Complete subvarieties of moduli spaces of algebraic curves

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Chapter 2

Explicit complete subvarieties of dimension \(d\) in \(M_{2d+1}\)

Abstract. We describe a construction of complete subvarieties of \(M_g\) with dimension of order \(\log_2(g)\).

In [25] Joe Harris describes a construction of higher dimensional subvarieties of \(M_g\), following a construction of Kodaira's: given a complete family \(\pi : X \to B\) of smooth curves of genus \(g\), one looks at the family of all curves of genus \(3g-1\) that are triple covers of fibers of \(\pi\) totally ramified over one point. This gives a family of \(\dim(B)+1\).

Starting with a fixed genus 2 curve \(C \to \text{Spec}(\mathbb{C})\) of genus 2, this construction gives rise to complete subvarieties of dimension \(d\) in \(M_g\), where \(g = \frac{3}{2}(3d-1)+2\). In this way we get complete subvarieties of \(M_g\) with dimension of order \(\log_3(g)\). (If one starts with \(\mathbb{P}^1\) or an elliptic curve, one gets families without moduli.)

In this chapter we improve on this result. We will construct subvarieties of \(M_g\) with dimension of order \(\log_2(g)\). To be more precise, we will exhibit complete subvarieties of dimension \(d\) in \(M_{2d+1}\). This construction was suggested in [17], but has not found its way into the literature yet.

Theorem 2.1 For every \(d \geq 1\) there exist in \(M_{2d+1}\) complete subvarieties of dimension \(d\).

In fact, we show a stronger statement, see Theorem 2.3. In showing the theorem above, we exhibit non-degenerate families of smooth curves \(X \to B\), for which the base \(B\) is complete and has dimension \(d\) and the genus of the fiber is \(2^{d+1}\). Non-degenerate means that the functorial map \(B \to M_{2d+1}\), sending \(b\) to \([X_b] \in M_{2d+1}\), generically has finite fibers. Our base field is \(\mathbb{C}\).

2.1 The basic step

The basic ingredient in our construction is the following lemma, which we will apply recursively.

Lemma 2.2 Suppose we are given the following data:

(i) a smooth algebraic curve \(\pi_0 : X_0 \to \text{Spec}(\mathbb{C})\);
(ii) a non-degenerate family \(\pi_1 : X_1 \to B_1\) of curves of genus \(g > 1\), smooth over a smooth base \(B_1\);
(iii) maps \(X_1 \to X_0\) and \(B_1 \to \text{Spec}(\mathbb{C})\) commuting with the \(\pi_i\), such that the induced map on the fibers of the families is a finite map of curves;
(iv) in \(X_0 \times X_0\) an irreducible, complete curve \(W_0\) not meeting the diagonal.

Then there exists a non-degenerate family \(\pi_2 : X_2 \to B_2\) of smooth curves of genus \(2g\), with smooth base \(B_2\) and \(\dim(B_2) = \dim(B_1)+1\). Moreover, there is a