map of families \( \pi_2 \rightarrow \pi_1 \) such that the induced map on the fibers is a finite map of curves. In particular, we obtain data as above, but with \((X_1, B_1)\) replaced by \((X_2, B_2)\):

\[
\begin{array}{ccc}
X_2 & \rightarrow & X_1 & \rightarrow & X_0 \\
\downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 \\
B_2 & \rightarrow & B_1 & \rightarrow & \text{Spec}(\mathbb{C})
\end{array}
\]

Finally, if \( B_1 \) is complete, then \( B_2 \) can be chosen to be complete as well.

**Proof.** Step 1 is the construction. In \( X_0 \times X_0 \) the curve \( W_0 \) parametrizes pairs of distinct points of \( X_0 \). Let \( W \) be an irreducible component of the inverse image of \( W_0 \) under the map \( X_1 \times B_1, X_1 \rightarrow X_0 \times X_0 \). Then \( W \) parametrizes pairs of distinct points in the fibers of \( X_1 \rightarrow B_1 \).

Let \( V \) be a desingularisation of \( W \). Set \( Y = X_1 \times B_1, V \). Then \( Y \rightarrow V \) is a family of curves, smooth over \( V \). It has two disjoint sections \( s_1, s_2 \) induced by the two projections of \( W \) onto \( X_1 \). With these sections \( Y \rightarrow V \) is a family of two-pointed curves. Note that \( Y \) is smooth since \( V \) is, and that the divisors \( s_i(V) \), \( i = 1, 2 \), meet the fibers of \( Y \rightarrow V \) transversally. Set \( \Gamma = s_1(V) + s_2(V) \in \text{Div}(Y) \).

For our construction we need a line bundle \( L \) on \( Y \) satisfying \( L^2 \cong O_Y(\Gamma) \). Such a bundle may not exist over the base \( V \). Pulling back along a finite étale cover \( V' \rightarrow V \) such an \( L \) will exist, as we presently show.

Indeed, let \( V_0 \) be the normalisation of \( W_0 \), and consider \( \Gamma_0 = X_0 \times V_0 \), the divisor associated to the corresponding sections of \( X_0 \times V_0 \rightarrow V_0 \). Clearly, \( \Gamma \) is the pullback of \( \Gamma_0 \). Let \( V_0' \rightarrow V_0 \) be (a component of) the finite étale cover associated to the kernel of \( \pi_1(V_0) \rightarrow H_1(V_0, \mathbb{Z}/2\mathbb{Z}) \). The class of the pullback of \( \Gamma_0 \) to \( X_0 \times V_0' \) is zero in \( H_2(X_0 \times V_0', \mathbb{Z}/2\mathbb{Z}) \), as Atiyah shows in [3]. So let \( V' = V \times V_0' \). Then the class of \( \Gamma' = \Gamma \times V' V' \subset Y' = Y \times V' \) is zero in \( H_2(Y', \mathbb{Z}/2\mathbb{Z}) \), and \( \Gamma' \) determines an even class in \( H_2(Y', \mathbb{Z}) \). Since \( \text{Pic}(Y') \) is an extension of the Neron-Severi group by a divisible group, there exists a line bundle \( L \) satisfying \( L^2 \cong O_{Y'}(\Gamma') \).

We continue the construction. By standard arguments, there exists in the total space of \( L \) a double cover \( X_2 \) of \( Y' \), ramified precisely along \( \Gamma' \). Setting \( B_2 = V' \) gives a new smooth family of algebraic curves \( \pi_2 : X_2 \rightarrow B_2 \). The fiber \( (X_2)_{b_2} \) is a double cover of \( (X_1)_{b_1} \), where \( b_1 \) is the image of \( b_2 \) in \( B_1 \), and this cover is ramified precisely over the tuple \( \text{Im}(b_2) \subset W \). By Riemann-Hurwitz, the fibers of \( \pi_2 \) have genus \( 2g \). The dimension of the base \( B_2 \) equals \( \dim(B_1) + 1 \). The construction is summarized in the following diagram.

\[
\begin{array}{c}
X_2 \\
\downarrow \\
Y' \\
\downarrow \\
B_2 = V' \\
\downarrow \\
X_1 \\
\end{array} \quad \begin{array}{c}
L \\
\downarrow \\
Y \rightarrow \mathbb{C} \times B_1 \rightarrow X_1 \\
\downarrow \\
W \subset X_1 \times B_1 \rightarrow \mathbb{C} \\
\downarrow \\
B_1
\end{array}
\]