{ even}\}$, and the polarization $\Xi$ then is $\Xi = \{L \in P \mid h^0(L) \geq 2\}$.

To prove the theorem, we show that, unless we are in case (1), dim Sing $\Xi < 0$. Since dim $P = 3$, this shows that in the cases (2) and (3) $(P, \Xi)$ is a Jacobian. If Sing$\Xi$ is a point $L \in \Xi$ – in which case $(P, \Xi)$ is a hyperelliptic Jacobian – we show how $L$ arises from the geometry of $Z$.

Note that the proof is an adaptation of the proof of Theorem 4.10 in [1]. Also note that case (1) is already treated by Beauville (see [1, p. 171]).

Start by assuming that Sing $\Xi$ is non-empty, say $L \in \text{Sing}(\Xi)$. We will show that in the cases (2) and (3) $L \in \Xi$ is a uniquely determined point. By [1, Lemma 4.1], either $h^0(L)$ is $\geq 4$ and even or $h^0(L) = 2$, and $s^*s \otimes t = s \otimes t^*$ for a basis $\{s, t\}$ of $H^0(\bar{W}, L)$, where $i: \bar{W} \to W$ is the involution interchanging the sheets of $\pi_W: \bar{W} \to W$.

First suppose that there is an $L \in \text{Sing}(\Xi)$, with $h^0(L) \geq 4$ and $h^0(L)$ even. Let $L_1$ be the pull-back of $L$ to $\bar{Z}$. Then $\deg(L_1) = 6$, and $h^0(L_1) \geq 4$. From Clifford's theorem it follows that $\bar{Z}$ is hyperelliptic. The hyperelliptic involution $\sigma$ of $\bar{Z}$ commutes with $i$. Moreover, $\sigma$ interchanges the fixed points of $i$, since all sections of $L_1$ come from sections of $L$. Hence $\sigma$ descends and $Z$ is hyperelliptic. If we denote the hyperelliptic involution of $Z$ also with $\sigma$, we see that $\sigma$ interchanges $p$, $p'$. So we are in case (1). This is the case in which $W$ has a so-called non-singular $g_2^1$. Non-singular means that the $g_2^1$ contains a divisor with non-singular support. It follows that $(P, \Xi)$ is a product of hyperelliptic Jacobians (see [1, p. 171]).

Secondly, suppose that there is an $L \in \text{Sing}(\Xi)$ with $h^0(L) = 2$, with a basis $\{s, t\}$ for $h^0(L)$ such that $s \otimes t^* = t^*s \otimes t$.

We first assume that $s$, $t$ have the property that either $s$ or $t$ is not zero at the singular point. By [1, Lemma 4.4], $L$ is of the form $(\pi_W)^*M(E)$. Here $M$ is a line bundle on $W$, $h^0(M) \geq 2$, $\deg((\pi_W)^*M(E)) = \deg(L) = \deg(\omega_W) = 6$. $M$ is non-singular, and $E$ is an effective divisor on $W$ with non-singular support such
that $(\pi_W)_*E \subset |\omega_W \otimes M^{-2}|$. Let $M_1$ be the pull-back of $M$ to $Z$. Then $M_1$ gives an $g^1_2$ or a $g^1_3$ on $Z$.

If $Z$ is hyperelliptic, a $g^1_3$ on $Z$ is the unique $g^1_2$ plus a point. We see that also in this case the two branch points $p$, $p'$ of $\tilde{Z} \to Z$ are interchanged by the hyperelliptic involution, and we are in case (1) again.

If $Z$ is not hyperelliptic, then $|M_1|$ is a $g^1_3$, $E = 0$, and the branch divisor $p + p'$ is contained in an element of the system $|M_1|$, say $p + p' \in |M_1|$. The relation $N_{M_1}(\pi_W)^*(M) = \omega_W$ gives by pull-back $N_{M_1}(\pi_Z)^*(M_1) = \omega_Z(p + p')$. It follows that $p + p' + 2p''$ is a canonical divisor, and $|M_1|$ is the pencil $|K_Z - p''|$. So we are in case (3) and $L = (\pi_W)^*M$.

Finally assume that the two sections $s$, $t$ of $L$ vanish simultaneously at the singular point. Let $L_1$ be the pull-back to $\tilde{Z}$ of $L$, and set $L_2 = L_1(-\tilde{p} - \tilde{p}')$. Then $Nm(L_2) = \omega_Z$, and $L_1$ has sections induced by $s$, $t$, which we will also denote by $s$, $t$, and which still satisfy $s \otimes t^* = t^*s \otimes t$. Then [1, Lemma 4.4] applies, giving that $L_2 = (\pi_Z)^*(M)$, where $M$ is a line bundle on $Z$, $h^0(M) \geq 2$. It follows that $Z$ is hyperelliptic and that $M$ is the line bundle associated with the unique $g^1_2$ on $Z$. So we are in case (3) and $L$ is an extension of $\pi_Z(M)(\tilde{p} + \tilde{p}')$ to $\tilde{W}$.

This finishes the first part of the proof. We will show in the second part that the above procedure can be reversed to obtain precisely one singularity of $\Xi$ if we are in case (2). If $p$ and $p'$ satisfy the condition of case (3) then we also obtain precisely one singularity of $\Xi$.

First assume that we are in case (2), i.e., $Z$ is hyperelliptic and that $p$, $p'$ are not interchanged by the hyperelliptic involution. The $g^1_2$ on $Z$ pulls back to a line bundle $L_2$ on $\tilde{Z}$. All extensions of the line bundle $L_1 = L_2(-\tilde{p} - \tilde{p}')$ to a line bundle $L$ on $\tilde{W}$ admit sections $s$, $t$ satisfying $s \otimes t^* = t^*s \otimes t$. There are precisely two extensions $L$ such that $Nm_L = \omega_W$, and only one of these is even (see [1, Prop. 3.5]). This takes care of the second case of the theorem, since we obtain indeed precisely one singularity $L$ of $\Xi$, showing that $(P, \Xi)$ is the Jacobian of a smooth, hyperelliptic curve.

Secondly, suppose that we are in case (3), and that we have a line bundle $M_1$ on $Z$, with $h^0_Z(M_1) = 2$ and $M_1 \cong \mathcal{O}_Z(p + p' + p'') \cong \mathcal{O}_Z(K_Z - p'')$. This $M_1$ has a unique extension $M$ to $W$ such that $h^0_W(M) = 2$. Then $Nm_W((\pi_W)^*M) = \omega_W$. For $Nm_W((\pi_W)^*M) = M^2$ is an extension of $M^2 = \omega_Z(p + p')$ to $W$ having a section non-zero at $p = p'$, and $\omega_W$ is the unique such extension.

Now $(\pi_W)^*M$ will yield an element of $\text{Sing}(\Xi)$ if $h^0_W((\pi_W)^*M) = 2$. We have the decomposition

$$H^0_Z((\pi_Z)^*M_1) \cong H^0_Z(M_1) \oplus H^0_Z(M_1 \otimes L_{Z/Z}^{-1})$$

which is the decomposition into even and odd parts with respect to the action of the involution $\iota$ on $H^0_Z((\pi_Z)^*M_1)$. We already have seen that all sections of $H^0_Z((\pi_Z)^*M_1) \cong H^0_W(M_1)$ extend to $\tilde{W}$. Also all sections of $H^0_Z((\pi_Z)^*M_1) \cong H^0_W(M_1 \otimes L_{Z/Z}^{-1})$ extend to $\tilde{W}$, since such a section is always zero at $\tilde{p}$ and $\tilde{p}'$. It
follows that $h^0_W((\pi_W)^*M) = h^0_Z((\pi_Z)^*M)$, and that $h^0_W((\pi_W)^*M) = 2$ if and only if $h^0_Z(M_1 \otimes L_{\mathcal{Z}/Z}^{-1}) = 0$. So we obtain precisely one singularity $(\pi_W)^*M$ of $\Sigma$ if and only if $h^0(M_1 \otimes L_{\mathcal{Z}/Z}^{-1}) = 0$. Note that $M_1 \otimes L_{\mathcal{Z}/Z}^{-1}$ is a theta characteristic: $M_1^2 \otimes L_{\mathcal{Z}/Z}^{-2} \cong \mathcal{O}_Z(p + p' + 2p'') \cong \omega_Z$ by assumption. 

3. The Degree of $\lambda$ and the Number of Hyperelliptic Fibres

Consider the family $\mathcal{P} \to B$ constructed in Section 1. Our aim in this section is to compute the degree of $\lambda = c_1((\pi_\mathcal{P})_*(\Omega_{\mathcal{P}/B}^{1}))$ and the number of hyperelliptic fibres of the family $\pi_\mathcal{P}: \mathcal{P} \to B$.

**Proposition 3.1.** The degree of $\lambda_{\mathcal{P}/B}$ equals $2^7$.

*Proof.* Denote by $p_B$ the projection $B \times C_3 \to C_3$, by $p_F$ the projection $F \times C_3 \to C_3$, and let $p_C$ be the composition $C \to B \times C_3 \xrightarrow{p_B} C_3$. The branch locus of $C \to B \times C_3$ is $\Gamma = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f}$. Here $\Gamma_{\pi_i \circ f}$ denotes the graph of $\pi_i \circ f$ in $B \times C_3$, $i = 1, 2$. We denote the ramification locus of $C \to B \times C_3$ by $\tilde{\Gamma} = \tilde{\Gamma}_{\pi_1 \circ f} + \tilde{\Gamma}_{\pi_2 \circ f} \subset C$. Let $\omega$ be a holomorphic 1-form on $C_3$. Let $\Omega = p_B^*\omega$ (respectively $\tilde{\Omega} = p_C^*\omega$) be its pull-back to $B \times C_3$ (respectively $C_3$), and consider it as a section of the line bundle $\omega_{B \times C_3/B}$ (respectively $\omega_{C_3}$).

We compute the intersection number $(\tilde{\Omega})^2$, where by $(\tilde{\Omega})$ we denote the divisor of $\tilde{\Omega}$. Clearly, $(\tilde{\Omega}) = p_C^*(\omega) + \tilde{\Gamma}$. So

$$(\tilde{\Omega})^2 = 2\tilde{\Gamma} \cdot p_C^*(\omega) + \tilde{\Gamma}^2$$

$$= 2\Gamma \cdot p_B^*(\omega) + \frac{1}{2}\Gamma^2.$$ 

Furthermore, we have

$$\Gamma \cdot p_B^*(\omega) = \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2}) \cdot p_F^*(\omega)$$

$$= 2\deg(f)\Gamma_{\pi_1} \cdot p_F^*(\omega)$$

$$= 2\deg(f)\deg(\pi_1)\deg(\omega)$$

$$= 16\deg(f),$$

and

$$\Gamma^2 = \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2})^2$$

$$= \deg(f)(2\Gamma_{\pi_1}^2 + 2\Gamma_{\pi_1} \cdot \Gamma_{\pi_2})$$

$$= \deg(f)(2\Gamma_{\pi_1}^2)$$

$$= 2\deg(f)((\pi_1 \times \text{id})^*\Delta_{C_3 \times C_3})^2$$

$$= -16\deg(f).$$
Hence
\[
(\tilde{\Omega})^2 = 32 \deg(f) - 8 \deg(f) = 24 \deg(f).
\]
Thus for the family \(\pi_C : C \to B\) we get \(\deg((\pi_C)_*(c_1(\omega_C/B)^2)) = 24 \deg(f)\). The degree of \(\lambda_{C/B}\) on \(B\) follows from the relation \(12\lambda = (\pi_C)_*(c_1(\omega_C/B)^2)\), which is a consequence of the Grothendieck–Riemann–Roch formula. Hence
\[
\deg(\lambda_{C/B}) = 2 \deg(f) = 2^7.
\]
We show that this is equal to the degree of the class \(\lambda_P/B\) corresponding to the family of Pryms \(\pi_P : P \to B\). By definition, \(\lambda_P/B\) is the first Chern class of the vectorbundle \((\pi_P)_*(\Omega^1_{P/B})\) on \(B\). We have the splitting \(\Omega^1_{\mathcal{J}C/B} = \Omega^1_{P/B} \oplus \Omega^1_{\text{Jac}(C_3)/B}\), so that
\[
(\pi_J)_*(\Omega^1_{\mathcal{J}C/B}) = (\pi_P)_*(\Omega^1_{P/B}) \oplus H^0(C_3, \omega_{C_3}).
\]
On the other hand, we have \((\pi_J)_*(\Omega^1_{\mathcal{J}C/B}) \cong (\pi_C)_*(\omega_{C/B})\). We conclude that \(\lambda_P/B = \lambda_C/B\). In particular \(\deg(\lambda_P/B) = 2^7\).

**PROPOSITION 3.2.** The number of hyperelliptic fibres of \(\pi_P : P \to B\) is \(2^8 \cdot 9\).

**Proof.** Let \(\mathcal{L}_b = \mathcal{L}|_{b \times C_3}\). Hence \(\mathcal{L}\) is the line bundle on \(B \times C_3\) used to construct the cover \(C \to B \times C_3\). The cover \(C_b/C_3\) is determined by the data \((C_3, f(\pi_1(b)) + f(\pi_2(b)), \mathcal{L}_b)\). By Theorem 2.1, \(\text{Prym}(C_b/C_3)\) is hyperelliptic if and only if there is a \(p \in C_3\) for which the divisor \(D = f(\pi_1(b)) + f(\pi_2(b)) + 2p\) is canonical and the theta characteristic \(\mathcal{L}_b^{-1}(D - p)\) is even.

The first condition depends only on the image \(f(b) \in F\). Let
\[
T = \{x \in F \mid \mathcal{O}_{C_3}(\pi_1(x) + \pi_2(x) + 2p) \cong \omega_{C_3} \text{ for some } p \in C_3\}.
\]
For \(x \in T\) let \(D\) be the unique canonical divisor \(\pi_1(b) + \pi_2(b) + 2p\). The set \(\mathcal{L}_b \setminus \{y \in f^{-1}(x)\}\) consists of all \(2^{2g} = 64\) distinct roots of \(\pi_1(x) + \pi_2(x)\) in \(\text{Pic}(C_3)\). It follows that the set \(\mathcal{L}_b^{-1}(D - p) \setminus \{x \in f^{-1}(x)\}\) consists of 64 distinct theta characteristics, of which 36 are even and 28 are odd. Hence \(h = 36 \cdot \#T\). The proposition now follows from the following lemma.

**LEMMA 3.3.** The cardinality of \(T\) equals 64.

**Proof.** Consider the curve \(\tilde{F} = \tilde{F}(t) = \{(x, p) \mid p \in \text{Supp}(K(x) - \pi_1(x) - \pi_2(x))\}\) in \(F \times C_3\). \(\tilde{F}\) is smooth, since \(F\) and \(C_3\) are. Clearly \(T\) is the branch locus of the projection \(\tilde{F} \to F\). We compute \(\#T\) by computing the genus of \(\tilde{F}\).

Let \(d\) be the degree of the projection of \(F\) onto \(F\) and \(d'\) be the degree of the projection of \(\tilde{F}\) onto \(C_3\). By definition \(d = 2\).
We compute $d'$ first by computing this degree not for $\tilde{F}(t)$ but for $\tilde{F}(0)$ instead. Note that $\tilde{F}(0) \subset C_3 \times C_3$ is reducible: $\tilde{F}(0) = \Delta + \Delta'$. Here $\Delta' \subset C_3 \times C_3$ denotes the graph of $i_\phi$, the involution interchanging the sheets of $\phi: C_3 \to E$. Then $\tilde{F}(0)$ consists of the disjoint union of the tangential correspondence

$$\{(p, q) \mid q \text{ on tangent line to } q \} \subset C_3 \times C_3$$

and the correspondence

$$\{(p, q) \mid p + i_\phi(p) + q + i_\phi(q) \in |K_{C_3}| \} \subset C_3 \times C_3.$$ 

It follows that $d' = 10 + 2 = 12$.

Denote by $E$ a 'horizontal' fibre $F \times \{c\}$ and by $F$ a 'vertical' fibre $\{y\} \times C_3$ in $F \times C_3$. Let $\Gamma_{\pi_i}$ be the graph of the maps $\pi_i: F \to C_3$, $i = 1, 2$. Then we have (see [5, p. 285]):

$$\tilde{F} \sim (d + 2)E + (d' + 4)F - \Gamma_{\pi_1} - \Gamma_{\pi_2} = 4E + 16F - \Gamma_{\pi_1} - \Gamma_{\pi_2}.$$ 

Here $\sim$ denotes linear equivalence. Furthermore, $\Gamma_{\pi_1}^2 = \Gamma_{\pi_2}^2 = 2 \cdot \Delta_{C_3}^2 = -8$ and $\Gamma_{\pi_1} \cdot \Gamma_{\pi_2} = 0$ by construction, and $K_{F \times C_3} \sim 4E + 16F$. From the adjunction formula it follows that $2p_a(\tilde{F}) = 2 = 96$, so in particular $\tilde{F}$ is irreducible.

Applying the Riemann–Hurwitz formula to the projection $\tilde{F} \to F$ yields the number of points of $T$. It is $\deg(K_{\tilde{F}}) - d \cdot \deg(K_{C_3}) = 96 - 2 \cdot 16 = 64$. \hfill $\square$

These results enable us to determine the intersection of the families $\mathcal{T} \to \tilde{B}$ with the hyperelliptic locus. In the Chow group $A^1(M_3)$ (with $Q$-coefficients) denote by $[H_3]_Q$ the $Q$-class of the hyperelliptic locus, and by $\lambda$ the class $c_1(\pi_*(\omega_{C_3/M_3}))$ (see [8]). In $A^1(M_3)$ we have that $[H_3]_Q = 9\lambda$ (see [6]).

Let $[\tilde{B}]_Q$ be the $Q$-class of the image of $\tilde{B}$ in $M_3$. Then $\deg([\tilde{B}]_Q \cdot [\lambda]_Q) = \deg(\lambda|_{\tilde{B}}) = 2 \deg(\lambda|_{\tilde{B}}) = 2^8$. The relation $[H_3]_Q = 9\lambda_Q$ implies that $\deg([\tilde{B}]_Q \cdot [H_3]_Q) = 9 \cdot 2^8$. We have $9 \cdot 2^8$ hyperelliptic fibres on $\tilde{B}$ by Corollary 3.3. If we let $b \in \tilde{B}$ be a point such that the fibre $T_b$ is hyperelliptic, it follows that the intersection multiplicity $i([\tilde{B}]_Q, [H_3]_Q; b) = 1$.

This multiplicity is computed on the base of the versal deformation space of $T_b$. The tangent space to the versal deformation space in the point corresponding to $T_b$ is $H^1(T_b, \Theta_b)$, where $\Theta_b$ is the tangent bundle of $T_b$. The tangent direction of $\tilde{B}$ in $H^1(T_b, \Theta_b)$ is the image of the Kodaira–Spencer map $T_{\tilde{B}, b} \to H^1(T_b, \Theta_b)$, and the tangent space to the hyperelliptic locus is the subspace of $H^1(T_b, \Theta_b)$ invariant under the hyperelliptic involution on $H^1(T_b, \Theta_b)$. So we conclude that the image of the Kodaira–Spencer map in $H^1(T_b, \Theta_b)$ is not contained in the invariant subspace of the hyperelliptic involution acting on $H^1(T_b, \Theta_b)$. However, we do not know how to prove this directly.
4. The Surjectivity of the Prym Map

Let $\mathcal{R}_{3,2}$ be the moduli space of tuples $(C, p + p', L)$, where $C$ is a smooth curve of genus 3, $p + p'$ is an effective divisor of degree 2 on $C$ such that $p \neq p'$, and $L$ is a line bundle on $C$ such that $L^2 \cong \mathcal{O}_C(p + p')$. Such a tuple $(C, p + p', L)$ determines a double cover of smooth curves $\tilde{C} \rightarrow C$ to which we can associate its Prym variety $\text{Prym}(C, p + p', L)$. This operation yields a morphism $\mathcal{P}_r: \mathcal{R}_{3,2} \rightarrow A_3$, where $A_3$ is the moduli space of principally polarized abelian varieties of dimension 3. We call this the Prym map.

Let $\mathcal{R}^{\text{ell}}_{3,2} \subset \mathcal{R}_{3,2}$ be the locus of tuples $(C, p + p', L)$ with the property that $C$ admits a map of degree 2 to an elliptic curve. We will denote the restriction of the Prym map to $\mathcal{R}^{\text{ell}}_{3,2}$ also by $\mathcal{P}_r$. The aim of this section is to show that $\mathcal{P}_r: \mathcal{R}^{\text{ell}}_{3,2} \rightarrow A_3$ is dominant. Note that the dimension of $\mathcal{R}_{3,2}$ is 8 and that both $\mathcal{R}^{\text{ell}}_{3,2}$ and $A_3$ are of dimension 6. Hence if the morphism $\mathcal{P}_r: \mathcal{R}^{\text{ell}}_{3,2} \rightarrow A_3$ is dominant, it is also generically finite.

We compute the codifferential of $\mathcal{P}_r$. After taking appropriate finite level structures, we may assume that $\mathcal{R}^{\text{ell}}_{3,2}$ and $A_3$ are smooth. Since $\mathcal{R}_{3,2}$ is an unramified cover of $\mathcal{M}_{3,2}$, we identify the cotangent spaces at the points $(C, p + p', L) \in \mathcal{R}_{3,2}$ and $(C, p + p') \in \mathcal{M}_{3,2}$. The identification of the cotangent space $T^*_{\mathcal{M}_{3,2}, (C, p + p')}$ with $H^0(C, \omega_C^2(p + p'))$ thus gives us $T^*_{\mathcal{R}_{3,2}, (C, p + p', L)} \cong H^0(C, \omega_C^2(p + p'))$.

Let $(A, \Theta)$ be a point of $A_3$. We identify its cotangent space $T^*_{A_3, (A, \Theta)}$ with $S^2T^*_A$, the symmetric square of the cotangent space to $A$ at the origin, by standard identifications. In our case $(A, \Theta) = \text{Prym}(C, p + p', L)$. From the definition of $\text{Prym}(\tilde{C}/C)$ as the odd part of $\text{Jac}(\tilde{C})$ (see [7]) and the splitting $H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes L)$ it follows that $T^*_{\text{Prym}(C, p + p', L), 0} \cong H^0(C, \omega_C \otimes L)$.

Using these identifications, the codifferential

$$\mathcal{P}_r^*: S^2H^0(C, \omega_C \otimes L) \rightarrow H^0(C, \omega_C^2(p + p'))$$

is the composition of two natural maps. It is the cup-product $S^2H^0(C, \omega_C \otimes L) \rightarrow H^0(C, \omega_C^2 \otimes L^2)$ followed by the identification $H^0(C, \omega_C^2 \otimes L^2) \cong H^0(C, \omega_C^2(p + p'))$ induced by the isomorphism $L^2 \cong \mathcal{O}_C(p + p')$.

The codifferential is computed in [1] for the Prym map $\mathcal{R}_{g+1} \rightarrow A_g$, where $\mathcal{R}_{g+1}$ is the moduli space of ‘admissible’ double covers. We obtain the above result by embedding $\mathcal{R}_{3,2}$ into $\mathcal{R}_4$ via $(C, p + p', L) \mapsto (C/(p \sim p'), L)$.

Now suppose that $C$ admits an elliptic involution, i.e. an automorphism of order 2 such that the quotient is an elliptic curve. Then $H^0(C, \omega_C^2)$ decomposes into $(\pm 1)$-eigenspaces $H^0(C, \omega_C^2)^\pm$. Embed $H^0(C, \omega_C^2(p + p'))$ into $H^0(C, \omega_C^2(p + p'))$. The cotangent space $T^*_{\mathcal{R}^{\text{ell}}_{3,2}, (C, p + p', L)}$ of $\mathcal{R}^{\text{ell}}_{3,2}$ at $(C, p + p', L)$ can be identified with the quotient $H^0(C, \omega_C^2(p + p'))/H^0(C, \omega_C^2)^-$. The codifferential of $\mathcal{R}^{\text{ell}}_{3,2} \rightarrow A_3$ in the point $(C, p + p', L)$ is the composition

$$\mathcal{P}_r^*: S^2H^0(C, \omega_C \otimes L) \rightarrow H^0(C, \omega_C^2(p + p'))$$
We now compute this codifferential in one specific point of $\mathcal{R}_{3,2}^{\text{ell}}$.

Let $C_{48}$ be the plane quartic curve given by the equation $X^4 + Y(Y^3 + Z^3) = 0$. It has an automorphism group of order 48 and admits the elliptic involution $X \mapsto -X, Y \mapsto Y, Z \mapsto Z$. This curve has four hyperflexes and 16 ordinary flexes. Let $q = (0, 1, -1), p = (0, 0, 1)$ and $p' = p'' = (1, \eta, 0)$, with $\eta$ a fourth root of $-1$. The points $p$ and $q$ are hyperflexes and $p' = p''$ is an ordinary flex. If we let $L$ be the line bundle $\mathcal{O}_{C_{48}}(2q - p'')$, the tuple $(C_{48}, p + p', L)$ is a point of $\mathcal{R}_{3,2}^{\text{ell}}$.

**Lemma 4.1.** The codifferential of the Prym map $\mathcal{P}_r: \mathcal{R}_{3,2}^{\text{ell}} \to A_3$ in the point $(C_{48}, p + p', L)$ has maximal rank.

**Proof.** The canonical embedding of $C_{48} \subset \mathbb{P}^2$ and the isomorphism $H^0(C_{48}, \omega \otimes L) \cong H^0(C_{48}, \omega^2(-2q - p''))$ enable us to write down an explicit basis for the space $H^0(C_{48}, \omega \otimes L)$. This gives six generators for the image of $S^2 H^0(C_{48}, \omega \otimes L)$ in $H^0(C_{48}, \omega^2(p + p'))$. Embedding $H^0(C_{48}, \omega^2(p + p')) \cong H^0(C_{48}, \omega^3(-2p''))$ in $H^0(C_{48}, \omega^3) \cong H^0(\mathbb{P}^2, O(3))$ we get six homogeneous forms of degree 3.

Also we embed $H^0(C_{48}, \omega^2(-2p''))$ in $H^0(\mathbb{P}^2, O(3))$ and write down a basis. In fact, if $L_{p''}$ is the equation of the flex line through $p''$ then $(XY L_{p''}, XZ L_{p''})$ is a basis. A straightforward calculation shows that these eight forms are linearly independent in $H^0(\mathbb{P}^2, O(3))$. \[ \Box \]

**Corollary 4.2.** The Prym map $\mathcal{P}_r: \mathcal{R}_{3,2}^{\text{ell}} \to A_3$ is dominant and generically finite.

**Corollary 4.3.** The generic curve of genus 3 occurs as a fibre of a family $T \to \tilde{B}$ appearing in Corollary 1.2.

**References**