Composing constraint solvers
Zoeteweij, P.

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We provide here the proofs of the Bounds consistency Theorems 5.8.2 and 5.8.3, and the MULTIPLICATION Theorem 5.9.1. This material is taken from unpublished joint work with Krzysztof Apt.

**Proof of the Bounds consistency Theorem 5.8.2.**

Let $\phi := (x \cdot y = z ; x \in D_x, y \in D_y, z \in D_z)$. Call a variable $u$ of $\phi$ **bounds consistent** if the bounds of its domain satisfy the condition of the bounds consistency (see Definition 5.8.1).

Given an integer interval $[l..h]$ denote by $[\downarrow l..\downarrow h]$ the corresponding real interval $[l,h]$. Suppose that $D_x = [l_x..h_x], D_y = [l_y..h_y], D_z = [l_z..h_z]$. To show that $\phi$ is closed under the applications of the MULTIPLICATION 1 rule it suffices to prove that

$$\{l_z,h_z\} \subseteq \text{int}(D_x \cdot D_y). \quad (A.1)$$

So take $c \in \{l_z,h_z\}$. By the bounds consistency of $z$ we have $c = a \cdot b$ for some $a \in D_x$ and $b \in D_y$. Since $D_x$ and $D_y$ are integer intervals we have $[a], [a] \in D_x$ and $[b], [b] \in D_y$. To prove (A.1), by the definition of $D_x \cdot D_y$, we need to find $a_1, a_2 \in D_x$ and $b_1, b_2 \in D_y$ such that

$$a_1 \cdot b_1 \leq c \leq a_2 \cdot b_2.$$ 

The choice of $a_1, a_2, b_1$ and $b_2$ depends on the sign of $a$ and of $b$ and is provided in the following table:

<table>
<thead>
<tr>
<th>condition</th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$a_2$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0$</td>
<td>$a$</td>
<td>$[b]$</td>
<td>$a$</td>
<td>$[b]$</td>
</tr>
<tr>
<td>$b = 0$</td>
<td>$[a]$</td>
<td>$b$</td>
<td>$[a]$</td>
<td>$b$</td>
</tr>
<tr>
<td>$a &gt; 0, b &gt; 0$</td>
<td>$[a]$</td>
<td>$[b]$</td>
<td>$[a]$</td>
<td>$[b]$</td>
</tr>
<tr>
<td>$a &gt; 0, b &lt; 0$</td>
<td>$[a]$</td>
<td>$[b]$</td>
<td>$[a]$</td>
<td>$[b]$</td>
</tr>
<tr>
<td>$a &lt; 0, b &gt; 0$</td>
<td>$[a]$</td>
<td>$[b]$</td>
<td>$[a]$</td>
<td>$[b]$</td>
</tr>
<tr>
<td>$a = 0, b &lt; 0$</td>
<td>$[a]$</td>
<td>$[b]$</td>
<td>$[a]$</td>
<td>$[b]$</td>
</tr>
<tr>
<td>$a &lt; 0, b &lt; 0$</td>
<td>$[a]$</td>
<td>$[b]$</td>
<td>$[a]$</td>
<td>$[b]$</td>
</tr>
</tbody>
</table>
To prove that \( \phi \) is closed under the applications of the MULTIPLICATION 2 and 3 rules it suffices to prove

\[
\{l_x, h_x\} \subseteq \text{int}(D_z/D_y) \quad \text{and} \quad \{l_y, h_y\} \subseteq \text{int}(D_z/D_x).
\]  

(A.2)

We need to distinguish a number of cases. The case analysis depends on the position of 0 w.r.t. each of the intervals \( D_x \) and \( D_y \). This leads to 9 cases, which by symmetry between \( x \) and \( y \) can be reduced to 6 cases. We present here the proofs for representative 3 cases.

**Case 1.** \( l_x \geq 0, \quad l_y \geq 0 \).

By the bounds consistency of \( x \) for some \( b \in [l_y, h_y] \) we have \( l_x \cdot b \in [l_z, h_z] \). Then \( b \leq h_y \) and \( l_z \geq 0 \), so \( l_x \cdot b \leq l_z \cdot h_y \). Also \( l_z \leq l_x \cdot b \), so

\[
l_z \leq l_x \cdot h_y.
\]

Next, by the bounds consistency of \( y \) for some \( a \in [l_x, h_x] \) we have \( a \cdot h_y \in [l_z, h_z] \). Then \( l_x \leq a \) and \( h_y \geq 0 \), so \( l_x \cdot h_y \leq a \cdot h_y \). Also \( a \cdot h_y \leq h_z \), so

\[
l_x \cdot h_y \leq h_z.
\]

So \( l_x \cdot h_y \in [l_z..h_z] \) and consequently by the definition of the integer intervals division

\[
l_x \in D_z/D_y \quad \text{and} \quad h_y \in D_z/D_x.
\]

By a symmetric argument

\[
h_x \in D_z/D_y \quad \text{and} \quad l_y \in D_z/D_x.
\]

**Case 2.** \( l_x \geq 0, \quad h_y \leq 0 \).

By the bounds consistency of \( x \) for some \( b \in [l_y, h_y] \) we have \( h_x \cdot b \in [l_z, h_z] \). Then \( b \leq h_y \) and \( h_x \geq 0 \), so \( h_x \cdot b \leq h_z \cdot h_y \). Also \( h_z \leq h_x \cdot b \), so

\[
l_z \leq h_x \cdot h_y.
\]

Next, by the bounds consistency of \( y \) for some \( a \in [l_x, h_x] \) we have \( a \cdot h_y \in [l_z, h_z] \). Then \( a \leq h_x \) and \( h_y \leq 0 \), so \( a \cdot h_y \geq h_x \cdot h_y \). Also \( h_x \geq a \cdot h_y \), so

\[
h_x \cdot h_y \leq h_z.
\]

So \( h_x \cdot h_y \in [l_z..h_z] \) and consequently by the definition of the integer intervals division

\[
h_x \in D_z/D_y \quad \text{and} \quad h_y \in D_z/D_x.
\]

Further, by the bounds consistency of \( x \) for some \( b \in [l_y, h_y] \) we have \( l_x \cdot b \in [l_z, h_z] \). Then \( l_y \leq b \) and \( l_x \geq 0 \), so \( l_x \cdot l_y \leq l_x \cdot b \). Also \( l_x \cdot b \leq h_z \), so

\[
l_x \cdot l_y \leq h_z.
\]
Next, by the bounds consistency of $y$ for some $a \in [l_x, h_x]$ we have $a \cdot l_y \in [l_z, h_z]$. Then $l_x \leq a$ and $l_y < 0$, so $l_x \cdot l_y \geq a \cdot l_y$. Also $a \cdot l_y \geq l_z$, so

$$l_z \leq l_x \cdot l_y.$$  

So $l_x \cdot l_y \in [l_z \ldots h_z]$ and consequently by the definition of the integer intervals division

$$l_x \in D_y/D_z \text{ and } l_y \in D_z/D_x.$$  

**Case 3.** $l_x < 0 < h_x$, $l_y \geq 0$.

The proof for this case is somewhat more elaborate. By the bounds consistency of $x$ for some $b \in [l_y, h_y]$ we have $l_x \cdot b \in [l_z, h_z]$. Then $l_y \leq b$ and $l_x < 0$, so $l_x \cdot l_y \geq l_x \cdot b$. But also $l_x \cdot b \geq l_z$, so

$$l_z \leq l_x \cdot l_y.$$  

Next, by the bounds consistency of $y$ for some $a \in [l_x, h_x]$ we have $a \cdot l_y \in [l_z, h_z]$. Then $l_x \leq a$ and $l_y \geq 0$, so $l_z \cdot l_y \leq a \cdot l_y$. But also $a \cdot l_y \leq h_z$, so

$$l_z \cdot l_y \leq h_z.$$  

So $l_z \cdot l_y \in [l_z \ldots h_z]$ and consequently by the definition of the integer intervals division

$$l_z \in D_y/D_z \text{ and } l_y \in D_z/D_x.$$  

Further, by the bounds consistency of $x$ for some $b \in [l_y, h_y]$ we have $h_x \cdot b \in [l_z, h_z]$. Then $l_y \leq b$ and $h_x > 0$, so $h_x \cdot l_y \leq h_x \cdot b$. But also $h_x \cdot b \leq h_z$, so

$$h_x \cdot l_y \leq h_z.$$  

Next, we already noted that by the bounds consistency of $y$ for some $a \in [l_x, h_x]$ we have $a \cdot l_y \in [l_z, h_z]$. Then $a \leq h_x$ and $l_y \geq 0$, so $a \cdot l_y \leq h_x \cdot l_y$. But also $l_z \leq a \cdot l_y$, so

$$l_z \leq h_x \cdot l_y.$$  

So $h_x \cdot l_y \in [l_z \ldots h_z]$ and consequently by the definition of the integer intervals division

$$h_x \in D_y/D_z.$$  

It remains to prove that $h_y \in D_z/D_x$. We showed already $l_z \cdot l_y \leq h_z$. Moreover, $l_x < 0$ and $l_y \leq h_y$, so $l_z \cdot h_y \leq l_x \cdot l_y$ and hence

$$l_z \cdot h_y \leq h_z.$$  

Also we showed already $l_z \leq h_x \cdot l_y$. Moreover $h_x > 0$ and $l_y \leq h_y$, so $h_x \cdot l_y \leq h_x \cdot h_y$ and hence

$$l_z \leq h_x \cdot h_y.$$
So if either \( l_z \leq l_x \cdot h_y \) or \( h_x \cdot h_y \leq h_z \), then either \( l_x \cdot h_y \in [l_z..h_z] \) or \( h_x \cdot h_y \in [l_z..h_z] \) and consequently \( h_y \in D_z/D_x \).

If both \( l_x \cdot h_y < l_z \) and \( h_z < h_x \cdot h_y \), then

\[
[l_z..h_z] \subseteq [l_x..h_x] \cdot h_y.
\]

In particular for some \( a \in D_z \) we have \( l_z = a \cdot h_y \), so \( h_y \in D_z/D_x \). as well.

This concludes the proof for this case. \( \square \)

**Proof of the Bounds consistency Theorem 5.8.3.**

We consider each variable in turn. We begin with \( x \). Suppose that \( D_x = [l_x..h_x] \).
\( \phi \) is closed under the applications of the **MULTIPLICATION 2** rule, so

\[
\{l_x, h_x\} \subseteq \text{int}(D_z/D_y).
\]  \hfill (A.3)

To show the bounds consistency of \( x \) amounts to showing

\[
\{l_x, h_x\} \subseteq D_z \otimes D_y.
\]  \hfill (A.4)

(Recall that given real intervals \( X \) and \( Y \) we denote by \( X \otimes Y \) their division, defined in Section 5.3.)

**Case 1.** \( \text{int}(D_z/D_y) = \mathbb{Z} \).

This implies that \( 0 \in D_z \cap D_y \), so by the definition of real intervals division

\[
D_z \otimes D_y = (-\infty, \infty).
\]

Hence (A.4) holds.

**Case 2.** \( \text{int}(D_z/D_y) \neq \mathbb{Z} \).

So \( \text{int}(D_z/D_y) \) is an integer interval, say \( int(D_z/D_y) = [l_{zy}..h_{zy}] \). Two subcases arise.

**Subcase 1.** \( D_z \otimes D_y \) is a, possibly open ended, real interval.

By (A.3) for some \( b_1, b_2 \in D_y \) and \( c_1, c_2 \in D_z \) we have

\[
l_{zy} \cdot b_1 = c_1.
\]

\[
h_{zy} \cdot b_2 = c_2.
\]

Let

\[
b := \min(b_1, b_2), \bar{b} := \max(b_1, b_2), \underline{c} := \min(c_1, c_2), \bar{c} := \max(c_1, c_2).
\]

So \( \{l_{zy}, h_{zy}\} \subseteq [c, \bar{c}] \otimes [b, \bar{b}] \). Also \( [c, \bar{c}] \otimes [b, \bar{b}] \subseteq D_z \otimes D_y \). Hence \( \{l_{zy}, h_{zy}\} \subseteq D_z \otimes D_y \) and consequently, by the assumption for this subcase, \( [l_{zy}, h_{zy}] \subseteq D_z \otimes D_y \). This proves (A.4) since by (A.3) \( \{l_x, h_x\} \subseteq [l_{zy}, h_{zy}] \).

**Subcase 2.** \( D_z \otimes D_y \) is not a, possibly open ended, real interval.
In what follows for an integer interval $D := [l..h]$ we write $D > 0$ if $l > 0$, $D < 0$ if $h < 0$. Also recall that $\langle D \rangle := \{x \in \mathbb{Z} \mid l < x < h\}$.

This subcase can arise only when $D_z > 0$ and $0 \notin \langle D_y \rangle$ or $D_z < 0$ and $0 \notin \langle D_y \rangle$, see [Rat96] (reported as Theorem 4.8 in [HJvEOl]), where the definition of the division of real intervals is considered.

Since $\phi$ is closed under the MULTIPICATION rule 3

$$D_y \subseteq \text{int}(D_z/D_z).$$

So $\text{int}(D_z/D_z) \neq \emptyset$ since by assumption $D_y$ is non-empty. Also, since $0 \notin D_z$, we have $\text{int}(D_z/D_z) \neq \mathbb{Z}$. So $\text{int}(D_z/D_z)$ is a non-empty integer interval such that $0 \in \langle \text{int}(D_z/D_z) \rangle$.

But $D_z > 0$ or $D_z < 0$, so if $D_z > 0$, then $\text{int}(D_z/D_z) > 0$ or $\text{int}(D_z/D_z) < 0$ and if $D_z < 0$, then $\text{int}(D_z/D_z) > 0$ or $\text{int}(D_z/D_z) < 0$, as well. So $0 \in \langle D_z \rangle$. Hence $0 \notin \langle D_z \rangle \cap \langle D_y \rangle$ while $0 \notin D_z$. This contradicts (5.5). So this subcase cannot arise.

The proof for the variable $y$ is symmetric to the one for the variable $x$.

Consider now the variable $z$. $\phi$ is closed under the applications of the MULTIPICATION 1 rule, so

$$D_z \subseteq \text{int}(D_z \cdot D_y).$$

Take now $c \in D_z$. Then there exist $a_1, a_2 \in D_x$ and $b_1, b_2 \in D_y$ such that $a_1 \cdot b_1 \leq c \leq a_2 \cdot b_2$. We can assume that both inequalities are strict, that is,

$$a_1 \cdot b_1 < c < a_2 \cdot b_2. \quad (A.5)$$

since otherwise the desired conclusion is established.

Let

$$a := \min(a_1, a_2), \bar{a} := \max(a_1, a_2), b := \min(b_1, b_2), \bar{b} := \max(b_1, b_2).$$

We now show that $a \in [a..\bar{a}]$ and $b \in [b..\bar{b}]$ exist such that $c = a \cdot b$. Since $[a..\bar{a}] \subseteq \overline{D_x}$ and $[b..\bar{b}] \subseteq \overline{D_y}$, this will establish the bounds consistency of $z$.

The choice of $a$ and $b$ depends on the signs of $a_1$ and $b_2$. When one of these values is zero, the choice is provided in the following table, where in each case on the account of (A.5) no division by zero takes place:

<table>
<thead>
<tr>
<th>condition</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = 0$</td>
<td>$c/b_2$</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$a_2 = 0$</td>
<td>$c/b_1$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$b_1 = 0$</td>
<td>$a_2$</td>
<td>$c/a_2$</td>
</tr>
<tr>
<td>$b_2 = 0$</td>
<td>$a_1$</td>
<td>$c/a_1$</td>
</tr>
</tbody>
</table>
It is straightforward to show that in each case the quotient belongs to the corresponding interval. For example, when \(a_1 = 0\) we need to prove that \(c/b_2 \in [a..a]\). By (A.5) \(a_2 \neq 0\). If \(a_2 > 0\), then again by (A.5), \(b_2 > 0\), so \(c/b_2 \in [0..a_2]\). In turn, if \(a_2 < 0\), then also by (A.5) \(b_2 < 0\), so, yet again by (A.5), \(c/b_2 \in [a_2..0]\).

When neither \(a_1\) nor \(b_2\) is zero, the choice of \(a\) and \(b\) has to be argued case by case.

**Case 1.** \(a_1 > 0\), \(b_2 > 0\).

Then by (A.5) \(b_1 < c/a_1\) and \(c/b_2 < a_2\). Suppose that both \(b_2 < c/a_1\) and \(c/b_2 < a_1\). Then \(a_1\cdot b_2 < c < a_1\cdot b_2\), which is a contradiction. So either \(c/a_1 \leq b_2\) or \(a_1 \leq c/b_2\), that is either \(c/a_1 \in [b_1..b_2]\) or \(c/b_2 \in [a_1..a_2]\).

**Case 2.** \(a_1 > 0\), \(b_2 < 0\).

Then by (A.5) \(b_1 < c/a_1\) and \(a_2 < c/b_2\). Suppose that both \(b_2 < c/a_1\) and \(a_1 < c/b_2\). Then \(a_1\cdot b_2 < c < a_1\cdot b_2\), which is a contradiction. So either \(c/a_1 \leq b_2\) or \(c/b_2 \leq a_2\), that is either \(c/a_1 \in [b_1..b_2]\) or \(c/b_2 \in [a_2..a_1]\).

**Case 3.** \(a_1 < 0\), \(b_2 > 0\).

Then by (A.5) \(c/a_1 < b_1\) and \(c/b_2 < a_2\). Suppose that both \(c/a_1 < b_2\) and \(c/b_2 < a_1\). Then \(a_1\cdot b_2 < c < a_1\cdot b_2\), which is a contradiction. So either \(b_2 \leq c/a_1\) or \(a_1 \leq c/b_2\), that is either \(c/a_1 \in [b_1..b_2]\) or \(c/b_2 \in [a_1..a_2]\).

**Case 4.** \(a_1 < 0\), \(b_2 < 0\).

Then by (A.5) \(c/a_1 < b_1\) and \(a_2 < c/b_2\). Suppose that both \(c/a_1 < b_2\) and \(a_1 < c/b_2\). Then \(a_1\cdot b_2 < c < a_1\cdot b_2\), which is a contradiction. So either \(b_2 \leq c/a_1\) or \(c/b_2 \leq a_1\), that is either \(c/a_1 \in [b_1..b_2]\) or \(c/b_2 \in [a_2..a_1]\).

So in each of the four cases we can choose either \(a := a_1\) and \(b := c/a_1\) or \(a := c/b_2\) and \(b := b_2\).

**Proof of the MULTIPLICATION Theorem 5.9.1.**

The weak interval division produces larger sets than the interval division. As a result the MULTIPLICATION rules 2w and 3w yield a weaker reduction than the original MULTIPLICATION rules 2 and 3. So it suffices to prove that \(\phi := \{x \cdot y = z : x \in D_x, y \in D_y, z \in D_z\}\) is closed under the applications of the MULTIPLICATION 1, 2 and 3 rules assuming that it is closed under the applications of the MULTIPLICATION 1, 2w and 3w rules. Suppose that \(D_x = [l_x..h_x], D_y = [l_y..h_y], D_z = [l_z..h_z]\). The assumption implies

\[
\{l_x, h_x\} \subseteq \text{int}(D_z : D_y) \tag{A.6}
\]

and

\[
\{l_y, h_y\} \subseteq \text{int}(D_z : D_x) \tag{A.7}
\]

The proof is by contradiction. Assume that (A.6) and (A.7) hold, while \(\phi\) is not closed under application of MULTIPLICATION 2 and 3. Without loss
of generality, suppose that \textit{MULTIPLICATION 2} is the rule that can make a further reduction. This is the case if

\[ \text{int}(D_z/D_y) \subseteq \text{int}(D_z : D_y). \]

By definition, the proper inclusion implies that \( l_y \geq 0 \) or \( h_y \leq 0 \). Assume \( l_y \geq 0 \), the case for \( h_y \leq 0 \) is similar. Let \( l'_y := \max(1, l_y) \), and let \( A := \{ l_z/l'_y, l_z/h_y, h_z/l'_y, h_z/h_y \} \), and \( B := \{ l_z/l_z, l_z/h_z, h_z/l_z, h_z/h_z \} \). A further implication of the proper inclusion is that one or both of \( l'_y \) and \( h_y \) do not have a multiple in \( D_z \): otherwise \( \min(A) \) and \( \max(A) \) would be elements of \( D_z/D_y \), and we would have \( \text{int}(D_z : D_y) = \text{int}(D_z/D_y) \). The cases for \( l'_y \) and \( h_y \) can be seen in isolation, and their proofs are similar, so here we only consider the case that \( l'_y \) does not have a multiple in \( D_z \). In what follows we can assume \( 0 \notin D_z \), since otherwise \( l'_y \) and \( h_y \) do have a multiple in \( D_z \).

**Case 1.** \( l_z > 0 \).

From (A.6) it follows that \( h_z \leq [\max(A)] \), which for the case \( l'_y, h_y, l_z, h_z > 0 \) that we consider here implies \( h_z \leq [h_z/l'_y] \). Because \( [l_z..h_z] \) does not contain a multiple of \( l'_y \), we have \( [h_z/l'_y] = [l_z/l'_y] \), so

\[
\frac{h_z}{l_z} \leq \frac{h_z}{l'_y}.
\]

A further consequence of (A.6) is that \( l_z/h_z > 0 \). From (A.7) it follows that \( l'_y \geq [\min(B)] \), which for \( l_z, l_z > 0 \) implies

\[
l'_y \geq \frac{l_z/h_z}{l_z/h_z} \geq l_z/h_z \geq [h_z/l'_y].
\]

Because \( l'_y \) is no divisor of \( l_z \), and both numbers are positive, we have \( [l_z/l'_y] \leq [l_z/l'_y] \), and consequently \( l'_y > l_z/(l_z/l'_y) \), leading to \( l'_y > l'_y \), which is a contradiction.

**Case 2.** \( h_z < 0 \).

Similarly, because \( l'_y, h_y > 0 \) and \( l_z, h_z < 0 \), it follows from (A.6) that \( l_z \geq \frac{[\min(A)]}{[l_z/l'_y]} \), and \( l_z, h_z < 0 \). Because \( [l_z..h_z] \) does not contain a multiple of \( l'_y \), we have \( [l_z/l'_y] = [h_z/l'_y] \), so

\[
l_z \geq [h_z/l'_y].
\]

We use this information in the following implication of (A.7):

\[
l'_y \geq [\min(B)] = [h_z/l'_y] \geq [h_z/l'_y]
\]

to get \( l'_y \geq h_z/[h_z/l'_y] \). Because \( [h_z/l'_y]/[h_z/l'_y] \), we have \( l'_y > h_z/(h_z/l'_y) \), leading to \( l'_y > l'_y \), which is a contradiction. \( \square \)