Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

Publication date
2005

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
CHAPTER 2

Explicit complete subvarieties of dimension $d$ in $M_{2d+1}$

Abstract. We describe a construction of complete subvarieties of $M_g$ with dimension of order $\log_2(g)$.

In [25] Joe Harris describes a construction of higher dimensional subvarieties of $M_g$, following a construction of Kodaira's: given a complete family $\pi : X \rightarrow B$ of smooth curves of genus $g$, one looks at the family of all curves of genus $3g - 1$ that are triple covers of fibers of $\pi$ totally ramified over one point. This gives a family of $\dim(B) + 1$. Starting with a fixed genus 2 curve $C \rightarrow \text{Spec}(\mathbb{C})$ of genus 2, this construction gives rise to complete subvarieties of dimension $d$ in $M_g$, where $g = \frac{3}{2}(3^d - 1) + 2$. In this way we get complete subvarieties of $M_g$ with dimension of order $\log_3(g)$. (If one starts with $\mathbb{P}^1$ or an elliptic curve, one gets families without moduli.)

In this chapter we improve on this result. We will construct subvarieties of $M_g$ with dimension of order $\log_2(g)$. To be more precise, we will exhibit complete subvarieties of dimension $d$ in $M_{2d+1}$. This construction was suggested in [17], but has not found its way into the literature yet.

Theorem 2.1 For every $d \geq 1$ there exist in $M_{2d+1}$ complete subvarieties of dimension $d$.

In fact, we show a stronger statement, see Theorem 2.3. In showing the theorem above, we exhibit non-degenerate families of smooth curves $X \rightarrow B$, for which the base $B$ is complete and has dimension $d$ and the genus of the fiber is $2^{d+1}$. Non-degenerate means that the functorial map $B \rightarrow M_{2d+1}$, sending $b$ to $[X_b] \in M_{2d+1}$, generically has finite fibers. Our base field is $\mathbb{C}$.

2.1 The basic step

The basic ingredient in our construction is the following lemma, which we will apply recursively.

Lemma 2.2 Suppose we are given the following data:

(i) a smooth algebraic curve $\pi_0 : X_0 \rightarrow \text{Spec}(\mathbb{C})$;

(ii) a non-degenerate family $\pi_1 : X_1 \rightarrow B_1$ of curves of genus $g > 1$, smooth over a smooth base $B_1$;

(iii) maps $X_1 \rightarrow X_0$ and $B_1 \rightarrow \text{Spec}(\mathbb{C})$ commuting with the $\pi_i$, such that the induced map on the fibers of the families is a finite map of curves;

(iv) in $X_0 \times X_0$ an irreducible, complete curve $W_0$ not meeting the diagonal.

Then there exists a non-degenerate family $\pi_2 : X_2 \rightarrow B_2$ of smooth curves of genus $2g$, with smooth base $B_2$ and $\dim(B_2) = \dim(B_1) + 1$. Moreover, there is a