Complete subvarieties of moduli spaces of algebraic curves

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Publication date
2005

Citation for published version (APA):
Introduction

In this chapter we will introduce the subject of this thesis: algebraic curves, their moduli spaces and complete subvarieties contained therein. In an informal way we will describe our main results on complete subvarieties of moduli spaces of curves. Finally we will discuss some open problems related to the subject of this thesis. For precise results one should consult Chapter 1 and the following chapters.

1 Algebraic curves and their moduli

Intuitively, an algebraic curve is a geometrical object that looks like a line, except that it is not necessarily straight; it may be curved or twisted. Normally, we imagine ourselves a curve embedded in some bigger space: in the plane, in threespace, or even in a higher dimensional space.

"Algebraic" means that the curve is given by polynomial equations. A plane curve can be given by one equation. For example, the plane curve defined by $y^2 = x(x - \frac{3}{2})(x - 1)$ is a so called elliptic curve (see Figure 1a). In threespace, two equations define in general a curve: each equation in threespace defines a surface, the intersection of two surfaces is a curve.

Of course there is a formal definition (an algebraic curve is a one-dimensional, irreducible, projective variety over a algebraically closed field $k = \overline{k}$). This definition contains an aspect we did not yet introduce: the field $k$, which contains the coefficients of the equations of the curve. If these coefficients are real numbers, we get a real curve inside $\mathbb{R}^n$. If we take these coefficients to be complex numbers, we get a complex curve inside $\mathbb{C}^n$, and in the case of $p$-adic numbers we get a $p$-adic curve.

![Figure 1a](image1a) (left) (The real part of) the genus 1 curve $y^2 = x(x - \frac{3}{2})(x - 1)$.

![Figure 1b](image1b) (right) A genus 1 curve over $\mathbb{C}$ depicted as a real, oriented surface.
Real curves are the curves we normally picture ourselves as curved or twisted lines in 2- or 3-space. Complex curves are harder to imagine, because the complex numbers do not lie on a line, they form a plane: the complex plane. Hence a complex curve normally is depicted as a surface, a curved or twisted surface. See Figure 1b.

On a real curve there can be special points: points which can be approached from different directions (nodes), or in which the direction of movement necessarily reverses (cusps). Such points are called singular (see Figure 2). The singular points can be computed from the equations of the curve. If a curve has no singular points, it is called "non-singular" or "smooth".

It is possible to get rid of the singular points of a curve. Via an algebro-geometric process called "desingularisation" one can change a singular curve into a unique non-singular one. The non-singular curve maps in a canonical way onto the singular one.

Figure 2a (left) The curve \( y^2 = x^3 \) has a cusp in the origin.
Figure 1b (right) The curve \( y^2 = x^2(x + 1) \) has a (simple) node in the origin.

There are several reasons for studying non-singular curves instead of singular ones. One reason is that one can classify non-singular curves via a topological invariant called the "genus". Roughly, given a non-singular algebraic curve, one considers it as a complex curve by viewing the coefficients as complex numbers; this complex curve can topologically be depicted as a closed real surface inside threespace. The number of holes of this surface is the genus. Of course there is a way to calculate the genus directly from the equations of the curve.

We will give an example. Consider the equation \( y^2 = x(x - 1)(x - a)(x - b)(x - c) \) with \( a, b, c \) different from each other and from 0, 1. This equation defines (in characteristic \( \neq 2 \)) a plane algebraic curve having one singular point (at infinity). After desingularization, the result is a nonsingular curve of genus 2.

In the example, we did not just exhibit one curve. By varying \( a, b \) and \( c \), one gets a family of curves of genus 2. Each \( a, b \) and \( c \) define a genus 2 curve, and moving from one value of \( a \) to another value, we move continuously from one curve to another curve. So each triple \( (a, b, c) \) defines a genus 2 curve, and one can prove that every curve of genus 2 occurs in such a way. In this way, the space of triples \( (a, b, c) \) is a parameter space for this family of curves.

In this threedimensional family, there are distinct triples which give rise to isomorphic curves. For example, the triples \( (a, b, c) \) and \( (b, a, c) \) give the same equation, so define the same curve. So this parameter space of triples \( (a, b, c) \) is in some sense "too large": it contains (almost) every curve more than once.
What if one “divides out” by all possible permutations? Igusa has shown in [30] that in that case one gets a parameter space in which every curve occurs precisely once. Such a parameter space is called a moduli space. This moduli space still has dimension three. Informally this is expressed as: “Curves of genus 2 vary in three dimensions”, or: “Curves of genus 2 have three essential parameters”.

The general case is similar to the example. If we fix a value for the genus, say 23, then one can construct a parameter space in which every non-singular curve of genus 23 occurs precisely once: the moduli space $M_{23}$ of smooth curves of genus 23. Every point in this parameter space represents uniquely a curve of genus 23. This is a big space: its dimension is 66. In general, for $g \geq 2$ the moduli space $M_g$ of smooth curves of genus $g$ has dimension $3g - 3$.

What about limits of non-singular curves? Are they necessarily non-singular? The answer is negative: a family of non-singular curves of a fixed genus over a pointed disk does not need to have a non-singular limit. (Even worse, the proof of the existence of a limit with reasonable singularities is quite hard.) The simplest example is the family of elliptic curves $y^2 = x(x-1)(x-\lambda)$. When $\lambda$ is 0 or 1, the corresponding curve is a singular curve. So limits of non-singular curves can be singular.

For the moduli space of smooth curves this means that it is not complete in the algebro-geometrical sense. In topological terms it means that the moduli space is not compact.

By adding points to the moduli space of smooth curves, corresponding to reasonably behaved singular limits, it is possible to compactify the moduli space of smooth curves. One gets a slightly bigger space, which is called the moduli space $\overline{M}_g$ of stable curves. It has the same dimension as $M_g$. The difference $\overline{M}_g \setminus M_g$ parametrizes singular curves with relatively mild singularities.

## 2 Complete subvarieties

Now we come to the main subject of this thesis: complete subvarieties of the moduli space of smooth algebraic curves over an algebraically closed field $k = \overline{k}$.

Complete subvarieties of the moduli space of smooth algebraic curves are subvarieties of the parameter space $M_g$. Each point of the subvariety represents an algebraic curve, so the subvariety itself represents a family of curves. The subvariety being complete means that the family is complete: all limits of the family should already be contained in the family itself, in particular all limits of the family are smooth. So a complete subvariety of $M_g$ corresponds to a compact family of smooth algebraic curves.

On the one hand, a complete subvariety of $M_g$ seems to represent a nice, natural concept: it is a global object parametrizing a family of varying smooth algebraic curves containing all its limits: no singular curves occur in such a family. On the other hand, algebro-geometric experience shows that this concept is far from natural: in almost all examples of families of algebraic curves one runs into singular limits.

Is it possible to avoid singular fibers in a global way? Joe Harris discussed these questions in [25]. The answer comes from the geometry of $M_g$. One result is that for $g \geq 3$, it is possible to construct complete 1-dimensional subvarieties of $M_g$. This
construction uses the so-called “Satake compactification” of $M_g$. From the construction it follows that these 1-dimensional complete subvarieties exist in abundance.

In the same article, Harris also exhibited higher dimensional complete subvarieties of $M_g$. He did this by constructing recursively explicit families of smooth algebraic curves over a complete base, in which the total space of the previous step serves as a base space of the next step. In this way Harris arrived at complete subvarieties of any dimension. However, as the dimension $d$ goes up, the genus of the curves in the family grows exponentially (the genus is of order $3^d$).

The central question in this field is: what is the maximal dimension of a complete subvariety of $M_g$. (This might depend on the characteristic of the base field.)

Why is this question relevant? If the maximal dimension of a complete subvariety of $M_g$ would be, say, $a(g)$, then any non-trivial family of stable curves of dimension bigger than $a(g)$ would necessarily degenerate, i.e., contain singular fibers. This would be a useful argument in algebraic geometry, where degeneration to singular fibers is a well-known technique.

With regard to this question, there is an important result by Steven Diaz. In [14] he gave an upper bound for the dimension of a complete subvariety of $M_g(C)$. Diaz showed that if $g \geq 2$ this dimension does not exceed $g - 2$.

3 Results

With regard to the central question mentioned above, this thesis contains a number of partial results, which we will present here. We will work over $C$, unless mentioned otherwise.

Our first result answers in some sense a question of Harris. In [25] Harris describes how to construct 1-dimensional complete subvarieties of $M_g \ (g \geq 3)$. He then notes that this construction is not very explicit and he asks: “I don’t believe anyone has ever written down a family of plane quartic curves, for example, corresponding to a complete curve in $M_3$.”

In Chapter 3 we present a construction of an explicit family of genus 3 curves, corresponding to a complete curve in $M_3(C)$ (see [49]). The curves in the family are not plane quartics, but curves associated to Prym varieties of 2 : 1 coverings of a fixed genus 3 curve ramified in two points. The positions of the two points vary, giving rise to the moduli. The nature of this construction is very geometrical, so the curves in this family are rather explicit.

Our second result is an improvement of Harris’s construction of higher dimensional complete subvarieties of $M_g$. The varieties Harris constructed in [25] had dimension of order $\log_3 g$.

In Chapter 2 we present a construction of complete subvarieties of $M_g$ of arbitrary dimension $d$, where $d$ is of order $\log_2 g$. To be more precise, we give a construction of complete subvarieties of dimension $d$ in the moduli space of curves of genus $2^{d+1}$. This construction can be generalized to obtain complete subvarieties of dimension $d$ in the moduli space of curves of genus $2^{d+1} + k$, for any $k \geq 0$. 
The third result improves on our previous result, which tells us that there is a complete surface in $M_8$. Whether or not there exists a complete surface in the moduli space of smooth curves of genus 4, 5, 6 or 7 was unknown.

In Chapter 4 we give a construction of a complete surface in the moduli space of curves of genus 6 ([50]). The construction only works if the characteristic is not equal to 0 and 2. So we found a complete surface in the moduli space of genus 6 curves in characteristic $p > 2$.

Our fourth result answers again a question of Joe Harris. In a series of lectures on rationally connected varieties [26], Harris asked for 1-dimensional families of non-rational curves without sections.

In Chapter 5 we will construct two examples answering Harris's question. The first is a family of stable curves over $\mathbb{P}^1$ which does not admit a section. The second is a complete family of smooth curves without a section.

4 Speculations

We will finish this introduction with some speculations. Still a lot is unknown about the existence of complete subvarieties of the moduli space of smooth curves.

All known examples of constructions of complete subvarieties of dimension $\geq 2$ of the moduli space of smooth curves lie in the locus $X$ of curves mapping non-trivially onto a curve of positive genus. This is a rather "small" sublocus of the moduli space. It has codimension $g - 1$.

There is no reason why complete subvarieties should lie in $X$. Indeed, the complete curves inside $M_g$ which are constructed using the Satake compactification ($g \geq 3$) do not lie inside $X$.

To construct explicit families of smooth curves over a complete base, we only know of one trick: taking ramified covers of a fixed base curve. New ideas are required to construct complete surfaces inside the moduli space of smooth curves outside $X$.

Even if we would know how to construct complete subvarieties of $M_g$ of dimension $d \geq 2$, we still would have other questions about the nature of these varieties. For instance, we know that — with the help of the Satake compactification — one can construct complete curves in $M_g$ through any given point. I.e., given any smooth algebraic curve, we can construct a complete 1-dimensional family of smooth curves containing this given curve.

For higher dimensional subvarieties Harris and Morrison phrased this as a question [27]. To be more concrete: given a general point of $M_g$, is it possible to construct in $M_g$ a complete surface containing this given point? Or, in terms of families: given any smooth algebraic curve, can one construct a 2-dimensional family of smooth curves containing this given curve?

Another speculation has to do with the characteristic of the base field. In Chapter 4 we constructed a complete surface in $M_6$ — but only in characteristic $p > 2$. Our
construction may work in characteristic 0, but it may also fail. The latter situation is not uncommon in algebraic geometry: in characteristic $p > 0$ one can have phenomena which in characteristic 0 do not exist.

Closely related to the problem of the existence of complete subvarieties in $M_g$ is the similar problem for $A_g$, the moduli space of principally polarized abelian varieties of dimension $g$. In this case there is a very striking result by Keel and Sadun [32]: in characteristic 0 there does not exist a complete subvariety of $A_g$ of codimension $g$. This had been conjectured by Oort [46]. This is a very remarkable result, because in characteristic $p > 0$ the situation is entirely different: there does exist a complete subvariety of codimension $g$, namely the locus of principally polarized abelian varieties of $p$-rank 0 [33].

So for $M_g$ we could speculate: is it possible that for some genus $g$ complete subvarieties of some dimension $d$ in $M_g$ do exist only in characteristic $p > 0$ and not in characteristic 0? To be more specific: could it be possible that for $g > 3$ in $M_g$ complete subvarieties of codimension $g - 2$ exist only in characteristic $p > 0$ and not in characteristic 0? See [23] for explicit examples.

Here we end our speculations. Note that these speculations contain very general and difficult questions, which we weren't able to solve.

In Chapter 7 of this thesis we will present some more detailed speculations on the existence of complete subvarieties of $M_g$. Specifically, we will consider the question of the existence of a complete surface in $M_4$. 