Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 1
Overview of known results

In this chapter we describe what is known about complete subvarieties of $M_g$, the moduli space of non-singular curves. The survey will be genealogical in nature: we will start with the basic results and we will finish with more recent developments. In the second part of this chapter we will study some results in detail.

In the results that we will present in this chapter, the characteristic of the base field plays a subtle role. All of the results will work in characteristic 0, i.e., over $C$. But since the constructions are geometrical in nature, they will work in any characteristic which is sufficiently big, i.e., not dividing the degrees of the maps which occur in the constructions.

1.1 Complete subvarieties: a survey

Mumford's Geometric Invariant Theory appeared in 1965 [38]. This book put the study of moduli spaces of curves as algebraic varieties on firm ground. Moduli spaces of curves have been studied since the nineteenth century ([47], [10]), but the techniques of modern algebraic geometry gave way to lots of new algebraic and geometrical results (see for example [13]).

The question on the existence of complete subvarieties of $M_g$ was addressed in 1974 by Oort in [45]. In 1985 Harris formulated a series of questions with respect to complete subvarieties of $M_g$ [25].

1.1.1 Constructions of complete curves in $M_g$

In 1966 Baily and Borel showed that the so-called Satake compactification is algebraic [4]. Using this Satake compactification one can construct lots of complete curves in $M_g$. It is not known to whom this idea can be attributed. Oort writes that Mumford pointed it out to him [45]. In the following we will describe the basic idea.

To every smooth curve $C$ one can associate its Jacobian $\text{Jac}(C)$, which is a principally polarized abelian variety. Assigning to the isomorphism class of a curve $C$ the isomorphism class of its Jacobian $\text{Jac}(C)$ defines a map from $M_g$ to $A_g$, which is called the Torelli map. Torelli's theorem assures us that this map is bijective on closed points.

There exists a projective compactification $A^*_g$ of $A_g$, called the Satake compactification. It admits a stratification as follows:

$$A^*_g = A_g \cup A_{g-1} \cup A_{g-2} \cup \ldots \cup A_1 \cup A_0.$$ 

Moreover, the Torelli map $M_g \to A_g$ can be extended to a map $\overline{M}_g \to A^*_g$ [42]. The image of $\overline{M}_g$ in $A^*_g$ is called the Satake compactification of $M_g$. 

The extended Torelli map $M_g \to A_g^*$ maps a stable curve to the Jacobian of its normalisation, in particular it forgets about the position of the nodes. The Torelli map $M_g \to A_g^*$ is injective on the closed points of $M_g$.

On the boundary of $M_g$ the Torelli map behaves differently. The boundary $\partial M_g = M_g \setminus M_g$ is a union of irreducible codimension-one components $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{[g/2]}$. Here $\Delta_0$ is the closure of the closure of the locus of irreducible curves of geometric genus $g - 1$ with one node. For $i \geq 1$, the general element of $\Delta_i$ is a curve having two components meeting in one point: an irreducible, smooth curve of genus $i$ and an irreducible smooth curve of genus $g - i$.

On these boundary components the dimensions of the fibers of the Torelli map are as follows:

- on $\Delta_0$, the fibers have dimension 1 for $g = 2$, for $g \geq 3$ they have dimension 2,
- on $\Delta_1$, the fibers have dimension 1 for $g \geq 3$,
- on $\Delta_2, \ldots, \Delta_{[g/2]}$, the fibers have dimension 2.

So for $g \geq 3$ the Torelli map contracts the boundary of $M_g$ with relative dimension at least one. Hence the boundary of $M_g$ in its Satake compactification has codimension at least two. Using the projectivity of $A_g^*$, we can intersect the image of $M_g$ with $3g - 4$ hyperplanes in general position to arrive at a projective one-dimensional subvariety lying completely in $M_g$. This proves the existence of projective curves in $M_g$ for $g \geq 3$.

In fact, the same construction shows that it is possible to construct a complete curve passing through any given point $[C]$ of $M_g$. In other words, given a smooth algebraic curve $C$, there exists a non-degenerate family of smooth curves $X \to B$ over a complete 1-dimensional base $B$, such that $C$ is isomorphic to one of the fibers.

**Remark 1.1** In fact, one can even show that complete curves exist in $M_g$ through any given finite set of points of $M_g$, using a Veronese embedding of the Satake compactification of $M_g$.

**Remark 1.2** In genus 2 the above argument breaks down (as it should), since the restriction of the extended Torelli map to $\Delta_1$ has relative dimension 0.

**Remark 1.3** By intersecting with $3g - 5$ hyperplanes in general position, the above argument gives for $g \geq 4$ a projective surface in $\bar{M}_g$, meeting only $\Delta_1$ and none of the other boundary divisors $\Delta_0, \Delta_2, \ldots, \Delta_{[g/2]}$.

**Remark 1.4** In [17], González-Díez and Harvey make this construction explicit for $g = 3$. They write down five explicit modular forms on $A_3$ which cut out a complete curve on the image of $M_3$.

Kodaira was the first to give an explicit construction of a complete family of smooth curves of genus $\geq 3$. In an article from 1967 he presents a complete curve of genus 129 in $M_6$ [34].

The starting point of this construction is a curve $D$ of genus $\geq 3$ with a fixed-point free involution $i : D \to D$. The genus of such a curve is odd, equal to $2h - 1$, where $h$ is the genus of the quotient curve $D/i$. Let $F = \{(x, ix) | x \in D\} \subset D \times D$. 
Then $F$ parametrizes moving pairs of distinct points on $D$, which will serve as branch points of a family of double covers of $D$.

An étale cover $f: F' \to F$, needed to overcome monodromy problems, serves as the base of a 1-dimensional family $S \to F'$. The total space is a surface $S$ which is a double cover of $F' \times D$ and which is ramified precisely over the image of the sections $\sigma_i: F' \to D$ determining the two moving branch points on $D$, with $\sigma_i(x) = (\pi_i(f(x)))$ ($i = 1, 2$). The choice of $F'$ guarantees that the double cover $S \to F' \times D$ exists.

Thus Kodaira obtains 1-dimensional, complete families $C \to B$ of double covers of a fixed smooth curve of genus $h$, ramified in two distinct points. The covers are smooth curves of genus $2(2h - 1)$, $h = 2, 3, 4, \ldots$ Moreover, Kodaira computes the tangent map of the functorial map $B \to M_g$, which turns out to be non-zero for $h \geq 2$. From this it follows that for $h \geq 2$ the image of $B$ in $M_g$ has maximal dimension, i.e., the family $C \to B$ is non-degenerate. (If $h = 1$ the tangent map is zero in all points of the base. In that case the image of $B$ in $M_g$ is a point and all fibers of the family $C \to B$ are isomorphic, i.e. this family is isotrivial.)

In the smallest possible case, $h = 2$, we have a family of curves of genus 6. In that case, the base curve turns out to be a 64:1 unramified (possibly reducible) cover of a genus 3 curve. This suggests that the base curve of a family of smooth curves is rather complicated.

In [17], González-Diez and Harvey construct for every $g \geq 4$ a complete family of smooth curves of genus $g$. Their starting point is a curve $D$ of genus 2 which admits a non-constant map $f: D \to E$ to an elliptic curve $E = (E, 0)$. Fix $2b$ pairwise distinct elements $t_1, \ldots, t_{2b} \in E$. In $E^{2b}$ we consider the translated small diagonal:

$$\delta_t = \{ (x + t_1, \ldots, x + t_{2b}) \mid x \in E \}.$$ 

Let $F = (f^{2b})^{-1}(\delta_t) \subset D^{2b}$. Let $\Delta_D \subset D^{2b}$ be the big diagonal. Then $F \cap \Delta_D = \emptyset$ in $D^{2b}$, since $\delta_t \cap \Delta_E = \emptyset$ in $E^{2b}$.

The curve $F$ parametrizes a family of moving $b$-tuples of distinct points on $D$ and will serve as the base of a family of branched covers of $D$. As in Kodaira’s construction, an étale cover $F' \to F$ is needed to overcome monodromy problems. See Subsection 1.2.2 for a detailed description.

Setting $b = 1, 2, 3, \ldots$, González-Diez and Harvey obtain families of double covers of $D$, ramified in $2b$ points. So they obtain a complete family $C \to B$ of smooth curves of genus $g$ for every $g \geq 4$.

**Remark 1.5** In the case that $b = 1$ it is easy to see that $(D \times D) \setminus \Delta$ contains projective curves. Namely, $\Delta^2 = 2 - 2g = -2$, so $\Delta$ can be contracted to a point. An explicit map which contracts $\Delta$ is the map $D \times D \to \text{Jac}(D)$ with $(x, y) \mapsto \mathcal{O}_D(x - y)$. 


1.1.2 Bounds on the dimension of complete subvarieties

In the previous section we gave examples of complete non-degenerate families of smooth curves. These examples give lower bounds for the dimension of a complete subvariety of $M_g$. In this section we discuss a theorem by Steven Diaz, which gives the best known upper bound of the dimension for such a variety. We assume $g \geq 2$.

**Theorem 1.6** ([14], [16], [35]) Let $V$ be a complete subvariety of $M_g(k)$, with $g \geq 2$ and $\text{char}(k)$ arbitrary. Then the dimension of $V$ is at most $g - 2$.

In characteristic 0, this result is due to Diaz [14], [16]. The generalization to arbitrary characteristic is due to Looijenga [35].

In his proof of Theorem 1.6, Diaz uses a stratification of $M_g$. Following an idea of Arbarello, he defines closed loci in $M_g$ as follows:

$$H_g(i,j) = \{ [C] \in M_g \mid \exists f : C \to \mathbb{P}^1 \text{ with } \deg(f) \leq i \text{ and } \#f^{-1}(\{0, \infty\}) \leq j \}.$$  

In this definition, the points in the inverse image are counted without multiplicity. Clearly, $H_g(i,j - 1) \subset H_g(i,j)$. Moreover, for fixed $i \geq g$ the $H_g(i,j)$ stratify $M_g$ since $H_g(i,1) = \emptyset$ and $H_g(g,g) = M_g$, because every curve of genus $g$ has a Weierstrass point, i.e., admits a map of degree $g$ to $\mathbb{P}^1$ with a point of total ramification. Diaz then shows that these strata have the following property.

**Lemma 1.7** ([14]) The locus $H_g(i,j) \setminus H_g(i,j - 1)$ does not contain a complete curve.

This lemma is proved using the theory of admissible covers, as developed by Harris-Mumford [28] and Diaz [15].

To prove Theorem 1.6 from this lemma, suppose we have a complete subvariety $V$ in $M_g$ of dimension $d$. We use the following stratification:

$$\emptyset = H_g(g,1) \subset H_g(g,2) \subset H_g(g,3) \subset \cdots \subset H_g(g,g-1) \subset H_g(g,g) = M_g.$$  

Now intersect $V$ with these strata. Lemma 1.7 shows that in each step of the resulting stratification of $V$ the codimension is at most 1. For if it were bigger, then we could intersect $V$ with hyperplanes to obtain a complete curve in $H_g(g,j) \setminus H_g(g,j-1)$ (here we use that $M_g$ is quasi-projective). Let $k \geq 2$ be the smallest integer such that $V \cap H_g(g,k) \neq \emptyset$. This intersection is complete, hence of dimension 0. Counting the steps in the stratification, we see that the dimension of $V$ is at most $g - 2$. This finishes the proof of Theorem 1.6.

Diaz based his proof on an idea of Arbarello from [2]. Arbarello defines:

$$W_k = \{ [C] \in M_g \mid \exists f : C \to \mathbb{P}^1 \text{ such that } \deg(f) \leq k \text{ and } \#f^{-1}(0) = 1 \}.$$  

That is, $W_k$ is the locus of curves having a $g_k^1$ with a point of total ramification. In this way Arbarello arrives at a shorter stratification:

$$\emptyset = W_1 \subset W_2 \subset W_3 \subset \cdots \subset W_{g-1} \subset W_g = M_g.$$  

Arbarello proves the strata $W_k$ are irreducible. Moreover, he claims that the differences $W_k \setminus W_{k-1}$ do not contain complete curves. If this were true, then this would give an alternative proof of Theorem 1.6. Unfortunately, Arbarello's proof of this claim contains a gap. Whether or not Arbarello's claim is true, is still an open question. It would be interesting to have an answer to this question, so we rephrase it in a more concrete way:
Question 1.8 Let $f : X \to B$ be a 1-dimensional, non-isotrivial (i.e., varying), complete family of triples $(C, D, p)$, where $C$ is a smooth curve of genus $g$, $D$ a $g_1^1$ on $C$ and $p$ a point of total ramification of $D$. Is it possible that all linear systems $D$ are base point free?

Since the publication of Diaz’s proof, the knowledge of $M_g$ has increased substantially. In particular, a lot more is known about the intersection theory of $M_g$ and $\overline{M}_g$.

In [35] Looijenga proves that any product of tautological classes of degree $d > g - 2$ is zero on $M_g$ [35](Theorem 1.1). In particular, it follows that $\kappa_1^{d-1}$ is zero on $M_g$. This gives another proof of Theorem 1.6 which is valid in all characteristics. For suppose that $M_g$ contains a complete subvariety $V$ of dimension $d > g - 2$. On $M_g$, $\kappa_1$ is ample, so $\kappa_1^d$ is a non-zero number on $V$. This contradicts Looijenga’s result from [35].

Remark 1.9 From the previous subsection, the largest known complete subvarieties of $M_g$ have dimension $2 \log g - 1$ (1.16). Note that this is very far from the Diaz’s bound.

1.1.3 Examples of higher-dimensional complete subvarieties

In Subsection 1.1.1 we discussed complete curves, i.e., 1-dimensional complete subvarieties of $M_g$, which exist for every $g \geq 3$. Higher dimensional subvarieties do also exist, because they can be constructed. In the literature one can find several constructions. We discuss some of them here.

Miller, in [36], constructs complete families of curves of arbitrary dimension, via double coverings. He generalizes Kodaira’s construction. His starting point is a curve $C$ of genus 3, carrying an involution which is fixed point free. The involution is used to parametrize pairs of distinct points on $C$. Via a base change, needed to extract square roots of the branch locus, a 1-dimensional family of double covers of $C$ is constructed, with fibers of genus 6.

Then one iterates this construction. On the fibers of the 1-dimensional family that one just has constructed, a fixed point free involution may not exist. After a suitable base change one can find an unramified double cover of the total space of the family, that itself is a family over the same base such that each fiber comes with a fixed point free involution. The fibers of the new family have genus 11. This new space of double covers parametrizes pairs of distinct points, which, after a base change needed for extracting square roots, are used as branching points of a complete 2-dimensional family of double covers, each fiber having genus 22. And so on.

The families constructed in this manner are non-degenerate, i.e., the natural map from the base to the moduli space of curves generically has finite fibers. For the 1-dimensional case, this was already proven by Kodaira in [34]. Kodaira computed the Kodaira Spencer map—the derivative of the map from the base of the family to $M_g$—in a point of the base and showed that is nonzero. The non-degeneracy for the higher dimensional case can be proven with induction on the dimension of the base. We refer to the proof of Lemma 2.2 for details.
Fact 1.10 Miller's construction from [36] results in a complete d-dimensional family of smooth curves of genus $\frac{1}{2}(2^{2d+1} + 1)$. In particular, this gives a complete surface in $M_{22}$ and a complete threefold in $M_{86}$.

Remark 1.11 Starting with a one-dimensional complete family of curves of genus 3, Miller's construction results in a complete non-degenerate d-dimensional family of smooth curves of genus $\frac{1}{2}(7 \cdot 4^{d-1} + 2)$. In particular, this gives a complete surface in $M_{10}$ and a complete threefold in $M_{38}$.

In [25], Harris improves on Miller's result. Harris refrains from taking double covers—instead he takes triple covers ramified in one point. This may seem a bit odd at first sight, but triple coverings ramified completely in one point do exist. However, these covers are not normal, and the genus of the base curve has to be at least one.

Given a complete family $X \to B$ of smooth curves of genus $g$, with $\text{dim}(B) = d$, Harris uses $X$ as the base of a new family, as follows: one wants to construct a $3:1$ cover $W \to X \times_B X$ which is totally ramified over $\Delta \subset X \times_B X$. Such a cover may not exist over $X$, but will exist over some cover $\tilde{X}$ of $X$. So over $\tilde{X} \times_B X$ one can find a cover $\tilde{W} \to \tilde{X} \times_B X$ which is totally ramified over $\tilde{\Delta}$. The family $\tilde{W} \to \tilde{X}$ is then a complete family with base of dimension $d + 1$. Using a curve of genus 2 as starting point, one obtains the following result.

These families constructed by Harris are non-degenerate. As in Miller's construction, in dimension one the non-degeneracy follows from a computation of the Kodaira Spencer map. Likewise, the non-degeneracy of the higher dimensional case follows by induction on the dimension of the base.

Fact 1.12 Starting with a single curve of genus 2, the construction of Harris in [25] gives d-dimensional families of smooth curves of genus $\frac{1}{2}(3^{d+1} + 1)$. In particular, this gives a complete surface in $M_{14}$ and a complete threefold in $M_{41}$.

Remark 1.13 Starting with a 1-dimensional complete family of curves of genus 3, the construction of Harris gives complete non-degenerate d-dimensional families of smooth curves of genus $\frac{1}{2}(5 \cdot 3^{d-1} + 1)$. In particular, this gives a complete surface in $M_{8}$ and a complete threefold in $M_{23}$.

Remark 1.14 Topologically, a triple cover ramified in one point can be constructed as follows. If $B$ is the base curve of genus $g$ and $p \in B$ the branch point, one looks at $G = \pi_1(B \setminus \{p\}, \ast)$. This is a free group on $2g$ generators $a_i, b_i, i = 1, \ldots, g$. A triple cover of $B$ ramified in $p$ corresponds to a $G$-action on three points on which $\prod_{i=1}^{g}[a_i, b_i]$ acts as a 3-cycle. If we want to do this construction in families, we get local systems of $G$-actions, which have sections after suitable pullbacks.

Remark 1.15 Algebraically, a triple cover of a curve $B$ ramified in one point $p \in B$ can be constructed as follows. First one constructs an unramified double cover $f : C \to B$. Write $f^{-1}(p) = p_1 + p_2$, and let $i$ denote the covering involution of $f : C \to B$. Let $\mathcal{L}$ be a line bundle on $C$ such that $\mathcal{L}^3 \cong \mathcal{O}(p_1 + 2p_2)$. Then $\mathcal{L}$ determines a cyclic $3:1$ cover $F \to C$, ramified completely in $p_1$ and $p_2$. We want $i$ to lift to an involution $\tilde{i}$ on $F$, for in that case the quotient $F/\tilde{i}$ is a $3:1$ cover of $B$ branched completely over $p$. 
A calculation shows that the existence of a lift of \( i \) to \( F \) is equivalent to \( Nm(f)(\mathcal{L}) = \mathcal{O}_B(p) \). Note that in any case, \( Nm(f)(\mathcal{L}) \otimes \mathcal{O}_B(-p) \in \text{Pic}(B) \). Since we can change \( \mathcal{L} \) by an element of \( \text{Pic}(C) \) and \( Nm(f) : \text{Pic}(C) \rightarrow \text{Pic}(B) \) is surjective, since \( Nm(f) \circ f^* \) is multiplication by 2 on \( \text{Pic}(B) \), we can find an \( \mathcal{L} \in \text{Pic}^1(C) \) with the desired property. The generalization of this construction to families \( X \rightarrow B \) is straightforward.

The final construction we want to discuss is the one by González-Díez and Harvey from [17], which is explained in detail in Chapter 2.

**Corollary 1.16** The construction of González-Díez and Harvey from [17] results in complete non-degenerate \( d \)-dimensional families of smooth curves of genus \( 2^{d+1} \). In particular, this gives a complete surface in \( M_8 \) and a complete threefold in \( M_{16} \).

Corollary 1.16 gives the best known existence results of complete subvarieties of dimension \( \geq 3 \). For surfaces there is a better result. In Chapter 4 we show that if the characteristic is odd, then already \( M_6 \) contains a complete surface.

### 1.2 Some calculations

In Subsection 1.1.1 we presented non-trivial examples of 1-dimensional families of smooth algebraic curves over a complete base. The genera of the base curves turned out to be quite high. In this section we will study two of these constructions in more detail. We will see that in those cases the base curve is in fact reducible, consisting of several (disjoint) curves of much lower genus.

#### 1.2.1 Kodaira’s construction revisited

Kodaira gave a construction of a complete family of smooth algebraic curves of genus 6 (Subsection 1.1.1 and [34]). Here we present a minimal version of Kodaira’s construction. In this section we need the characteristic of the base field to be different from 2.

**Theorem 1.17** The base of the complete family of genus 6 curves constructed by Kodaira in [34] is reducible. A refined construction gives a base which is a disjoint union of irreducible smooth curves of genus 9.

**Proof.** Let \( D \) be a smooth curve of genus 3 with a free involution \( i \). Let \( D' \) be the quotient curve \( D/i \) and \( \pi : D \rightarrow D' \) the quotient map. In \( D' \times D \) consider the divisor \( \Sigma = \{(x, y) \mid \pi(y) = x\} \). Each point of \( \Sigma \) determines an ordered pair of inverse images of a point in \( D' \). We will construct a family of double covers of \( D \), ramified precisely in these pairs of points.

The curve \( D' \) maps to \( \text{Pic}^2(D) \) via \( x \mapsto \mathcal{O}_D(\pi^{-1}(x)) \). One can check that the divisor class of \( \Sigma \) becomes even over the fiber product \( D' \times_{\text{Pic}^2D} \text{Pic}^1D \), where \( \text{Pic}^1D \rightarrow \text{Pic}^2D \) is the squaring map. Setting \( B = D' \times_{\text{Pic}^2D} \text{Pic}^1D \), we can construct
a double cover of $D \times B$ ramified precisely along the pullback of $\Sigma$. This surface mapping to $B = D' \times_{\text{Pic}^2 D} \text{Pic}^1 D$ is (a slight improvement of) Kodaira's family of smooth genus 6 curves.

So the fiber product $B$ will be the base of the complete family of curves we are constructing. This smooth curve might be reducible, i.e., might consist of a number of disjoint components. To compute this number, we study the following diagram:

$$
\begin{array}{c}
\text{Pic}^1(D) \\
\downarrow L \rightarrow L^0 \downarrow \\
D' \rightarrow \text{Pic}^2(D).
\end{array}
$$

We are interested in the topological obstruction to lifting the map $D' \rightarrow \text{Pic}^2(D)$ to $\text{Pic}^1(D)$. The obstruction is precisely the image of the fundamental group $\pi_1(D', \ast)$ in $\pi_1(\text{Pic}^2(D), \ast)/\pi_1(\text{Pic}^1(D), \ast) \cong \text{Jac}(D)[2] \cong (\mathbb{Z}/2\mathbb{Z})^6$. We need the following lemma.

**Lemma 1.18** The image of $\pi_1(D', \ast)$ in $\pi_1(\text{Pic}^2(D), \ast)/\pi_1(\text{Pic}^1(D), \ast)$ is isomorphic to the image of $\text{Jac}(D')[2]$ in $\text{Jac}(D)[2]$ under $\pi^*$. 

**Proof.** Choosing a Weierstrass point $p \in D'$ as a base point, we get a commutative diagram

$$
\begin{array}{c}
\text{Pic}^1(D) \cong \text{Pic}^0(D) \\
\downarrow L \rightarrow L^0 \downarrow \\
D' \rightarrow \text{Pic}^2(D) \cong \text{Pic}^0(D).
\end{array}
$$

The map $D' \rightarrow \text{Pic}^0(D)$ factors via the canonical map $D' \rightarrow \text{Jac}(D')$ (using the base point $p$) and thus the image of $\pi_1(D', \ast)$ is at most the image of $\text{Jac}(D')[2]$. That it is not smaller follows from the fact that on the genus 2 curve $D'$, every 2-torsion point of its Jacobian can be written as a difference of two Weierstrass points. 

Since $D \rightarrow D'$ is unramified, $\text{Jac}(D') \rightarrow \text{Jac}(D)$ has as kernel a subgroup of order 2. Therefore, the image of $\pi_1(D', \ast)$ in $\pi_1(\text{Pic}^2(D), \ast)/\pi_1(\text{Pic}^1(D), \ast) \cong (\mathbb{Z}/2\mathbb{Z})^6$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. As a consequence, the fiber product $D' \times_{\text{Pic}^2 D} \text{Pic}^1 D$ is a disjoint union of eight components, each mapping with degree 8 to $D'$.

To finish the proof of Theorem 1.17, note that Lemma 1.18 implies that the base $B = D' \times_{\text{Pic}^2 D} \text{Pic}^1 D$ is reducible. Its components are unramified coverings of $D'$ of degree 8. By the Hurwitz-Zeuthen formula, these are curves of genus 9.
1.2.2 The construction of González-Díez and Harvey revisited

We will give a more detailed description of the construction of González-Díez and Harvey, who construct complete families of smooth curves of genus $g$, for every $g \geq 4$ (Subsection 1.1.1 and [17]). We will study the genus 4 case in more detail. As in the Kodaira example, we can show that the base curve is reducible. In the following the characteristic of the base field is different from 2.

**Theorem 1.19** The base of the complete family of genus 4 curves constructed by González-Díez and Harvey in [17] is reducible. A refined construction gives a base which is a disjoint union of irreducible smooth curves of genus 9.

**Proof.** Let $E$ be an elliptic curve and $u$, $v$ two distinct points on $E$. The linear system $|u + v|$ determines a $2:1$ map $E \rightarrow \mathbf{P}^1$, which has four ramification points $w_0, w_1, w_2, w_3$. The square roots of $u + v \in \text{Pic}^2(E)$ are all effective, given by the $w_i$. Choose one, say $w_0$. Let $w_0 = 0$ be the origin of $E$.

Let $f : D \rightarrow E$ be the $2:1$ covering branched in the two points $u$ and $v$ that is determined by the square root $w_0$. Denote the corresponding ramification points in $D$ by $\tilde{u}$, $\tilde{v}$. Then $f^*(0) = f^*(w_0) \sim \tilde{u} + \tilde{v}$. This determines the hyperelliptic $g_2^1$ on $D$. Moreover, the Weierstrass points of $D$ are given by $f^{-1}(w_i), i = 1, 2, 3$. By the Hurwitz-Zeuthen formula $D$ is a curve of genus 2.

Let $t \in E$ be a point different from 0. Our construction will depend on $E$, $u$, $v$ and $t$. A priori these data will be general, but at some point we will specialize $u$, $v$ and $t$.

Let $F \subset D \times D$ be the pullback via $f \times f$ of the translated diagonal $\Delta_t$:

$$\Delta_t = \{(x, x + t) | x \in E\} \subset E \times E.$$

Then $F \subset D \times D \setminus \Delta$. The projections on the first and second coordinate give maps $\pi_1, \pi_2 : F \rightarrow D$, which are of degree 2: $\pi_1$ is branched over $f^{-1}(u - t)$ and $f^{-1}(v - t)$, $\pi_2$ is branched over $f^{-1}(u + t)$ and $f^{-1}(v + t)$. So for general $t$, the curve $F$ in $D \times D$ is a non-singular curve of genus 5.
Section 1.2. Some calculations

We need the following lemma.

**Lemma 1.20** In the diagram

\[
\begin{array}{c}
\text{Pic}^1(D) \\
\downarrow_{L \rightarrow L^{g2}} \\
F \xrightarrow{\pi_1+\pi_2} \text{Pic}^2(D)
\end{array}
\]

the image of \( \pi_1(F, \ast) \) in \( \pi_1(\text{Pic}^2(D), \ast)/\pi_1(\text{Pic}^1(D), \ast) \) equals \( E[2] \), the kernel of multiplication by 2 on \( E \).

**PROOF.** The following diagram commutes:

\[
\begin{array}{ccc}
\text{Pic}^1(D) \times D & \xrightarrow{(L,d) \mapsto (L-d,d)} & \text{Pic}^0(D) \times D \\
\downarrow_{(\times 2, \text{id}_D)} & & \downarrow_{(\times 2, \text{id}_D)} \\
F \xrightarrow{\pi_1+\pi_2} \text{Pic}^2(D) \times D & \xrightarrow{(L,d) \mapsto (L-2d,d)} & \text{Pic}^0(D) \times D
\end{array}
\]

This shows that the fiber product of diagram (1) equals the fiber product of the following diagram.

\[
\begin{array}{c}
\text{Pic}^0(D) \\
\downarrow_{L \rightarrow L^{g2}} \\
F \xrightarrow{\pi_1-\pi_2} \text{Pic}^0(D).
\end{array}
\]

So the topological obstruction of lifting the map \( F \rightarrow \text{Pic}^2(D) \) to \( F \rightarrow \text{Pic}^1(D) \) in (1) can be computed from (2). Therefore we compute the image of \( \pi_1(F, \ast) \) in \( \pi_1(\text{Pic}^0(D), \ast)/\text{sq}_n(\pi_1(\text{Pic}^0(D), \ast)) \cong \text{Jac}(D)[2] \).

Let \( \text{Nm} = \text{Nm}(f) : \text{Pic}^0(D) \rightarrow \text{Pic}^0(E) \) be the norm map induced by \( f : D \rightarrow E \). By definition the image of \( F \) in diagram (2) equals \( \text{Nm}^{-1}(-t) \). This is an elliptic curve; it is the (translated) Prym variety associated to \( f : D \rightarrow E \) (see [39]). Denote this curve by \( E' \).

The monodromy group of (2) then equals \( E'[2] \). But the 2-torsion subgroup \( E'[2] \) is mapped isomorphically onto the 2-torsion subgroup of \( \text{Nm}^{-1}(0) \cong E' \) via the following injection (see [39]):

\[
f^* : E = \text{Pic}^0(E) \rightarrow \text{Pic}^0(D).
\]

We finish the proof of Theorem 1.19. By Lemma 1.20 the base of the family \( C \rightarrow B \) constructed by González-Díez and Harvey is reducible. More precisely, the family exists already over a 4 : 1 cover of (the normalization of) \( F \). This means that for general \( u, v \in E \) and general \( t \in E \) the base of this family has genus 17.

But if we specialize \( u, v \in E \) and \( t \in E \) we can do better. Choose \( u, v \) and \( t \) in such a way that \( u - v \in E[2] \) and \( t = u - v \). Then \((\bar{u}, \bar{v})\) and \((\bar{v}, \bar{u})\) are singular points of \( F \), locally given by \( x^2 = y^2 \). In this case \( F \) has two simple nodes, and the normalization \( N \) of \( F \) has genus 3. The base of an irreducible component of \( C \rightarrow B \) is then a 4 : 1 cover of \( N \), so has genus 9 by the Hurwitz-Zeuthen formula. 

\[\square\]
We will finish this section by calculating the number of hyperelliptic fibers and the number of fibers carrying a so-called vanishing theta-null (a base point free $g^1_3$ with $2g^1_3 = K$). We will do these calculations on the irreducible families $C \to B$ from Theorem 1.19 in which $t \in E$ is general. Note that for these families, the base is irreducible, but the genus of the base may be greater than 9.

We need the following lemmas.

**Lemma 1.21** Let $(p, q) \in F$ and let $g : C \to D$ be a double cover of the genus 2 curve $D$, branched in the two points $p$ and $q$, and determined by a square root $\mathcal{L}$ of $\mathcal{O}_D(p + q)$. Suppose that $t \not\in E[2]$. Then $C$ is hyperelliptic if and only if $\mathcal{L}$ is effective, i.e., $\mathcal{L} \cong \mathcal{O}_D(r)$ for some point $r \in D$ (in which case $p + q$ is a divisor in the hyperelliptic $g^1_2$ on $D$).

**Proof.** If we assume that $\mathcal{L}$ is effective, then the equivalence $g^*(r) \sim \tilde{p} + \tilde{q}$ immediately yields a $g^1_2$ on $C$, i.e., $C$ is hyperelliptic.

On the other hand, if $C$ is hyperelliptic, then the hyperelliptic involution $i$ commutes with the involution $j$ associated to the map $g : C \to D$. The ten fixed points of $i$ are the Weierstrass points of $C$. They map to the six Weierstrass points of $D$.

Either $\tilde{p}$ and $\tilde{q}$ are fixed by $i$ or interchanged by $i$. The first possibility is excluded by the assumption that $t \not\in E[2]$. For in that case $p$ and $q$ would be Weierstrass points, so that $p - q$ would be in $D[2]$, implying $t \in E[2]$.

So the ramification points $\tilde{p}$ and $\tilde{q}$ are interchanged by $i$, and $\tilde{p} + \tilde{q}$ is a divisor in the $g^1_2$ on $C$. Likewise, $p + q$ is a divisor of the $g^1_2$ on $D$. The ten Weierstrass points of $C$ map onto five Weierstrass points of $D$. This leaves one Weierstrass point $r$ on $D$, having the property that $2r \sim p + q$ and $f^*(r) \sim \tilde{p} + \tilde{q}$. Set $\mathcal{L} = \mathcal{O}_D(r)$. □

**Lemma 1.22** Let $(p, q) \in F$ and let $g : C \to D$ be a double cover of the genus 2 curve $D$, branched in the two points $p$ and $q$, and determined by a square root $\mathcal{L}$ of $\mathcal{O}_D(p + q)$. Suppose $C$ is not hyperelliptic. Then $C$ has a vanishing theta null if and only if

$$\mathcal{L} \in \{ \mathcal{O}_D(q + a - a') \mid a \in D \} \subset \text{Pic}^1(D).$$

Here $'$ denotes the hyperelliptic involution.

**Proof.** Since $C$ is not hyperelliptic, $\mathcal{L}$ is not effective (see Lemma 1.21). Then $K \otimes \mathcal{L}$ is base point free and has $h^0 = 2$. This means we can find a divisor $q + a + b \in [K \otimes \mathcal{L}]$. Hence $\mathcal{L} \sim \mathcal{O}_D(q + a - b')$. Pulling back to $C$ gives $\tilde{p} + \tilde{q} + f^*(b') \sim 2\tilde{q} + f^*(a)$, i.e.,

$$\tilde{p} + f^*(b') \sim \tilde{q} + f^*(a).$$

So this equivalence gives one $g^1_3$ on $C$. If we add the left hand side and the right hand side, we get $\tilde{p} + \tilde{q} + f^*(a + b')$. This equals $K_C$ if and only if $a + b'$ is a canonical divisor on $D$, i.e., $a = b$. So the $g^1_3$ obtained is a vanishing theta null if and only if $\mathcal{L}$ is of the form $\mathcal{O}_D(q + a - a')$. □
Proposition 1.23 If $t \in E$ is general, the irreducible families from Theorem 1.19 have twelve hyperelliptic fibers.

Proof. In the construction, the curve $F \subset D \times D \setminus \Delta$ plays a central role. It admits natural maps to $\text{Pic}^0(D)$ and $\text{Pic}^2(D)$. First we will focus on these maps separately. In the final stage of the proof we will study the link between these maps.

The construction of Theorem 1.19 gives a commutative diagram:

$$
\begin{array}{ccc}
F' & \to & E'' \subset \text{Pic}^0(D) \\
\downarrow & & \downarrow \times 2 \\
F & \xrightarrow{\pi_1-\pi_2} & E' \subset \text{Pic}^0(D).
\end{array}
$$

Here $F'$ is the base of an irreducible component of the families constructed in Theorem 1.19. It is the fiber product of $F$ and $E''$ over $E'$, where $E'$ and $E''$ are elliptic curves. More precisely, $E' = \text{Nm}_{D/E}^{-1}(-t) \subset \text{Pic}^0(D)$ is the image of $F \to \text{Pic}^0(D)$ and $E''$ is a component of the inverse image of $E'$ under the squaring map $\times 2 : \text{Pic}^0(D) \to \text{Pic}^0(D)$.

Similarly, we have a commutative diagram:

$$
\begin{array}{ccc}
F' & \to & \text{Pic}^1(D) \\
\downarrow & & \downarrow \times 2 \\
F & \xrightarrow{\pi_1+\pi_2} & \text{Pic}^2(D).
\end{array}
$$

The map $F \to \text{Pic}^2(D)$ factorises as $F \subset D^2 \to D^{(2)} \to \text{Pic}^2(D)$. Since $t$ is general, $F \subset D \times D$ is not symmetric. Furthermore, $D^{(2)} \to \text{Pic}^2(D)$ is an isomorphism outside the canonical system $|K_D|$, which is contracted to a point. So $F \to \text{Pic}^2(D)$ is birational onto its image.

However, $F \to \text{Pic}^2(D)$ is not an isomorphism onto its image. We claim that there are eight pairs $(a, b)$ on $F$ which, via the sum map, are mapped to the hyperelliptic $g^2 \subset \text{Pic}^2(D)$. I.e., $\text{Im}(F) \subset \text{Pic}^2(D)$ has an eightfold point.

To see this, suppose that $(a, b) \in F$ and $a + b \in |K_D|$. Then $\text{Nm}_{D/E}(b - a) = t$ and $\text{Nm}_{D/E}(a + b) = \text{Nm}_{D/E}(K_D) = k$. Therefore $2\text{Nm}_{D/E}(a) = k - t$ and $\text{Nm}_{D/E}(2b) = k + t$. This gives four possibilities for $\text{Nm}_{D/E}(a) \in E$; the same for $\text{Nm}_{D/E}(b)$. Correspondingly there are eight possibilities for $a$ and $b$ in $D$. This results in eight points $(a, b)$ in $F$ such that $a + b = a + a' \sim K_D$. Note that $\{ \text{Nm}_{D/E}(b) | \exists a : (a, b) \in F, a + b \sim K_D \}$ is an $E[2]$-orbit in $E$. We will need this later on.

Note that on $F$ we have eight hyperelliptic pairs $(a, b)$. The inverse images in $F'$ of such a pair consists of four points. Such a point is completely determined by its image $(a, b)$ in $F$ and by its image in $\text{Pic}^1(D)$ which is a square root of $a + b \sim K_D$ (diagram 4). By diagram 3, the inverse image in $F'$ over a point $(a, b) \in F$ is a $E'[2]$-orbit. On the other hand, the square roots in $\text{Pic}^1(D)$ of $K_D$ form a $\text{Pic}^0(D)[2]$-orbit.

The central question is: How do the $8 \times 4 = 32$ roots of $K_D$ on $F'$ compare to the 16 roots of $K_D$ in $\text{Pic}^1(D)$? We claim that on $F'$ we obtain the full $\text{Pic}^0(D)[2]$-orbit twice.
To link the maps from $F$ to $\text{Pic}^0(D)$ and to $\text{Pic}^2(D)$, we have the following diagram.

\[
\begin{align*}
F' & \longrightarrow \text{Pic}^1(D) \times D \quad (L,d) \mapsto (L-d,d) \\
\downarrow & \downarrow (\times 2, \text{id}_D) \\
F & \longrightarrow \text{Pic}^2(D) \times D \quad (L,d) \mapsto (L-2d,d) \\
\end{align*}
\]

To prove the claim, consider two different points of $F'$ both mapping to $K_D$. Let $(a, b)$ and $(c, d)$ in $F$ be the corresponding points in $F$, and $L$ respectively $M$ be the corresponding roots in $\text{Pic}^0(D)$, so that $2L \sim a-b$ and $2M \sim c-d$. By construction, $L$ and $M$ differ by an element of $E'[2]$.

The corresponding roots of $K_D$ are $L+b$ and $M+d$. The difference $(L+b)-(M+d)$ is an element of $\text{Pic}^0(D)[2]$. The set of all possible differences is a subgroup $G$ of $\text{Pic}^0(D)[2]$. Obviously, $E'[2] \subseteq G$. But $\text{Nm}_{D/E}(L+b-M-d) = \text{Nm}_{D/E}(b-d)$, since $L-M \in E'[2]$. Moreover, $\text{Nm}_{D/E}(b-d) \in E[2]$ and, as noted above, $\text{Nm}_{D/E}$ maps all possible differences $b-d$ surjectively onto $E[2]$. So we see that $\text{Nm}_{D/E}: G \subseteq \text{Pic}^0(D)[2] \rightarrow E[2]$ is surjective and that its kernel contains $E'[2]$. Hence $G = \text{Pic}^0(D)[2]$.

This means that on $F'$ we get every root of $K_D$ at least once. Since we have 32 roots, we obtain every root in fact twice. So on $F'$ we obtain a total of $2 \cdot 6 = 12$ effective roots of $K_D$.

**Proposition 1.24** If $t \in E$ is general, the irreducible families constructed above contain 16 fibers with a non-degenerate vanishing theta null.

**Proof.** We need the same setup as in the proof of Proposition 1.23. In particular we focus on the map $F \rightarrow \text{Pic}^0(D)$ (diagram 3). Moreover, $E' = \text{Nm}_{D/E}^{-1}(-t)$, and for $s \in E$ with $2s = -t$ we have $E'' = \text{Nm}_{D/E}^{-1}(s)$.

Using Lemma 1.22, the fibers with a vanishing theta null correspond to the intersection of $E''$ and the curve $\{ a - a' \mid a \in D \}$ in $\text{Pic}^0(D)$. The norm of $a - a'$ equals $2\text{Nm}_{D/E}(a)$ and has to be equal to $s$. Hence for $\text{Nm}_{D/E}(a)$ we have four possibilities on $E$. This gives eight possibilities for $a \in D$, which give eight intersection points on $E''$ (since $t \in E$ is general).

We claim that over these eight intersection points the map $F' \rightarrow E''$ is unbranched. Since $E'' \rightarrow E'$ is étale, it suffices to understand the branching of $F \rightarrow E''$. As one can check, this map has degree two: it identifies $(u, v) \in F$ with $(u', v') \in F$. Its branch points are the points $b - b' \in \text{Pic}^0(D)$ which map to $-t \in E$. So $2\text{Nm}_{D/E}(b) = -t$, which gives eight branch points. It is straightforward to check that via the multiplication by 2 map $E'' \rightarrow E$ none of the eight intersection points $a - a'$ on $E''$ maps the eight branch points $b - b'$ on $E'$. This gives 16 points on $F'$ corresponding to fibers with a nontrivial vanishing theta null.

**Theorem 1.25** In the construction above, for $p, q \in D$ and $t \in E$ general, the degree of $\lambda$ on the base of each irreducible component of the families equals 4.
Section 1.2. Some calculations

Proof. We recall part of the construction. In the diagram below $C \to F'$ is a complete family of smooth double covers of $D$ over the irreducible complete base $F'$. The curve $F'$ is a $4 : 1$ étale cover of $F$.

\[
\begin{array}{ccc}
C & \downarrow \rho & \rightarrow & F' \times D & \rightarrow & F \times D & \rightarrow & D \\
F' & \downarrow & \rightarrow & F & \rightarrow & F
\end{array}
\]

The surface $C$ is a double ramified cover of $F' \times D$. The branch locus on $F' \times D$ is a union $\Sigma = \Sigma_1 + \Sigma_2$ of the images of the two sections $F' \to F' \times D$ induced by the two projections of $F$ to $D$. The corresponding ramification locus on $C$ is denoted by $\tilde{\Sigma}$.

Let $\omega$ be a 1-form on $D$, let $\Omega$ be the pullback to $F' \times D$ and $\tilde{\Omega}$ the pullback to $C$. Then $\tilde{\Omega}$ is a section of $H^0(C, K_C/F')$. Denote by $(\Omega)$ the divisor of the 1-form $\Omega$. Then $(\tilde{\Omega}) = \rho^*(\Omega) + \tilde{\Sigma}$.

We will compute the intersection number $(\tilde{\Omega})^2$, which usually is denoted by $\kappa_1$. The degree of $\lambda$ on $F'$ will follow from the relation $\kappa_1 = 12\lambda - \delta$, which holds on $\bar{M}_g$.

We have

\[
(\tilde{\Omega})^2 = (\rho^*(\Omega) + \tilde{\Sigma})^2 = 2\rho^*(\Omega)\tilde{\Sigma} + \tilde{\Sigma}^2 = 2\Omega\Sigma + \frac{1}{2}\Sigma^2.
\]

A straightforward calculation gives

\[
\Omega \cdot \Sigma = 2 \deg(F' \to F) \deg(K_D) \deg(\pi_i : F \to D) = 32
\]

and

\[
\Sigma^2 = 2\Delta^2 \deg(F' \to F) \deg(\pi_i : F \to D) = -32,
\]

where $\Delta^2$ is the self-intersection of $\Delta$ in $D \times D$. Note that $\Sigma_1 \Sigma_2 = 0$, because $F \cap \Delta = \emptyset$. Since $F^2 = 0$ one might be tempted to think that $\Sigma_i^2 = 0$. But $\Sigma_i$ is a pullback of $\Delta \subset D \times D$, not of $F$. For details we refer to the proof of Proposition 3.4. Resuming, we get $\tilde{\Omega}^2 = \kappa_1 = 48$, hence $\lambda_{F'} = 4$.

Remark 1.26 We perform a check. Hyperelliptic fibers have 10 vanishing theta nulls (determined by the 10 Weierstrass points), while a non-hyperelliptic fiber has only one vanishing theta null. This gives a total of $136 = 12 \cdot 10 + 16$ intersection points with the locus of curves with a vanishing theta null, for $t \in E$ general.

The locus of curves with a vanishing theta null is a divisor on $M_4$, and its class in $\text{Pic}(M_4)$ equals $34\lambda$ (see [20], p. 423). Restricted to the base of the family, this divisor has degree $34 \cdot 4 = 136$, which checks.