CHAPTER 2

Explicit complete subvarieties of dimension $d$ in $M_{2d+1}$

Abstract. We describe a construction of complete subvarieties of $M_g$ with dimension of order $\log_2(g)$.

In [25] Joe Harris describes a construction of higher dimensional subvarieties of $M_g$, following a construction of Kodaira's: given a complete family $\pi : X \to B$ of smooth curves of genus $g$, one looks at the family of all curves of genus $3g - 1$ that are triple covers of fibers of $\pi$ totally ramified over one point. This gives a family of $\dim(B) + 1$. Starting with a fixed genus 2 curve $C \to \text{Spec}(\mathbb{C})$ of genus 2, this construction gives rise to complete subvarieties of dimension $d$ in $M_g$, where $g = \frac{3}{2}(3d - 1) + 2$. In this way we get complete subvarieties of $M_g$ with dimension of order $\log_3(g)$. (If one starts with $\mathbb{P}^1$ or an elliptic curve, one gets families without moduli.)

In this chapter we improve on this result. We will construct subvarieties of $M_g$ with dimension of order $\log_2(g)$. To be more precise, we will exhibit complete subvarieties of dimension $d$ in $M_{2d+1}$. This construction was suggested in [17], but has not found its way into the literature yet.

Theorem 2.1 For every $d \geq 1$ there exist in $M_{2d+1}$ complete subvarieties of dimension $d$.

In fact, we show a stronger statement, see Theorem 2.3. In showing the theorem above, we exhibit non-degenerate families of smooth curves $X \to B$, for which the base $B$ is complete and has dimension $d$ and the genus of the fiber is $2^{d+1}$. Non-degenerate means that the functorial map $B \to M_{2d+1}$, sending $b$ to $[X_b] \in M_{2d+1}$, generically has finite fibers. Our base field is $\mathbb{C}$.

2.1 The basic step

The basic ingredient in our construction is the following lemma, which we will apply recursively.

Lemma 2.2 Suppose we are given the following data:

(i) a smooth algebraic curve $\pi_0 : X_0 \to \text{Spec}(\mathbb{C})$;
(ii) a non-degenerate family $\pi_1 : X_1 \to B_1$ of curves of genus $g > 1$, smooth over a smooth base $B_1$;
(iii) maps $X_1 \to X_0$ and $B_1 \to \text{Spec}(\mathbb{C})$ commuting with the $\pi_i$, such that the induced map on the fibers of the families is a finite map of curves;
(iv) in $X_0 \times X_0$ an irreducible, complete curve $W_0$ not meeting the diagonal.

Then there exists a non-degenerate family $\pi_2 : X_2 \to B_2$ of smooth curves of genus $2g$, with smooth base $B_2$ and $\dim(B_2) = \dim(B_1) + 1$. Moreover, there is a