Complete subvarieties of moduli spaces of algebraic curves

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The degree of $\lambda$ on $B_2$ is also a multiple of $g - 1$. Since $\lambda$ is ample on $M_2$ [40], the image of $B_2$ in $M_2$ is degenerate.

If we allow more ramification points, then the first argument fails, but the other two arguments still hold. In fact, the following much stronger statement holds true.

**Theorem 2.4** Let $X \to B$ be family of smooth curves of genus $g$ over a irreducible base $B$. Suppose that every fiber is a double cover of a fixed elliptic curve $(E, 0)$, ramified in $2k$ points. Then $X \to B$ is isotrivial.

**Proof.** After a base change, we may assume that $X \to B$ is a double cover of the constant family $B \times E \to B$. Let $R \subset X$ be the ramification divisor of $X \to B \times E$. Denote its image in $B \times E$ by $T$. The curve $T$ is a multisection of $B \times E \to B$. After a base change, $T$ is a unions of sections $s_1, \ldots, s_{2k} : B \to B \times E$.

To be more precise, let $T_{B}^{2k}$ be the $2k$-fold fiber product of $T$ over $B$. Remove from this product all the diagonals (the locus of points where two or more coordinates are the same) and take the closure of the remainder inside $T_{B}^{2k}$. This is a curve which parametrizes distinct $2k$-tuples in the fiber of $T \to B$. After base changing via this curve, the ramification locus is a union of sections $s_1, \ldots, s_{2k} : B \to B \times E$. By assumption the sections $s_1, \ldots, s_{2k}$ are disjoint. We claim:

the sections $s_1, \ldots, s_{2k}$, considered as maps $B \to E$, differ by a constant, i.e., $s_i = s_j + t_{ij}$ for a fixed $t_{ij} \in E$.

To prove the claim, consider the differences $s_i - s_j : B \to E$. Since $s_i$ and $s_j$ are disjoint, these are maps $B \to E \setminus \{0\}$. Since $E \setminus \{0\}$ is affine, these differences must be constant.

Now we can finish the proof of Theorem 2.4. From the second claim it follows that the fibers of $X \to B$ are double covers $X_b \to E$, of which the branch loci differ only by a translation of $E$, i.e., all branch loci are isomorphic.

In general, given a prescribed branch locus, there are precisely $2^{2g}$ different isomorphism classes of double branched covers of a fixed curve $C$ of genus $g$. So the fibers of $X \to B$ fit in at most four different isomorphism classes, and the image of $B \to M_k$ is finite. Hence $X \to B$ is isotrivial. $\square$