Complete subvarieties of moduli spaces of algebraic curves

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2.2 A refinement

With a minor modification of Lemma 2.2, we can sharpen Theorem 2.1 to obtain a stronger statement.

**Theorem 2.3** For every \( d \geq 1 \) and every \( g \geq 2^{d+1} \) there exist complete subvarieties of \( M_g \) of dimension \( d \).

**Proof.** The idea is to modify Lemma 2.2. We choose \( X_0 \) such that it admits a map onto an elliptic curve \( E \). Instead of a complete curve in \( X_0 \times X_0 \) we take for any \( k \geq 1 \) a complete curve in \( X_0^{2k} \) not meeting the big diagonal \( \Delta = \{(x_1, \ldots, x_{2k}) : x_i = x_j \text{ for some } i \neq j \} \). Such a curve parametrizes \( 2k \)-tuples of distinct points on \( X_0 \). It is possible to find such a curve. Indeed, inside \( E^{2k} \) take \( \{(e, e+t_1, \ldots, e+t_{2k}) : e \in E \} \) with \( t_i \neq t_j \) for \( i \neq j \). I.e., we translate the small diagonal \( \{(e, e, \ldots, e) : e \in E \} \) along the coordinate directions in such a way that it does not meet the big diagonal. Then simply take the inverse image of this curve via the map \( X_0^{2k} \to E^{2k} \).

Now one can redo the construction of Lemma 2.2, taking double covers ramified not in 2 points, but in \( 2k \) points. This will result in a family \( \pi_2 : X_2 \to B_2 \) with \( \dim(B_2) = \dim(B_1) + 1 \) with the genus of the fibers of \( \pi_2 : X_2 \to B_2 \) being equal to \( 2g + k - 1 \).

To prove the theorem, take \( g \geq 2^{d+1} \). Write \( g = 2^{d+1} + m \) with \( m \geq 0 \). We start with a bi-elliptic base curve \( X_0 \) of genus 2. We apply the construction of Lemma 2.2 \( (d-1) \) times, to arrive at a \( (d-1) \)-dimensional family of smooth curves of genus \( 2^d \). With the help of the described modification of Lemma 2.2 we construct in the final step a \( d \)-dimensional family of double covers of curves of genus \( 2^d \) ramified in \( 2m + 2 \) distinct points, to arrive at a family \( \pi_d : X_d \to B_d \) of smooth curves of genus \( g = 2^{d+1} + m \).

The image of the functorial map \( B_d \to M_g \) is a complete subvariety of \( M_g \) of dimension \( d \), as is seen easily. \( \square \)

2.3 Minimality of the construction

In the construction described above we start with a (bi-elliptic) genus 2 curve \( X_1 = X_0 \), given rise to a 1-dimensional family of smooth curves of genus 4. If we would have started with a elliptic curve \( X_1 = X_0 \), then the result is a complete family \( X_2 \to B_2 \) of smooth curves of genus 2. This may seem an improvement of our results, but this family of genus 2 curves is degenerate. There are several ways to see this:

- The moduli space \( M_2 \) of smooth genus 2 curves is affine. For this reason, it does not contain complete curves. Hence the image of \( B_2 \) in \( M_2 \) must be a point (assuming that the base is irreducible), and all fibers are pairwise isomorphic.
- If one calculates the Kodaira-Spencer map of \( X_2 \to B_2 \) in a point \( B_2 \), then this derivative turns out to be a multiple of \( g - 1 \), where \( g \) is the genus of the base \( X_1 = X_0 \). So the tangent map of \( B_2 \to M_2 \) is zero in any point of \( B_2 \).