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### Complete subvarieties of moduli spaces of algebraic curves

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## CHAPTER 2

# Explicit complete subvarieties of dimension $d$ in $M_{2^{d+1}}$

**Abstract.** We describe a construction of complete subvarieties of  $M_g$  with dimension of order  $\log_2(g)$ .

In [25] Joe Harris describes a construction of higher dimensional subvarieties of  $M_g$ , following a construction of Kodaira's: given a complete family  $\pi : X \rightarrow B$  of smooth curves of genus  $g$ , one looks at the family of all curves of genus  $3g - 1$  that are triple covers of fibers of  $\pi$  totally ramified over one point. This gives a family of  $\dim(B) + 1$ . Starting with a fixed genus 2 curve  $C \rightarrow \text{Spec}(\mathbf{C})$  of genus 2, this construction gives rise to complete subvarieties of dimension  $d$  in  $M_g$ , where  $g = \frac{3}{2}(3^d - 1) + 2$ . In this way we get complete subvarieties of  $M_g$  with dimension of order  $\log_3(g)$ . (If one starts with  $\mathbf{P}^1$  or an elliptic curve, one gets families without moduli.)

In this chapter we improve on this result. We will construct subvarieties of  $M_g$  with dimension of order  $\log_2(g)$ . To be more precise, we will exhibit complete subvarieties of dimension  $d$  in  $M_{2^{d+1}}$ . This construction was suggested in [17], but has not found its way into the literature yet.

**Theorem 2.1** For every  $d \geq 1$  there exist in  $M_{2^{d+1}}$  complete subvarieties of dimension  $d$ .

In fact, we show a stronger statement, see Theorem 2.3. In showing the theorem above, we exhibit non-degenerate families of smooth curves  $X \rightarrow B$ , for which the base  $B$  is complete and has dimension  $d$  and the genus of the fiber is  $2^{d+1}$ . Non-degenerate means that the functorial map  $B \rightarrow M_{2^{d+1}}$ , sending  $b$  to  $[X_b] \in M_{2^{d+1}}$ , generically has finite fibers. Our base field is  $\mathbf{C}$ .

### 2.1 The basic step

The basic ingredient in our construction is the following lemma, which we will apply recursively.

**Lemma 2.2** Suppose we are given the following data:

- (i) a smooth algebraic curve  $\pi_0 : X_0 \rightarrow \text{Spec}(\mathbf{C})$ ;
- (ii) a non-degenerate family  $\pi_1 : X_1 \rightarrow B_1$  of curves of genus  $g > 1$ , smooth over a smooth base  $B_1$ ;
- (iii) maps  $X_1 \rightarrow X_0$  and  $B_1 \rightarrow \text{Spec}(\mathbf{C})$  commuting with the  $\pi_i$ , such that the induced map on the fibers of the families is a finite map of curves;
- (iv) in  $X_0 \times X_0$  an irreducible, complete curve  $W_0$  not meeting the diagonal.

Then there exists a non-degenerate family  $\pi_2 : X_2 \rightarrow B_2$  of smooth curves of genus  $2g$ , with smooth base  $B_2$  and  $\dim(B_2) = \dim(B_1) + 1$ . Moreover, there is a