Complete subvarieties of moduli spaces of algebraic curves

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map of families $\pi_2 \to \pi_1$ such that the induced map on the fibers is a finite map of curves. In particular, we obtain data as above, but with $(X_1, B_1)$ replaced by $(X_2, B_2)$:

\[
\begin{array}{ccc}
X_2 & \rightarrow & X_1 & \rightarrow & X_0 \\
\downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 \\
B_2 & \rightarrow & B_1 & \rightarrow & \text{Spec}(\mathbb{C})
\end{array}
\]

Finally, if $B_1$ is complete, then $B_2$ can be chosen to be complete as well.

**PROOF.** STEP 1 is the construction. In $X_0 \times X_0$ the curve $W_0$ parametrizes pairs of distinct points of $X_0$. Let $W$ be an irreducible component of the inverse image of $W_0$ under the map $X_1 \times B_1, X_1 \to X_0 \times X_0$. Then $W$ parametrizes pairs of *distinct* points in the fibers of $X_1 \to B_1$.

Let $V$ be a desingularisation of $W$. Set $Y = X_1 \times B_1 V$. Then $Y \to V$ is a family of curves, smooth over $V$. It has two disjoint sections $s_1, s_2$ induced by the two projections of $W$ onto $X_1$. With these sections $Y \to V$ is a family of two-pointed curves. Note that $Y$ is smooth since $V$ is, and that the divisors $s_i(V)$, $i = 1, 2$, meet the fibers of $Y \to V$ transversally in $Y$. Set $\Gamma = s_1(V) + s_2(V) \in \text{Div}(Y)$.

For our construction we need a line bundle $L$ on $Y$ satisfying $L^2 \cong O_Y(\Gamma)$. Such a bundle may not exist over the base $V$. Pulling back along a finite étale cover $V' \to V$ such an $L$ will exist, as we presently show.

Indeed, let $V_0$ be the normalisation of $W_0$, and consider $\Gamma_0 \subset X_0 \times V_0$, the divisor associated to the corresponding sections of $X_0 \times V_0 \to V_0$. Clearly, $\Gamma$ is the pullback of $\Gamma_0$. Let $V'_0 \to V_0$ be (a component of) the finite étale cover associated to the kernel of $\pi_1(V_0) \to H_1(V_0, \mathbb{Z}/2\mathbb{Z})$. The class of the pullback of $\Gamma_0$ to $X_0 \times V'_0$ is zero in $H_2(X_0 \times V'_0, \mathbb{Z}/2\mathbb{Z})$, as Atiyah shows in [3]. So let $V' = V \times V_0 V'_0$. Then the class of $\Gamma' = \Gamma \times_{V'} V' \subset Y' = Y \times_{V} V'$ is zero in $H_2(Y', \mathbb{Z}/2\mathbb{Z})$, and $\Gamma'$ determines an even class in $H_2(Y', \mathbb{Z})$. Since $\text{Pic}(Y')$ is an extension of the Neron-Severi group by a divisible group, there exists a line bundle $L$ satisfying $L^2 \cong O_{Y'}(\Gamma')$.

We continue the construction. By standard arguments, there exists in the total space of $L$ a double cover $X_2$ of $Y'$, ramified precisely along $\Gamma'$. Setting $B_2 = V'$ gives a new smooth family of algebraic curves $\pi_2 : X_2 \to B_2$. The fiber $(X_2)_{b_2}$ is a double cover of $(X_1)_{b_1}$, where $b_1$ is the image of $b_2$ in $B_1$, and this cover is ramified precisely over the tuple $\text{Im}(b_2) \in W$. By Riemann-Hurwitz, the fibers of $\pi_2$ have genus $2g$. The dimension of the base $B_2$ equals $\dim(B_1) + 1$. The construction is summarized in the following diagram.

\[
\begin{array}{ccc}
X_2 & \rightarrow & L \\
\downarrow & & \downarrow \\
Y' & \rightarrow & Y = V \times_{B_1} X_1 \\
\downarrow & & \downarrow \\
B_2 = V' & \rightarrow & V & \rightarrow & W \subset X_1 \times_{B_1} X_1 & \rightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \rightarrow & B_1
\end{array}
\]