Complete subvarieties of moduli spaces of algebraic curves

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Chapter 2. Explicit complete subvarieties of dimension $d$ in $M_{2d+1}$

STEP 2. We claim that the map $B_2 \to M_{2g}$ is non-degenerate. First note that the general fibers of $B_2 \to B_1$ parametrize double coverings of a fixed genus $g$ curve. The restriction of $B_2 \to M_{2g}$ to such a fiber is non-degenerate. This follows from a direct calculation of the Kodaira-Spencer map, as in [34] (here we need that $g > 1$).

Secondly, suppose $Z \subset B_2$ is irreducible, one-dimensional, and contained in the general fiber of the map to $M_{2g}$. Then we have maps of families:

$$
\begin{align*}
X_2 \times_{B_2} Z & \to X_1 \times_{B_1} \text{Im}(Z) \\
\downarrow & \downarrow \\
Z & \to \text{Im}(Z) \subset B_1
\end{align*}
$$

Since $Z$ is not be contained in a fiber of $B_2 \to B_1$, $\text{Im}(Z) \subset B_1$ is one-dimensional. As $Z$ maps to a point in $M_{2g}$, the fibers of $X_2 \times_{B_2} Z \to Z$ are all isomorphic. Since $X_1 \to B_1$ is non-degenerate, this would give a curve $F$ of genus $2g$ (the fiber of $X_2 \times_{B_2} Z \to Z$) doubly covering infinitely many, pairwise non-isomorphic curves of genus $g$. This is absurd, since $F$ has a finite automorphism group. $\square$

**Proof of Theorem 2.1.** To obtain the required families, let $X_0$ be a smooth genus $2$ curve. Consider the map $X_0 \times X_0 \to \text{Jac}(X_0)$, sending $(P, Q)$ to $[P - Q]$. This map is birational, and $\Delta$ is blown down to the origin. Let $W_0$ be the inverse image of a curve in $\text{Jac}(X_0)$ which does not contain the origin. Then $W_0$ is a complete curve in $X_0 \times X_0$ not meeting the diagonal.

Applying Lemma 2.2 to $X_1 = X_0$ yields a complete, non-degenerate, one-dimensional family $X_2 \to B_2$ of genus $4$ curves. Applying it to $X_2 \to B_2$ yields a complete, non-degenerate, two-dimensional family $X_3 \to B_3$ of genus $8$ curves. In this way we obtain for any $d$ non-degenerate families of smooth curves of genus $2^{d+1}$ over a complete base of dimension $d$. $\square$