Complete subvarieties of moduli spaces of algebraic curves

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Chapter 2. Explicit complete subvarieties of dimension \( d \) in \( M_{2d+1} \)

STEP 2. We claim that the map \( B_2 \to M_{2g} \) is non-degenerate. First note that the general fibers of \( B_2 \to B_1 \) parametrize double coverings of a fixed genus \( g \) curve. The restriction of \( B_2 \to M_{2g} \) to such a fiber is non-degenerate. This follows from a direct calculation of the Kodaira-Spencer map, as in [34] (here we need that \( g > 1 \)).

Secondly, suppose \( Z \subset B_2 \) is irreducible, one-dimensional, and contained in the general fiber of the map to \( M_{2g} \). Then we have maps of families:

\[
\begin{align*}
X_2 \times_{B_2} Z & \to X_1 \times_{B_1} \text{Im}(Z) \\
Z & \to \text{Im}(Z) \subset B_1
\end{align*}
\]

Since \( Z \) is not be contained in a fiber of \( B_2 \to B_1 \), \( \text{Im}(Z) \subset B_1 \) is one-dimensional. As \( Z \) maps to a point in \( M_{2g} \), the fibers of \( X_2 \times_{B_2} Z \to Z \) are all isomorphic. Since \( X_1 \to B_1 \) is non-degenerate, this would give a curve \( F \) of genus \( 2g \) (the fiber of \( X_2 \times_{B_2} Z \to Z \)) doubly covering infinitely many, pairwise non-isomorphic curves of genus \( g \). This is absurd, since \( F \) has a finite automorphism group.

\[ \Box \]

**Proof of Theorem 2.1.** To obtain the required families, let \( X_0 \) be a smooth genus 2 curve. Consider the map \( X_0 \times X_0 \to \text{Jac}(X_0) \), sending \( (P, Q) \) to \([P - Q]\). This map is birational, and \( \Delta \) is blown down to the origin. Let \( W_0 \) be the inverse image of a curve in \( \text{Jac}(X_0) \) which does not contain the origin. Then \( W_0 \) is a complete curve in \( X_0 \times X_0 \) not meeting the diagonal.

Applying Lemma 2.2 to \( X_1 = X_0 \) yields a complete, non-degenerate, one-dimensional family \( X_2 \to B_2 \) of genus 4 curves. Applying it to \( X_2 \to B_2 \) yields a complete, non-degenerate, two-dimensional family \( X_3 \to B_3 \) of genus 8 curves. In this way we obtain for any \( d \) non-degenerate families of smooth curves of genus \( 2^{d+1} \) over a complete base of dimension \( d \).

\[ \Box \]