Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

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STEP 2. We claim that the map $B_2 \to M_{2g}$ is non-degenerate. First note that the general fibers of $B_2 \to B_1$ parametrize double coverings of a fixed genus g curve. The restriction of $B_2 \to M_{2g}$ to such a fiber is non-degenerate. This follows from a direct calculation of the Kodaira-Spencer map, as in [34] (here we need that $g > 1$).

Secondly, suppose $Z \subset B_2$ is irreducible, one-dimensional, and contained in the general fiber of the map to M_{2g}. Then we have maps of families:

$$\begin{array}{ccc} X_2 \times_{B_2} Z & \longrightarrow & X_1 \times_{B_1} \mathrm{Im}(Z) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathrm{Im}(Z) \subset B_1 \end{array}$$

Since Z is not be contained in a fiber of $B_2 \to B_1$, $\mathrm{Im}(Z) \subset B_1$ is one-dimensional. As Z maps to a point in M_{2g}, the fibers of $X_2 \times_{B_2} Z \to Z$ are all isomorphic. Since $X_1 \to B_1$ is non-degenerate, this would give a curve F of genus $2g$ (the fiber of $X_2 \times_{B_2} Z \to Z$) doubly covering infinitely many, pairwise non-isomorphic curves of genus g. This is absurd, since F has a finite automorphism group. □

PROOF OF THEOREM 2.1. To obtain the required families, let X_0 be a smooth genus 2 curve. Consider the map $X_0 \times X_0 \to \mathrm{Jac}(X_0)$, sending (P, Q) to $[P - Q]$. This map is birational, and Δ is blown down to the origin. Let W_0 be the inverse image of a curve in $\mathrm{Jac}(X_0)$ which does not contain the origin. Then W_0 is a complete curve in $X_0 \times X_0$ not meeting the diagonal.

Applying Lemma 2.2 to $X_1 = X_0$ yields a complete, non-degenerate, one-dimensional family $X_2 \to B_2$ of genus 4 curves. Applying it to $X_2 \to B_2$ yields a complete, non-degenerate, two-dimensional family $X_3 \to B_3$ of genus 8 curves. In this way we obtain for any d non-degenerate families of smooth curves of genus 2^{d+1} over a complete base of dimension d. □