Appendix B

Appendix to chapter 2

B.1 Proofs

B.1.1 Proof of Lemma 1

The expected trading profit of a speculator becoming informed about the idiosyncratic risk factor of firm $i$ is

$$
\left(1 - \frac{\phi_i}{\theta_i}\right) \left(1 - q_a\frac{\phi_a}{\theta_a}\right) \cdot 0 + \left(1 - \frac{\phi_a}{\theta_a}\right) \left(q_aq_l(R_{H,i} - R_{L,i}) + R_{L,i} - R_{L,i}\right)
$$

$$+ q_a\frac{\phi_a}{\theta_a} \left[q_l(R_{H,i} - R_{L,i}) + R_{L,i} - R_{L,i}\right]
$$

if $A_i = 0$ and

$$
\left(1 - \frac{\phi_i}{\theta_i}\right) \left(1 - q_a\frac{\phi_a}{\theta_a}\right) \cdot 0 + \left(1 - \frac{\phi_a}{\theta_a}\right) \left(-q_aq_l(R_{H,i} - R_{L,i}) - R_{L,i} + q_a(R_{H,i} - R_{L,i}) + R_{L,i}\right)
$$

$$+ q_a\frac{\phi_a}{\theta_a} \left[-q_l(R_{H,i} - R_{L,i}) - R_{L,i} + R_{H,i}\right]
$$

if $A_i = 1$. Simplifying these terms and taking the unconditional expectation yields the term in Lemma 1. The analogue holds for the incentives to become informed about the common risk factor $A_a$, except that the summation is over all securities.

B.1.2 Proof of Proposition 1

For the proof, note that welfare is a smooth function of $\Delta R$. There are four cases to distinguish depending on whether the information acquisition equations (2.2) and (2.3) are binding:

$(\phi_a > 0, \phi_l > 0)$: solve the FOC (equation 2.5) to get the interior solution $\Delta R_{SP}^{int}$. It goes to infinite as $\left(c_a\frac{\theta_c}{\theta_l} + c_l\frac{\theta_l}{\theta_c}\right)(K - x_L) \rightarrow c_l\frac{\theta_l}{\theta_cq_l(1-q_l)} + c_a\frac{\theta_a}{\theta_cq_c(1-q_c)}$. Thus, if $\left(c_a\frac{\theta_c}{\theta_l} + c_l\frac{\theta_l}{\theta_c}\right)(K - x_L) > c_l\frac{\theta_l}{\theta_cq_l(1-q_l)} + c_a\frac{\theta_a}{\theta_cq_c(1-q_c)}$, the optimal solution is $\Delta R$ maximal. By assumption $\Delta R_{max} >$
\( \Delta R \): For existence we need that \( \Delta R > 0 \). For existence we need that \( \Delta R > 0 \). We would like to minimize \( \Delta R \). If on the other hand \( \Delta R < 0 \), then welfare is concave in \( \Delta R \) around \( \Delta R = 0 \). But this violates \( \Delta R > 0 \), so it is not possible that \( \Delta R < 0 \).

Next, I show that the interior solution is in fact a local maximum. If \( \Delta R > 0 \), then welfare is concave in \( \Delta R \) around \( \Delta R = 0 \), because the second derivative is negative as I will show:

\[
\frac{\partial^2 W}{\partial \Delta R^2} = \frac{\partial^2 \phi_a}{\partial \Delta R^2} \left[ 1 - \frac{q_a}{\theta_a} \left( 1 - \frac{1 - q_l}{\theta_l} \right) \right] \left( K - x_L \right) c_a - \frac{\partial^2 \phi_a}{\partial \Delta R^2} \left[ 1 - \frac{q_l}{\theta_l} \left( 1 - \frac{1 - q_a}{\theta_a} \phi_a \right) \right] \left( K - x_L \right) - c_L - 2(K - x_L) \frac{\partial^2 \phi_a}{\partial \Delta R^2} \frac{\partial \phi_a}{\partial \Delta R} \frac{\partial \phi_a}{\partial \Delta R}.
\]
Using $\frac{\partial^2 W}{\partial \Delta R^2} = -\frac{\partial W}{\partial \Delta R} \frac{2}{\Delta R}$, this yields

\[
\frac{\partial^2 W}{\partial \Delta R^2} = -2 \frac{\partial^2 \phi_a}{\partial \Delta R^2} \left[ \frac{1}{\theta_a} - \frac{1}{\theta_l} \theta_l \phi_l \right] (K - x_L) - c_a
\]

\[
+ \frac{\partial^2 \phi_l}{\partial \Delta R^2} \left[ \frac{1}{\theta_l} - \frac{1}{\theta_a} \phi_a \right] (K - x_L) - c_l
\]

\[
- 2(K - x_L) \frac{1}{\theta_l} - \frac{1}{\theta_a} \phi_a \frac{\partial \phi_l}{\partial \Delta R} \frac{\partial \phi_a}{\partial \Delta R}
\]

The term in curly brackets is equal to the FOC (2.5) and therefore 0, so welfare is concave around the interior solution.

Let us compare the interior solution to the other three possible corner solutions. The claim is that the interior solution is optimal if it exists. Note that in the cases where ($\phi_a > 0, \phi_l = 0$) or ($\phi_a = 0, \phi_l > 0$), welfare is increasing in $\Delta R$. But then the interior solution must be better because $\overline{\Delta R}_l < \Delta R_{\text{Interior}}$ resp. $\overline{\Delta R}_a < \Delta R_{\text{Interior}}$ and the fact that welfare is smooth in $\Delta R$. Is it possible that choosing $\Delta R$ minimal such that ($\phi_a = 0, \phi_l = 0$) is better? No, as the difference between welfare under the interior solution and under no information only depends on linear terms containing $\phi_l^*$ and $\phi_a^*$, without a constant (thus not affine).

If $0 < \Delta R_{\text{SP}} < \max[\overline{\Delta R}_l, \overline{\Delta R}_a]$, then the interior solution does not exist. If $\overline{\Delta R}_l > \overline{\Delta R}_a$, then only cases ($\phi_a > 0, \phi_l = 0$) or ($\phi_a = 0, \phi_l = 0$) can exist, which means the optimal solution must either be $\Delta R_l$ or $\Delta R_{\text{min}}$. If $K - x_L > \frac{\theta_a c_a}{1-q_a}$, then the optimal solution is therefore $\Delta R_l$, otherwise $\Delta R_{\text{min}}$. The reasoning is the same when $\overline{\Delta R}_l > \overline{\Delta R}_a$.

If the optimal solution is not within the corner solutions, then naturally it must be the corner solution closest since the objective function is smooth.

**B.1.3 Proof of Proposition 2**

If $K - x_L < \frac{\theta_a c_a}{1-q_a}$, then the FOC (equation 2.6) is negative regardless of $\phi_a$ (the equilibrium $\Delta R$). So in that case the unique equilibrium must be $\Delta R = \Delta R_{\text{min}}$.

**Simple Equilibrium** If however $K - x_L \geq \frac{\theta_a c_a}{1-q_a}$, then potentially firms would like have some information. FOC 2.6 is a quadratic equation,

\[
\Delta R_{\text{Int}}^2 \theta_a c_a - \Delta R_{\text{Int}}^2 [\overline{\Delta R}_l((1-q_l)q_a(K - x_L) - \theta_l c_l) + \theta_a c_a \overline{\Delta R}_a]
\]

\[
- \frac{1}{2}(1-q_l)\overline{\Delta R}_l(1-q_a)\overline{\Delta R}_a(K - x_L) = 0 \quad (B.1)
\]

Solving for the $\Delta R$ yields the term in the proposition. The determinant is always positive therefore a solution always exists. It is also easy to verify that the first term of $\Delta R_{\text{Int}}^2$ is smaller than the second, thus the sign in front of the second term can not be negative.

For the interior solution to be an equilibrium, we need that $\Delta R_{\text{Int}}^2 > \max[\overline{\Delta R}_l, \overline{\Delta R}_a]$ such that $\phi_a^* > 0$ and $\phi_l^* > 0$ and that it is not profitable for a firm to deviate. At the
interior solution, it is easy to show that firms are always at a local maximum so any (marginal) deviation \( \Delta R_i' > \bar{R}_i \) from \( \Delta R^{CE}_{\text{Int}} \) is not profitable. For a firm not to deviate to \( \Delta R_i' < \bar{R}_i \), we need that

\[
\Pi_{\Delta R^{CE}_{\text{Int}}} - \Pi_{\Delta R_i'} = \phi^{\text{int}}_i \left( \frac{1 - q_i}{\theta_i} \left( 1 - \frac{1 - q_a}{\theta_a} \phi^{\text{int}}_a \right) (K - x_L) - c_i \right)
- \phi^{\text{int}}_a c_a \left( 1 - \frac{\Delta R_i'}{\Delta R^{CE}_{\text{Int}}} \right) \geq 0 \quad \text{(B.2)}
\]

where \( \phi^{\text{int}}_i \) and \( \phi^{\text{int}}_a \) are the amount of speculators becoming informed in equilibrium under no deviation. Rearranging terms and simplifying yields

\[
\phi^{\text{int}}_a c_a \left[ \frac{\Delta R^{CE}_{\text{Int}}}{\Delta R_i} - 1 - \left( 1 - \frac{\Delta R_i'}{\Delta R^{CE}_{\text{Int}}} \right) \right] \geq 0.
\]

The condition is least likely to hold when \( \Delta R_i' \) is chosen minimally, \( \Delta R_{\text{min}} \). The term in square brackets is then exactly the condition in the proposition.

We also need that \( \Delta R^{CE}_{\text{Int}} > \bar{R}_i \) and \( \Delta R^{CE}_{\text{Int}} > \bar{R}_a \). Rewriting condition \( \Delta R^{CE}_{\text{Int}} > \bar{R}_a \), it turns out that it is equivalent to \((1 - q_i)(K - x_L) \geq \theta_i c_i\), so it always holds. Condition \( \Delta R^{CE}_{\text{Int}} > \bar{R}_i \) always holds if the no-deviation condition \( \frac{\Delta R^{CE}_{\text{Int}}}{\Delta R_i} - 1 \geq 1 - \frac{\Delta R_{\text{min}}}{\Delta R^{CE}_{\text{Int}}} \) holds. To see this, note that if \( \Delta R_{\text{min}} > \bar{R}_i \), then together with the no-deviation condition this directly implies that \( \Delta R^{CE}_{\text{Int}} > \bar{R}_i \). If on the other hand \( \Delta R_{\text{min}} < \bar{R}_i \), then condition \( \frac{\Delta R^{CE}_{\text{Int}}}{\Delta R_i} - 1 \geq 1 - \frac{\Delta R_{\text{min}}}{\Delta R^{CE}_{\text{Int}}} \) is equivalent to \( \Delta R^{CE}_{\text{Int}} \geq \bar{R}_i \left( 1 + \sqrt{1 - \frac{\Delta R_{\text{min}}}{\Delta R_i}} \right) \). From that condition it automatically follows that \( \Delta R^{CE}_{\text{Int}} > \bar{R}_i \).

**Mixed Equilibrium** When \((1 - q_i)(K - x_L) \geq \theta_i c_i\) and \( \frac{\Delta R^{CE}_{\text{Int}}}{\Delta R_i} - 1 < 1 - \frac{\Delta R_{\text{min}}}{\Delta R^{CE}_{\text{Int}}} \) we will show that the mixed equilibrium described in the proposition is indeed an equilibrium and that it is unique. For it to be an equilibrium, we need that firms issuing \( \bar{R}_i \left( 1 + \sqrt{1 - \frac{\Delta R_{\text{min}}}{\Delta R_i}} \right) \) and firms issuing \( \Delta R_{\text{min}} \) do not deviate at the margin and that both types earn the same profits. Note that the equilibrium can only exist if \( \Delta R_{\text{min}} < \bar{R}_i \).

Then condition \( \frac{\Delta R^{CE}_{\text{Int}}}{\Delta R_i} - 1 < 1 - \frac{\Delta R_{\text{min}}}{\Delta R^{CE}_{\text{Int}}} \) becomes \( \Delta R^{CE}_{\text{Mix}} < \bar{R} \equiv \bar{R}_i \left( 1 + \sqrt{1 - \frac{\Delta R_{\text{min}}}{\Delta R_i}} \right) \).

Firms issuing \( \Delta R_{\text{min}} \) do not deviate at the margin because for them \( \phi^{\text{int}}_i = 0 \) (due to \( \Delta R_{\text{min}} < \bar{R}_i \)), thus they only care about the adverse selection costs stemming from information about the common risk factor induced by the other firms.

The condition for all firms to earn the same profits is:

\[
c_a \phi_a |_{\Delta R = \Delta R^{CE}_{\text{Mix}}} \left[ \frac{\bar{R}}{\Delta R_i} - 1 - \left( 1 - \frac{\Delta R_{\text{min}}}{\Delta R_i} \right) \right] = 0.
\]

Solving this quadratic equation for \( \bar{R} \) yields \( \bar{R} = \bar{R}_i \left( 1 + \sqrt{1 - \frac{\Delta R_{\text{min}}}{\Delta R_i}} \right) \).

\( \Delta R^{CE}_{\text{Mix}} \) is pinned down by the local optimality condition of the firms issuing \( \bar{R} \). We can rewrite FOC (equation 2.6) with \( \frac{\theta_i c_i}{q_a q_i (1 - q_i) \bar{R}^{2}} \) in place of \( \frac{\partial \Pi}{\partial \Delta R_i} \) and \( \theta_a (1 -
\[
\frac{c_a}{q_a(1-q_a)\Delta R_{\text{Mix}}} \text{ in place of } \phi_a,
\]

\[
\Delta R^2_{\text{Mix}} \Delta R_I [(1-q_l)q_a(K-x_L) - \theta I c_I] - \Delta R^2_{\text{Mix}} \left[ \theta_a c_a \Delta R^2 - \Delta R_a(1-q_a)\Delta R_I(1-q_l)(K-x_L) \right] + \theta_a c_a \Delta R_a \Delta R^2 = 0.
\]

Solving this equation yields

\[
\Delta R^2_{\text{Mix}} = \frac{1}{2\Delta R_I [(1-q_l)q_a(K-x_L) - \theta I c_I]} \left[ \theta_a c_a \Delta R^2 - \Delta R_a(1-q_a)\Delta R_I(1-q_l)(K-x_L) \right] \pm \sqrt{\left[ \theta_a c_a \Delta R^2 - \Delta R_a(1-q_a)\Delta R_I(1-q_l)(K-x_L) \right]^2 - 4\Delta R_I [(1-q_l)q_a(K-x_L) - \theta I c_I] \theta_a c_a \Delta R_a \Delta R^2}.
\]

I will prove that the discriminant is always positive. Thus we require that

\[
\left[ \theta_a c_a \Delta R^2 - \Delta R_a(1-q_a)\Delta R_I(1-q_l)(K-x_L) \right]^2 > 4\Delta R_I [(1-q_l)q_a(K-x_L) - \theta I c_I] \theta_a c_a \Delta R_a \Delta R^2.
\]

Using equation B.1 together with \( \Delta R^2_{\text{Mix}} < \Delta R \), we also know that

\[
\Delta R \theta_a c_a - (1-q_l)\Delta R_I (1-q_a)\Delta R_a (K-x_L) > \Delta R [\Delta R_I (1-q_l)q_a (K-x_L) - \theta I c_I] + \theta_a c_a \Delta R_a.
\]

(B.3)

Thus we can underestimate the LHS of the determinant condition,

\[
\Delta R \left[ \Delta R_I [(1-q_l)q_a (K-x_L) - \theta I c_I] + \theta_a c_a \Delta R_a \right]^2 > 4\Delta R_I [(1-q_l)q_a (K-x_L) - \theta I c_I] \theta_a c_a \Delta R_a \Delta R^2.
\]

Reformulating this condition yields

\[
[ \Delta R_I (1-q_l)q_a (K-x_L) - \theta I c_I] - \theta_a c_a \Delta R_a] > 0,
\]

which always holds.

One can also show that the sign in front of the square root must be negative.

The fraction \( f^\ast \) of firms issuing \( \Delta R \) is then pinned down by

\[
\Delta R^2_{\text{Mix}} = f^\ast \Delta R + (1-f^\ast)\Delta R_{\text{min}}
\]

\[
\leftrightarrow f^\ast = \frac{\Delta R^2_{\text{Mix}} - \Delta R_{\text{min}}}{\Delta R - \Delta R_{\text{min}}}
\]

Next I show that \( f \in [0,1] \) if and only if \( \frac{\Delta R_{\text{min}}}{\Delta R_{\text{min}}} - 1 < 1 - \frac{\Delta R_{\text{min}}}{\Delta R_{\text{min}}} \). This implies that a mixed equilibrium only exists in that region. For \( f < 1 \) to hold, we require that \( \Delta R^2_{\text{Mix}} < \Delta R \). Solving this inequality yields condition B.3, the condition that holds if and only if \( \frac{\Delta R_{\text{min}}}{\Delta R_{\text{min}}} - 1 < 1 - \frac{\Delta R_{\text{min}}}{\Delta R_{\text{min}}} \) (the inequality sign flips twice when \( (1-q_l)q_a (K-x_L) < \theta I c_I \)).

Condition \( f > 0 \) is equivalent to \( \Delta R^2_{\text{Mix}} > \Delta R_{\text{min}} \). But firms can not issue a safer security than \( \Delta R_{\text{min}} \), so it is impossible that \( f < 0 \). A mixed equilibrium with three or more different strategies can not exist because there are only two local maxima (one for \( \Delta R < \Delta R_I \) and one for \( \Delta R > \Delta R_I \)). This implies that the mixed equilibrium in the proposition is unique.
B.1.4 Proof of Proposition 3

If \( \frac{1}{\theta}(1 - q_l)(K - x_L) < c_l \), then from proposition 2 it follows that firms issue that \( \Delta R_{\text{min}} \) in the private market equilibrium. Region 1 is defined exactly as the parameter values where \( \Delta R_{\text{min}} \) is also socially optimal (proposition 1), thus no intervention by the planner is necessary. Since region 2 is just the difference between the region defined by \( \frac{1}{\theta}(1 - q_l)(K - x_L) < c_l \) and region 1, the socially optimal security has \( \Delta R > \Delta R_{\text{min}} \). Thus the planner would like to increase information production.

Next take the case when \( \frac{1}{\theta}(1 - q_l)(K - x_L) \geq c_l \). We need to show that the socially optimal security is riskier than in equilibrium when \( c_a \) small and safer when \( c_a \) large. Since both the socially optimal and equilibrium security are smooth in \( c_a \), then we can conclude that there must be a cutoff \( c_a^* \) which defines the regions 3 and 4 together with \( \frac{1}{\theta}(1 - q_l)(K - x_L) \geq c_l \). First I show that the socially optimal security is riskier than in equilibrium when \( c_a \) small. If \( \frac{1}{\theta}(1 - q_l)(K - x_L)q_a < c_l \), then the socially optimal security is \( \Delta R_l \) whereas in equilibrium firms issue \( \Delta R_{\text{min}} \) (follows from their FOC (2.6)). If on the other hand \( \frac{1}{\theta}(1 - q_l)(K - x_L)q_a > c_l \), then the socially optimal security is \( \Delta R_{\text{max}} \) for a \( c_a \gg 0 \), whereas in equilibrium firms only issue \( \Delta R_{\text{max}} \) if \( c_a \to 0 \). To see this, note that From the first part of proposition 1, \( \Delta R_{\text{max}} \) is socially optimal whenever \( \theta_l\Delta R_l c_l + \theta_a\Delta R_a c_a \leq \left( \Delta R_l (1 - q_l)q_a + \Delta R_a (1 - q_a)q_l \right)(K - x_L) \). The condition can be rewritten as

\[
c_a \left( \frac{c_a - q_a(1 - q_a)}{q_a} - \frac{K - x_L}{q_a} \right) \leq \Delta R_l \left[ (1 - q_l)q_a(K - x_L) - \theta_l c_l \right].
\]

The right hand side is positive. Thus the condition always holds for \( \frac{1}{\theta_a q_a (1 - q_a)(K - x_L)} > c_a \).

Now I show that the socially optimal security is safer than in the private equilibrium when \( c_a \) is large. Note that when \( c_a \) is large, then \( \Delta R_a > \Delta R_l \). We can also show that \( \Delta R_{\text{Int}}^{SP} < \Delta R_a \). Rewriting that inequality yields

\[
\overline{\Delta R}_l [2(1 - q_a)(1 - q_l)(K - x_L) + (1 - q_l)q_a(K - x_L) - \theta_l c_l] < \overline{\Delta R}_a [\theta_a c_a - (1 - q_a)q_l(K - x_L)],
\]

which holds if \( c_a \) large enough. Thus we have that \( \Delta R_{\text{Int}}^{SP} = \overline{\Delta R}_a \). In the private market equilibrium the solution is interior if \( \frac{1}{\theta_l}(1 - q_l)(K - x_L) \geq c_l \). Rewriting the inequality yields exactly \( \Delta R_{\text{Int}}^{CE} > \overline{\Delta R}_a \) yields \( \frac{1}{\theta_l}(1 - q_l)(K - x_L) > c_l \), thus it holds.