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Complete subvarieties of moduli spaces of algebraic curves

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map of families $\pi_2 \to \pi_1$ such that the induced map on the fibers is a finite map of curves. In particular, we obtain data as above, but with (X_1, B_1) replaced by (X_2, B_2):

$$\begin{array}{ccccc} X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 \\ B_2 & \longrightarrow & B_1 & \longrightarrow & \mathrm{Spec}(\mathbf{C}) \end{array}$$

Finally, if B_1 is complete, then B_2 can be chosen to be complete as well.

PROOF. STEP 1 is the construction. In $X_0 \times X_0$ the curve W_0 parametrizes pairs of distinct points of X_0. Let W be an irreducible component of the inverse image of W_0 under the map $X_1 \times_{B_1} X_1 \to X_0 \times X_0$. Then W parametrizes pairs of *distinct* points in the fibers of $X_1 \to B_1$.

Let V be a desingularisation of W. Set $Y = X_1 \times_{B_1} V$. Then $Y \to V$ is a family of curves, smooth over V. It has two disjoint sections s_1, s_2 induced by the two projections of W onto X_1. With these sections $Y \to V$ is a family of two-pointed curves. Note that Y is smooth since V is, and that the divisors $s_i(V)$, $i = 1, 2$, meet the fibers of $Y \to V$ transversally in Y. Set $\Gamma = s_1(V) + s_2(V) \in \mathrm{Div}(Y)$.

For our construction we need a line bundle L on Y satisfying $L^2 \cong \mathcal{O}_Y(\Gamma)$. Such a bundle may not exist over the base V. Pulling back along a finite étale cover $V' \to V$ of V such an L will exist, as we presently show.

Indeed, let V_0 be the normalisation of W_0, and consider $\Gamma_0 \subset X_0 \times V_0$, the divisor associated to the corresponding sections of $X_0 \times V_0 \to V_0$. Clearly, Γ is the pullback of Γ_0. Let $V_0' \to V_0$ be (a component of) the finite étale cover associated to the kernel of $\pi_1(V_0) \to H_1(V_0, \mathbf{Z}/2\mathbf{Z})$. The class of the pullback of Γ_0 to $X_0 \times V_0'$ is zero in $H_2(X_0 \times V_0', \mathbf{Z}/2\mathbf{Z})$, as Atiyah shows in [3]. So let $V' = V \times_{V_0} V_0'$. Then the class of $\Gamma' = \Gamma \times_V V' \subset Y' = Y \times_V V'$ is zero in $H_2(Y', \mathbf{Z}/2\mathbf{Z})$, and Γ' determines an even class in $H_2(Y', \mathbf{Z})$. Since $\mathrm{Pic}(Y')$ is an extension of the Neron-Severi group by a divisible group, there exists a line bundle L satisfying $L^2 \cong \mathcal{O}_{Y'}(\Gamma')$.

We continue the construction. By standard arguments, there exists in the total space of L a double cover X_2 of Y', ramified precisely along Γ'. Setting $B_2 = V'$ gives a new smooth family of algebraic curves $\pi_2 : X_2 \to B_2$. The fiber $(X_2)_{b_2}$ is a double cover of $(X_1)_{b_1}$, where b_1 is the image of b_2 in B_1, and this cover is ramified precisely over the tuple $\mathrm{Im}(b_2) \in W$. By Riemann-Hurwitz, the fibers of π_2 have genus $2g$. The dimension of the base B_2 equals $\dim(B_1) + 1$. The construction is summarized in the following diagram.

$$\begin{array}{cccccccc} X_2 \longrightarrow L & & & & & & & \\ \searrow \quad \downarrow & & & & & & & \\ Y' & \longrightarrow & Y = V \times_{B_1} X_1 & & & & & \\ \downarrow & & \downarrow & & & & & \\ B_2 = V' & \longrightarrow & V & \longrightarrow & W \subset X_1 \times_{B_1} X_1 & \longrightarrow & X_1 & \\ & & & & \downarrow & & \downarrow & \\ & & & & X_1 & \longrightarrow & B_1 & \end{array}$$