Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 2

Explicit complete subvarieties of dimension d in $M_{2^{d+1}}$

Abstract. *We describe a construction of complete subvarieties of M_g with dimension of order* $\log_2(g)$.

In [25] Joe Harris describes a construction of higher dimensional subvarieties of M_g, following a construction of Kodaira's: given a complete family $\pi : X \to B$ of smooth curves of genus g, one looks at the family of all curves of genus $3g-1$ that are triple covers of fibers of π totally ramified over one point. This gives a family of $\dim(B)+1$. Starting with a fixed genus 2 curve $C \to \mathrm{Spec}(\mathbf{C})$ of genus 2, this construction gives rise to complete subvarieties of dimension d in M_g, where $g = \frac{3}{2}(3^d - 1) + 2$. In this way we get complete subvarieties of M_g with dimension of order $\log_3(g)$. (If one starts with $\mathbf{P}^1$ or an elliptic curve, one gets families without moduli.)

In this chapter we improve on this result. We will construct subvarieties of M_g with dimension of order $\log_2(g)$. To be more precise, we will exhibit complete subvarieties of dimension d in $M_{2^{d+1}}$. This construction was suggested in [17], but has not found its way into the literature yet.

Theorem 2.1 *For every $d \geq 1$ there exist in $M_{2^{d+1}}$ complete subvarieties of dimension d.*

In fact, we show a stronger statement, see Theorem 2.3. In showing the theorem above, we exhibit non-degenerate families of smooth curves $X \to B$, for which the base B is complete and has dimension d and the genus of the fiber is 2^{d+1}. Non-degenerate means that the functorial map $B \to M_{2^{d+1}}$, sending b to $[X_b] \in M_{2^{d+1}}$, generically has finite fibers. Our base field is $\mathbf{C}$.

2.1 The basic step

The basic ingredient in our contruction is the following lemma, which we will apply recursively.

Lemma 2.2 *Suppose we are given the following data:*

(i) a smooth algebraic curve $\pi_0 : X_0 \to \mathrm{Spec}(\mathbf{C})$;

(ii) a non-degenerate family $\pi_1 : X_1 \to B_1$ of curves of genus $g > 1$, smooth over a smooth base B_1;

(iii) maps $X_1 \to X_0$ and $B_1 \to \mathrm{Spec}(\mathbf{C})$ commuting with the π_i, such that the induced map on the fibers of the families is a finite map of curves;

(iv) in $X_0 \times X_0$ an irreducible, complete curve W_0 not meeting the diagonal.

Then there exists a non-degenerate family $\pi_2 : X_2 \to B_2$ of smooth curves of genus $2g$, with smooth base B_2 and $\dim(B_2) = \dim(B_1)+1$. Moreover, there is a