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Causality and Tsirelson's bounds

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We study the properties of no-signaling correlations that cannot be reproduced by local measurements on entangled quantum states. We say that such correlations violate Tsirelson bounds. We show that if these correlations are obtained by some reversible unitary quantum evolution U , then U cannot be written in the product form $U_A \otimes U_B$. This implies that U can be used for signaling and for entanglement generation. This result is completely general and in fact can be viewed as a characterization of Tsirelson bounds. We then show how this result can be used as a tool to study Tsirelson bounds and we illustrate this by rederiving the Tsirelson bound of $2\sqrt{2}$ for the Clauser-Horn-Shimony-Holt inequality, and by deriving a new Tsirelson bound for qutrits.

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I. INTRODUCTION

Quantum non-locality is manifest when measurements are carried out on two entangled particles in spatially separated regions. As first shown by Bell [1], the correlations obtained in such an experiment cannot be reproduced using classical local models, often called “local hidden variable” (LHV) models. Nonlocality is a fascinating chapter of physics and has attracted much attention since its discovery because it relates two fundamental aspects of nature, special relativity and quantum mechanics.

Because information cannot travel faster than the speed of light, and since the measurements are carried out in spatially separated regions, such a setup cannot be used to transmit information from one site to the other. This constraint is expressed formally by the “no-signaling conditions” we describe explicitly below.

The amount of nonlocality present in such correlations can be studied quantitatively by introducing the concept of “Bell expression.” This is an expression which is bounded by a certain value for LHV models, but can exceed this value in the case of quantum correlations. We illustrate this by the well known Clauser-Horn-Shimony-Holt (CHSH) expression [2]. The CHSH expression is bounded by 2 for LHV models. But local measurements carried out on entangled quantum systems can reach the value of $2\sqrt{2}$. Tsirelson [3] showed that this is the maximum value attainable by local measurements on entangled quantum systems.

However, correlations exist which obey the no-signaling conditions and for which the CHSH expression reaches the value of 4 (this is the maximum value consistent with positivity of the probabilities). Popescu and Rohrlich [4] were the first to study these maximally nonlocal correlations as

objects of interest in their own right. This led them to raise a fundamental question: why is nature not maximally nonlocal? Providing an answer to this question would presumably deepen our understanding of relativity and quantum mechanics.

The CHSH expression is not the only way to study the nonlocality of quantum correlations. One can generalize it in many ways, for instance by changing the quantum states, by changing the measurements, and in particular letting the measurements have more settings and/or more outcomes, or by having more entangled parties. In all cases the nonlocality of the quantum correlations can be studied using Bell inequalities which generalize the CHSH inequality. For any such Bell inequality there will be three, generally distinct, characteristic values: the maximal value attainable by LHV models, the maximal value attainable by local measurements on entangled quantum state, and the maximal value compatible with the no-signaling conditions. In the present work, by analogy with the CHSH expression, we will call “Tsirelson's bound” the maximal value of any Bell expression attainable by local measurements on entangled quantum states.

The boundary of the space of correlations which can be attained using a given model (LHV theory, quantum mechanics) presents a fundamental character. But it is very difficult to find this boundary in general. In the case of the LHV models, finding the boundary can be formulated as the mathematical problem of finding the facets of a polytope of which the vertices are known [5,6].

In the case of the boundary of the space of quantum correlations, Tsirelson's seminal work has been extended in a number of ways. The specific case of the CHSH inequality has been further studied in Refs. [4,7–10]. In the case of two settings and two outcomes at each site all Tsirelson bounds are known for inequalities of the CHSH type (correlation inequalities), see Refs. [5,11] for reviews, but not in the general case. Very little is known when the number of settings and the number of outcomes increases (see however the numerical approach of Ref. [12]), although the quantum corre-

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lations appear to have a very interesting structure already for dimension 3, see, for instance, Refs. [13,14]. Very few results are also known in the case of more than two parties [15].

Thus there clearly remains a lot to be learned about the boundary of the space of correlations obtainable by local measurements on entangled quantum states. Better understanding this boundary is of fundamental interest as mentioned above, but is also of practical interest as understanding this boundary may enable the derivation of better experimental tests of quantum nonlocality, and may suggest better ways in which quantum nonlocality can help in information processing tasks such as communication complexity [16].

All these earlier works on Tsirelson's bound proceeded essentially in the same way, namely, they first imposed that the measurements carried out on the entangled states are local, and then tried to find, given this condition, what is the maximal value attainable by a Bell inequality. Here we shall adopt another approach which at first sight may seem surprising, even paradoxical, but which is very much in the spirit of the work of Popescu and Rohrlich. Indeed we shall explore what are the consequences of requiring the coexistence of the following apparently contradictory conditions: (1) the existence of correlations that obey the no-signaling condition and which violate Tsirelson's bound and (2) the validity of quantum mechanics. The only way to make these apparently contradictory requirements compatible is to drop the locality condition: it must be that during the measurement process some communication took place between the parties. Hence one is dealing with a single global, rather than several local, measurements.

The communication that takes place during the measurement is, however, "hidden." Indeed the correlations obey the no-signaling condition. Thus although communication was necessary to produce the correlations, the correlations themselves cannot be used to communicate. The question of finding Tsirelson bounds can thus be reformulated as the question of finding whether or not, within the framework of quantum mechanics, such "hidden" communication is necessary to obtain the correlations.

Our first result is to show the (surprising) fact that, if the whole measurement process including the hidden communication which takes place during the measurement is a unitary reversible process, then this hidden communication can always be "revealed." By this we mean that if the measurement process is implemented by some unitary evolution U , then by carrying out local quantum operations the parties can in fact signal and generate entanglement. More specifically this is realized by carrying out a coherent superposition of different measurements. This result is completely general. In fact it characterizes the boundary of the space of correlations obtainable by carrying out local measurements on entangled quantum states. This first result is closely related to earlier work devoted to characterizing the constraints that locality and causality impose on quantum operations, see Refs. [17–19]. It is also closely related to—but more general than—work of Dieks [10] who also emphasized the causality conditions implicit in derivations of Tsirelson bounds.

Because of its generality one can turn the above remark into a tool to study Tsirelson bounds. This is done as follows:

take some correlations and suppose that the quantum measurement process that gave rise to them is unitary and reversible. Let the parties carry out a coherent superposition of different measurements and try to show that such a superposition measurement generates entanglement, or allows communication. If you succeed you have shown that these correlations exceed Tsirelson's bound. Thus whereas previous studies of Tsirelson bounds approached the bound from below by exhibiting correlations that could be attained using local quantum measurements, here we have a method of approaching the bounds from above, by showing that some correlations cannot be obtained using local measurements on entangled quantum states. The two approaches are thus complementary.

We apply this method to several examples. First we study the simple example of the correlations considered by Popescu and Rohrlich that maximally violate the CHSH inequality and show how in this case the unitary U would allow signaling and entanglement generation. We also generalize this example to the case where the number of measurement settings and the number of measurement outcomes is d . Then we rederive Tsirelson's bound for the CHSH inequality. Finally we study a generalization of the CHSH inequality involving three measurement settings for each party and three outcomes for each measurement. We derive a Tsirelson bound in this case although, to our knowledge, none was known previously in this case. We also point out an important discussion at the end of Sec. III devoted to the interpretation of our approach, and in particular its relation to the program initiated by Popescu and Rohrlich whose aim is to understand why the quantum correlations are not maximally nonlocal.

II. NONLOCAL CORRELATIONS

Consider the following situation: there are two parties, Alice and Bob, who each choose an input (their measurement setting), x and y , and produce an output (their measurement outcome), a and b . Upon repeating this experiment many times one can describe it synthetically by the set of probability distributions $P(a, b|x, y)$ (the probabilities of the outputs conditional on the inputs). We impose that the correlations obey the no-signaling conditions

$$\sum_b P(a, b|x, y) = P(a|x) \text{ is independent of } y,$$

$$\sum_a P(a, b|x, y) = P(b|y) \text{ is independent of } x \quad (1)$$

which express the fact that Alice's (Bob's) output cannot provide her (him) with any information about Bob's (Alice's) input.

Of course if Alice's and Bob's measurements, starting from the choice of inputs x or y and ending with the production of the output a or b , take place in spatially separated locations, then the no signaling conditions must be obeyed. But as mentioned in the introduction we shall also consider correlations which require some communication between Alice and Bob. If the no signaling conditions are not obeyed then it is immediately obvious that communication is required to obtain the correlations $P(a, b|x, y)$ and the whole

question of Bell inequalities and Tsirelson bounds becomes irrelevant. Thus we always impose the no signaling conditions.

Let us suppose that the above measurement process can be described entirely within the context of the quantum formalism. Without loss of generality we can describe it as follows: initially the parties start with the state

$$|\Psi_{xy}^0\rangle = |0\rangle_A |0\rangle_B |x\rangle_A |y\rangle_B |\psi\rangle_{AB}, \quad (2)$$

where $|x\rangle_A |y\rangle_B$ are the quantum states containing Alice's and Bob's inputs (i.e., they specify the measurement settings), $|0\rangle_A |0\rangle_B$ are the initial states of Alice's and Bob's output registers, as well as the initial state of any local ancillas they may use, $|\psi\rangle_{AB}$ is the entangled state Alice and Bob share. The subscripts indicate to which subspace, Alice's, Bob's, or both, the states belong.

In order to produce their outputs, Alice and Bob carry out some unitary operation U to obtain the state

$$|\Psi_{xy}^1\rangle = U|\Psi_{xy}^0\rangle = \sum_{a,b} |a\rangle_A |b\rangle_B |\varphi_{xy}^{ab}\rangle_{AB}. \quad (3)$$

Here $|a\rangle_A$ and $|b\rangle_B$ are the final states of Alice's and Bob's output registers. If Alice and Bob carry out local measurements, then $U = U_A \otimes U_B$ must have a product form. If Alice and Bob carry out some communication during the measurement, then U will in general not be a product.

In all cases the un-normalized states $|\varphi_{xy}^{ab}\rangle_{AB}$ satisfy the orthogonality relations

$$\sum_{a,b} \langle \varphi_{xy}^{ab} | \varphi_{x'y'}^{ab} \rangle = \delta_{xx'} \delta_{yy'}. \quad (4)$$

Using the above notation, the probabilities of finding outcomes a and b conditional on inputs x and y is

$$P(a,b|x,y) = \langle \varphi_{xy}^{ab} | \varphi_{xy}^{ab} \rangle. \quad (5)$$

The space of correlations $P(a,b|x,y)$ which can be obtained using local measurements, i.e., using product operations $U = U_A \otimes U_B$, is easily shown to be a convex set. Hence it can be characterized by linear inequalities

$$\mathbf{L} \cdot \mathbf{P} = \sum_{a,b,x,y} L_{abxy} P(a,b|x,y) \leq L_{QM}. \quad (6)$$

Here L_{QM} is the largest value of $\mathbf{L} \cdot \mathbf{P}$ obtainable using local measurements

$$L_{QM} = \max_{\psi_{AB}, U=U_A \otimes U_B} \sum_{x,y,a,b} L_{abxy} \langle \varphi_{xy}^{ab} | \varphi_{xy}^{ab} \rangle. \quad (7)$$

The set of inequalities $\mathbf{L} \cdot \mathbf{P} \leq L_{QM}$ constitute the Tsirelson bounds.

III. CHARACTERIZING NONLOCAL CORRELATIONS IN TERMS OF SIGNALING

Let us now prove our first result. We consider some correlations $P(a,b|x,y)$ that obey the no signaling condition. These correlations must satisfy positivity $P(a,b|x,y) \geq 0$ and normalization $\sum_{a,b} P(a,b|x,y) = 1$. Hence there exists

some transformation of the form (2) $\rightarrow$ (3), realized by some U , that implements the measurements leading to these correlations. We will show that if these correlations violate a Tsirelson bound of the form Eq. (6), then the transformation U can be used to generate entanglement and to carry out classical communication. This result holds for all values of the number of measurement settings and measurement outcomes.

To prove this we reason as follows. In view of definition (7), U cannot be written as a tensor product $U \neq U_A \otimes U_B$. However, it is easy to show that any unitary that cannot be written as a tensor product can generate entanglement. The idea of the argument is to consider the action of U_{AB} on the state $|\psi\rangle_{A'ABB'} = N \sum_i |i\rangle_{A'} |i\rangle_A \sum_j |j\rangle_B |j\rangle_{B'}$ where N is a normalization constant. It is easy to see that the state $|\psi\rangle_{A'ABB'} = I_{A'} \otimes U_{AB} \otimes I_{B'} |\psi\rangle_{A'ABB'}$ completely characterizes the transformation U_{AB} . If $U_{AB} = U_A \otimes U_B$, then there is no entanglement between systems $A'A$ and BB' in state $|\psi\rangle$. Conversely if state $|\psi\rangle$ contains no entanglement between systems $A'A$ and BB' , then the transformation U_{AB} associated to ψ_f is a tensor product. Hence separability of ψ_f and tensor product character of U_{AB} are equivalent statements. This implies that any unitary which cannot be written as a tensor product can generate entanglement by acting with it on the maximally entangled state $|\psi\rangle_{A'ABB'}$. We now refer to recent work of Bennett *et al.* [18] in which it is shown that any unitary that can generate entanglement necessarily also allows signaling. This concludes the proof.

Note that for the above proof to hold it is essential to allow the parties to manipulate coherently the registers, $|x\rangle$ and $|y\rangle$, which specify which measurement is going to be realized. Specifically they must be able to prepare superpositions of different register states, and to locally entangle these registers with other systems. We shall see this at work in the examples considered below.

At this point there is an important remark to make. The above argument supposed that there is no decoherence. Decoherence can of course be included in the description of Sec. II as follows: first one adds to the description of the initial state Eq. (2) the initial state of the environment; second during the evolution the state of the measuring devices can get entangled with the environment; third at the end of the evolution one traces over the environment. But if there is such an environment which is inaccessible to the parties, it may be that the entanglement which necessarily exists if the correlations violate a Cirel'son bound, is in fact localized in the environment, and hence is inaccessible. Thus our first result only holds if no decoherence occurs, i.e., if the evolution is reversible.

This remark does not change anything to the applications we consider in Sec. IV and V, but it is an important remark from the point of view of interpretation. We suspect that reversibility plays a crucial role in understanding Tsirelson's bounds. Indeed one can rephrase the program initiated by Popescu and Rohrlich as, "What is the minimum set of axioms which imply Tsirelson's bound?" Popescu and Rohrlich showed that causality is not enough. However, adding reversibility may constrain the correlations much more. This idea is supported by the fact that Hardy recently proposed a set of axioms which imply quantum mechanics, and revers-

ibility is the crucial one that differentiates classical from quantum mechanics [21]. Our result above seems to suggest that both causality and reversibility could be enough to rule out the extremal correlations of Popescu and Rohrlich, and maybe even recover Tsirelson's bound. We do not know how to prove this, but we feel it is a very interesting avenue of research.

IV. SIGNALING WITH PERFECT CORRELATIONS

As discussed in Refs. [5,6], for a fixed number of measurement settings and outcomes, the correlations which obey the no signaling conditions (1) constitute a polytope of which the quantum correlations (those obtained using only entanglement and local quantum measurements) and classical correlations (those obtained using only local hidden variable models) are proper subsets. The most nonlocal correlations are the extremal points, i.e., the vertices, of the no signaling polytope.

The correlations considered by Popescu and Rohrlich are extremal no signaling correlations in the above sense. Explicitly, if we suppose that $a, b, x, y \in \{0, 1\}$, they take the form $P^{PR}(a, b|x, y) = 1/2$ if $a \oplus b = xy$, $P^{PR}(a, b|x, y) = 0$ otherwise.

In what follows we shall consider a generalization of the Popescu-Rohrlich correlations to the case where the number of measurement settings is d and the number of measurement outcomes is d . Specifically take $a, b, x, y \in \{0, \dots, d-1\}$, then these correlations take the form

$$P(a, b|x, y) = \begin{cases} 1/d & \text{if } a - b = xy \text{ mod } d, \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

[in the condition that appears in Eq. (8) the addition and multiplication are modulo d]. We note that the correlations (8) are extremal points of the set of no signaling correlations with $a, b, x, y \in \{0, \dots, d-1\}$. These correlations coincide with the Popescu-Rohrlich correlations for $d=2$. (Note that this is not the only generalization of the Popescu-Rohrlich correlations to higher dimensions. For instance, in Ref. [6] a generalization to the case $x, y \in \{0, 1\}, a, b \in \{0, \dots, d-1\}$ was considered. Our method seems more difficult to apply in this case, and we therefore consider here the correlations (8).)]

The argument we now present is loosely inspired by Ref. [20]. Equation (8) implies that in Eq. (3) the sum over a, b is restricted to $a - b = xy \text{ mod } d$. Let us now show that in this case the unitary transformation U , that implements the correlations from Eq. (8), allows signaling and generation of entanglement. To this end suppose that after realizing U the parties carry out the local unitary transformations $e^{i2\pi\hat{a}/d}$ and $e^{-i2\pi\hat{b}/d}$ where $e^{i2\pi\hat{a}/d}|a\rangle_A = e^{i2\pi a/d}|a\rangle_A$, $e^{-i2\pi\hat{b}/d}|b\rangle_B = e^{-i2\pi b/d}|b\rangle_B$. Because $a - b = xy \text{ mod } d$, the resulting state can be written as

$$e^{i2\pi(\hat{a}-\hat{b})/d}U|\Psi^0\rangle = e^{i2\pi xy/d} \sum_{\substack{a,b \\ a-b=xy \text{ mod } d}} |a\rangle_A |b\rangle_B |\varphi_{xy}^{ab}\rangle_{AB}.$$

Finally the parties carry out the inverse of the transformation U to obtain

$$|\Psi_{xy}^f\rangle = U^\dagger e^{i2\pi(\hat{a}-\hat{b})/d}U|\Psi^0\rangle = e^{i2\pi xy/d} |0\rangle_A |0\rangle_B |x\rangle_A |y\rangle_B |\psi\rangle_{AB}. \quad (9)$$

They have recovered the original state up to the phase $e^{i2\pi xy/d}$.

To signal, the parties carry out the above series of transformations on a superposition of inputs. Indeed if Bob uses a uniform superposition of input states the evolution is

$$\frac{1}{\sqrt{d}} \sum_y |\Psi_{xy}^f\rangle_B = |0\rangle_A |0\rangle_B |x\rangle_A \left(\frac{1}{\sqrt{d}} \sum_y e^{i2\pi xy/d} |y\rangle_B \right) |\psi\rangle_{AB}. \quad (10)$$

The states $|z\rangle = (1/\sqrt{d}) \sum_y e^{i2\pi zy/d} |y\rangle_B$ are orthogonal (they are the Fourier transform of $|y\rangle_B$). Hence by measuring his input variable in this basis Bob can learn Alice's input, i.e., Alice has transmitted $\log_2 d$ bits of information to Bob.

One can show in a similar way that the transformation U also allows generation of entanglement. Indeed if both parties use a coherent superposition of input states the final state is

$$\frac{1}{d} \sum_{x,y} |\Psi_{xy}^f\rangle_A = |0\rangle_A |0\rangle_B \frac{1}{d} \sum_{x,y} e^{i2\pi xy/d} |x\rangle_A |y\rangle_B |\psi\rangle_{AB}. \quad (11)$$

One easily checks that the final state contains $\log d$ ebits more entanglement than the initial state.

In summary by combining the transformation U , its inverse $U^\dagger$, and the local transformation $e^{i2\pi\hat{a}/d}$ and $e^{-i2\pi\hat{b}/d}$, the parties can either transmit $\log d$ bits of classical information or can generate $\log d$ ebit of entanglement. This would be impossible if U was local and therefore shows that the correlations (8) violate a Tsirelson bound.

V. NOISY CORRELATIONS AND TSIRELSON BOUNDS

The above example illustrates our main result in the case of extremal, i.e., maximally nonlocal, correlations. We now generalize it to the case where the correlations are noisy. We will suppose that the correlations are noisy versions of the extremal correlations (8). Our aim is to show that if the amount of noise is too small, then the unitary transformation U that implements the measurement is nonlocal and in particular allows the parties to communicate. In what follows we will rederive Tsirelson bound of $2\sqrt{2}$ for the CHSH inequality and also obtain a Tsirelson bound for correlations of the form (8) in the case $d=3$.

In order to measure the degree of non-locality of quantum correlations we will use a Bell expression of the form (6). We write it as $\mathcal{L}^d(\mathbf{P}) = \mathbf{L}^d \cdot \mathbf{P}$ with

$$L_{abxy}^d = \begin{cases} 1/d^2 & \text{if } a - b = xy \text{ mod } d, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

This Bell expression $\mathcal{L}^d$ gives the probability, averaged over the inputs, that the outputs satisfy the relation $a - b = xy \text{ mod } d$, hence $0 \leq \mathcal{L}^d \leq 1$. Note that $\mathcal{L}^2$ (i.e., $d=2$) is related to the CHSH expression by the rescaling $\mathcal{L}^2 = \mathcal{L}^{\text{CHSH}}/8 + 1/2$.

The reason for using $\mathcal{L}^d$ is that it allows us to compare the different values of d , since it is always bounded between 0 and 1. We will denote by $\mathcal{L}^d(LHV)$ the values of $\mathcal{L}^d$ obtainable in a local hidden variable model and by $\mathcal{L}^d(QM)$ the values of $\mathcal{L}^d$ obtainable using local measurements on entangled quantum states. In the case $d=2$ one has $\mathcal{L}^2(LHV) \leq 3/4=0.75$ and $\mathcal{L}^2(QM) \leq 1/2 + 1/2\sqrt{2} \approx 0.853$ [this is Tsirelson bound (rescaled) which we rederive below]. In the case $d=3$ it is not difficult to show, by enumerating all deterministic local classical strategies, that $\mathcal{L}^3(LHV) \leq 2/3 \approx 0.66$ and we show below that $\mathcal{L}^3(QM) \leq 1/3 + 2/3\sqrt{3} \approx 0.72$ (although we do not know whether this bound can be attained). This shows that as d increases from 2 to 3 it is increasingly difficult to satisfy the relation $a-b=xy \bmod d$, both classically and quantum mechanically. We have so far not been able to generalize these results to higher values of d .

Our argument proceeds essentially as above. We suppose the parties start with state $|\psi_{xy}^0\rangle$ given in Eq. (2). The main new ingredient is that before applying U the parties first copy their input into a separate register on which U does not act (this will provide us with a useful orthogonality relation later). This is done using the unitary transformation which in the computational basis realizes the copy operation. That is Alice carries out the local unitary operation U_A^{copy} which acts as $U_A^{\text{copy}}|0\rangle_A^{\text{copy}}|x\rangle_A = |x\rangle_A^{\text{copy}}|x\rangle_A$, where $|0\rangle_A^{\text{copy}}$ is the initial state of the copy system. (Note that this operation only copies perfectly in the computational basis, it does not copy superpositions perfectly, hence it is not in contradiction with the quantum no-cloning theorem.) Similarly Bob carries out U_B^{copy} which acts as $U_B^{\text{copy}}|0\rangle_B^{\text{copy}}|y\rangle_B = |y\rangle_B^{\text{copy}}|y\rangle_B$.

Thus we have

$$\begin{aligned} & U_A^{\text{copy}} \otimes U_B^{\text{copy}} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy}^0\rangle \\ &= |x\rangle_A^{\text{copy}} |y\rangle_B^{\text{copy}} |\Psi_{xy}^0\rangle \\ &= |x\rangle_A^{\text{copy}} |y\rangle_B^{\text{copy}} |0\rangle_A |0\rangle_B |x\rangle_A |y\rangle_B |\psi\rangle_{AB}. \end{aligned} \quad (13)$$

The parties now act with U (We suppose that U does not act on the copied inputs) to yield

$$\begin{aligned} & UU_A^{\text{copy}} \otimes U_B^{\text{copy}} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy}^0\rangle \\ &= |x\rangle_A^{\text{copy}} |y\rangle_B^{\text{copy}} \sum_{a,b} |a\rangle_A |b\rangle_B |\varphi_{xy}^{ab}\rangle_{AB}. \end{aligned} \quad (14)$$

Since U does not perfectly reproduce the correlations (8) all values of a, b can appear in Eq. (14). For simplicity of notation we shall denote hereafter

$$|\varphi_{xy}^k\rangle = \sum_{\substack{a,b \\ a-b-xy=k \bmod d}} |a\rangle_A |b\rangle_B |\varphi_{xy}^{ab}\rangle_{AB}. \quad (15)$$

The parties now carry out the local operations $e^{i2\pi\hat{a}/d} \otimes e^{-i2\pi\hat{b}/d}$ to obtain

$$\begin{aligned} & e^{i2\pi\hat{a}/d} \otimes e^{-i2\pi\hat{b}/d} UU_A^{\text{copy}} \otimes U_B^{\text{copy}} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy}^0\rangle \\ &= e^{i2\pi xy/d} |x\rangle_A^{\text{copy}} |y\rangle_B^{\text{copy}} \sum_k e^{i2\pi k/d} |\varphi_{xy}^k\rangle. \end{aligned} \quad (16)$$

Finally the parties carry out the inverse operations $U^\dagger$ and $U_A^{\text{copy}\dagger} \otimes U_B^{\text{copy}\dagger}$ to obtain

$$\begin{aligned} |\Psi_{xy}^f\rangle &= U_A^{\text{copy}\dagger} \otimes U_B^{\text{copy}\dagger} U^\dagger e^{i2\pi\hat{a}/d} \otimes e^{-i2\pi\hat{b}/d} \times UU_A^{\text{copy}} \\ &\otimes U_B^{\text{copy}} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy}^0\rangle. \end{aligned} \quad (17)$$

Our aim is to find the maximal value of the Bell expression $\langle \mathcal{L}^d \rangle = (1/d^2) \sum_{\substack{x,y,a,b \\ a-b-xy=0 \bmod d}} P(a,b|x,y)$ for which the transformation U can be implemented without communication, i.e., for which $U = U_A \otimes U_B$ is local. To this end we shall investigate for what values of $\mathcal{L}^d$ the transformation $U_A^{\text{copy}\dagger} \otimes U_B^{\text{copy}\dagger} U^\dagger e^{i2\pi\hat{a}/d} \otimes e^{-i2\pi\hat{b}/d} UU_A^{\text{copy}} \otimes U_B^{\text{copy}}$ does not necessarily imply that Alice and Bob can signal. We suppose that initially Bob prepares his input in a coherent superposition $(1/\sqrt{d}) \sum_y |y\rangle_B$ whereas Alice prepares her input in state $|x\rangle_A$. After carrying out the operations $U_A^{\text{copy}\dagger} \otimes U_B^{\text{copy}\dagger} U^\dagger e^{i2\pi\hat{a}/d} \otimes e^{i2\pi\hat{b}/d} UU_A^{\text{copy}} \otimes U_B^{\text{copy}}$ Bob measures his input in the basis $|z\rangle_B = (1/\sqrt{d}) \sum_y e^{i2\pi zy/d} |y\rangle_B$. The probability of finding z given that Alice prepared x is

$$P(z|x) = \left(\frac{1}{\sqrt{d}} \sum_{y'} \langle \Psi_{xy'}^f | \right) \Pi_z \left(\frac{1}{\sqrt{d}} \sum_y |\Psi_{xy}^f\rangle \right), \quad (18)$$

where $\Pi_z = |z\rangle_B \langle z|$ projects onto state $|z\rangle_B$ and acts as the identity on the rest of the Hilbert space. We use the fact that

$$\begin{aligned} \Pi_z &\geq \left(\frac{1}{\sqrt{d}} \sum_{y'} e^{-i2\pi zy'/d} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy'}^0\rangle \right) \\ &\times \left(\frac{1}{\sqrt{d}} \sum_y e^{i2\pi zy/d} \langle \Psi_{xy}^0 | \langle 0|_A^{\text{copy}} \langle 0|_B^{\text{copy}} \right), \end{aligned}$$

where the $\geq$ sign means that the difference of the two operators is a positive operator. This implies that

$$\begin{aligned} P(z|x) &\geq \left| \frac{1}{d} \sum_{yy'} e^{-i2\pi zy'/d} \langle \Psi_{xy'}^0 | \langle 0|_A^{\text{copy}} \langle 0|_B^{\text{copy}} U^{\text{copy}\dagger} U^\dagger e^{i2\pi(\hat{a}-\hat{b})/d} \right. \\ &\times UU_A^{\text{copy}} |0\rangle_B^{\text{copy}} |0\rangle_A^{\text{copy}} |\Psi_{xy}^0\rangle \left. \right|^2 \\ &= \left| \frac{1}{d} \sum_y e^{i2\pi y(x-z)/d} \sum_k e^{i2\pi k/d} \langle \varphi_{xy}^k | \varphi_{xy}^k \rangle \right|^2, \end{aligned} \quad (19)$$

where we denote $U^{\text{copy}} = U_A^{\text{copy}} \otimes U_B^{\text{copy}}$ and we have used Eq. (13) and ${}^{\text{copy}}\langle y' | y \rangle_B^{\text{copy}} = \delta_{y'y}$. The reason for introducing the copy operations $U_{A(B)}^{\text{copy}}$ is now clear: without them we would have obtained a similar expression to Eq. (19), but with a double sum over y and y' rather than the single sum over y .

If the transformation U is no signaling, it must be that Bob cannot learn what Alice's input was. This implies that $P(z|x) = P(z|x')$ for all z, x', x . We now combine this with the normalization condition $\sum_z P(z|x) = 1$ to obtain the condition

$$\sum_x P(x|x) = 1.$$

Replacing $P(x|x)$ in this equality by the bound (19) yields the inequality

$$1 \geq \sum_x \left| \frac{1}{d} \sum_y \sum_k e^{i2\pi k/d} \langle \varphi_{xy}^k | \varphi_{xy}^k \rangle \right|^2. \quad (20)$$

We now use that if $\sum_{j=1}^d |x_j|^2 \leq 1$, then $\sum_{j=1}^d |x_j| \leq \sqrt{d}$, to obtain

$$\sqrt{d} \geq \sum_x \left| \frac{1}{d} \sum_y \sum_k e^{i2\pi k/d} \langle \varphi_{xy}^k | \varphi_{xy}^k \rangle \right|. \quad (21)$$

Using Eqs. (5) and (15) we can rewrite this as

$$\sqrt{d} \geq \sum_x \left| \frac{1}{d} \sum_y \sum_k e^{i2\pi k/d} \sum_{a,b} P(a,b|xy) \right|. \quad (22)$$

This is a Tsirelson inequality: any correlations that violate it cannot be reproduced using local measurements on entangled quantum states. We will now show that the inequality (22) is tight in the case of $d=2$: it is equivalent to the bound $2\sqrt{2}$ on the CHSH inequality. We will also show that in the case $d=3$ it gives a nontrivial bound on $\mathcal{L}^3$, as announced above. However, in the case $d=4$ it apparently does not give an interesting bound, presumably because it does not incorporate enough of the no signaling constraints.

First we simplify Eq. (22) by using $|x| \geq |\operatorname{Re}(x)| \geq \operatorname{Re}(x)$ to obtain

$$\sqrt{d} \geq \sum_x \frac{1}{d} \sum_y \sum_k \cos(2\pi k/d) \sum_{a,b} P(a,b|xy). \quad (23)$$

We now specialize to the case $d=2$. Equation (23) then becomes

$$2\sqrt{2} \geq \sum_x \sum_y \left(\sum_{a,b} P(a,b|xy) - \sum_{a,b} P(a,b|xy) \right). \quad (24)$$

which is Tsirelson's bound for the CHSH expression.

In the case $d=3$, Eq. (23) becomes

$$3\sqrt{3} \geq \sum_{x=0}^2 \sum_{y=0}^2 \left(\sum_{a,b} P(a,b|xy) - \frac{1}{2} \sum_{a,b} P(a,b|xy) - \frac{1}{2} \sum_{a,b} P(a,b|xy) \right). \quad (25)$$

We now use that $\sum_{a,b} P(a,b|xy) = 1$ to eliminate the last two terms and obtain

$$\frac{9}{2} + 3\sqrt{3} \geq \frac{3}{2} \sum_{x=0}^2 \sum_{y=0}^2 \sum_{a,b} P(a,b|xy). \quad (26)$$

Upon multiplication of Eq. (26) by $2/3^3$ one obtains $\mathcal{L}^3(QM) \leq \frac{1}{3} + 2/3\sqrt{3}$ as announced.

VI. CONCLUSION

In the present work we have studied the properties of no signaling correlations that cannot be reproduced using local measurements on entangled quantum states. Such correlations we say violates Tsirelson bounds. We supposed that the correlations are obtained by some reversible unitary quantum evolution U . Because the correlations violate a Tsirelson bound the evolution U cannot be written in the product form $U_A \otimes U_B$. We show that this implies that U can be used for signaling and for entanglement generation. This approach is very general and in fact can be viewed as a complete characterization of Tsirelson bounds. We show that it can be used as a tool to study Tsirelson bounds and we illustrate this by rederiving the Tsirelson bound of $2\sqrt{2}$ for the CHSH inequality, and by deriving a new Tsirelson bound for qutrits.

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