Complete subvarieties of moduli spaces of algebraic curves

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2.2 A refinement

With a minor modification of Lemma 2.2, we can sharpen Theorem 2.1 to obtain a stronger statement.

Theorem 2.3 *For every $d \geq 1$ and every $g \geq 2^{d+1}$ there exist complete subvarieties of M_g of dimension d.*

PROOF. The idea is to modify Lemma 2.2. We choose X_0 such that it admits a map onto an elliptic curve E. Instead of a complete curve in $X_0 \times X_0$ we take for any $k \geq 1$ a complete curve in X_0^{2k} not meeting the big diagonal $\Delta = \{ (x_1, \ldots, x_{2k}) \colon x_i = x_j \text{ for some } i \neq j \}$. Such a curve parametrizes $2k$-tuples of distinct points on X_0. It is possible to find such a curve. Indeed, inside E^{2k} take $\{ (e, e+t_1, \ldots, e+t_{2k}) \colon e \in E \}$ with $t_i \neq t_j$ for $i \neq j$. I.e., we translate the small diagonal $\{ (e, e, \ldots, e) \colon e \in E \}$ along the coordinate directions in such a way that it does not meet the big diagonal. Then simply take the inverse image of this curve via the map $X_0^{2k} \to E^{2k}$.

Now one can redo the construction of Lemma 2.2, taking double covers ramified not in 2 points, but in $2k$ points. This will result in a family $\pi_2 : X_2 \to B_2$ with $\dim(B_2) = \dim(B_1) + 1$ with the genus of the fibers of $\pi_2 : X_2 \to B_2$ being equal to $2g + k - 1$.

To prove the theorem, take $g \geq 2^{d+1}$. Write $g = 2^{d+1} + m$ with $m \geq 0$. We start with a bi-elliptic base curve X_0 of genus 2. We apply the construction of Lemma 2.2 $(d-1)$ times, to arrive at a $(d-1)$-dimensional family of smooth curves of genus 2^d. With the help of the described modification of Lemma 2.2 we construct in the final step a d-dimensional family of double covers of curves of genus 2^d ramified in $2m+2$ distinct points, to arrive at a family $\pi_d : X_d \to B_d$ of smooth curves of genus $g = 2^{d+1} + m$.

The image of the functorial map $B_d \to M_g$ is a complete subvariety of M_g of dimension d, as is seen easily. □

2.3 Minimality of the construction

In the construction described above we start with a (bi-elliptic) genus 2 curve $X_1 = X_0$, given rise to a 1-dimensional family of smooth curves of genus 4. If we would have started with a *elliptic* curve $X_1 = X_0$, then the result is a complete family $X_2 \to B_2$ of smooth curves of genus 2. This may seem an improvement of our results, but this family of genus 2 curves is degenerate. There are several ways to see this:

- The moduli space M_2 of smooth genus 2 curves is *affine*. For this reason, it does not contain complete curves. Hence the image of B_2 in M_2 must be a point (assuming that the base is irreducible), and all fibers are pairwise isomorphic.
- If one calculates the *Kodaira-Spencer map* of $X_2 \to B_2$ in a point B_2, then this derivative turns out to be a multiple of $g - 1$, where g is the genus of the base $X_1 = X_0$. So the tangent map of $B_2 \to M_2$ is zero in any point of B_2.