Complete subvarieties of moduli spaces of algebraic curves

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- The degree of λ on B_2 is also a multiple of $g-1$. Since λ is ample on M_2 [40], the image of B_2 in M_2 is degenerate.

If we allow more ramification points, then the first argument fails, but the other two arguments still hold. In fact, the following much stronger statement holds true.

Theorem 2.4 *Let $X \to B$ be family of smooth curves of genus g over a irreducible base B. Suppose that every fiber is a double cover of a fixed elliptic curve $(E, 0)$, ramified in $2k$ points. Then $X \to B$ is isotrivial.*

PROOF. After a base change, we may assume that $X \to B$ is a double cover of the constant family $B \times E \to B$. Let $R \subset X$ be the ramification divisor of $X \to B \times E$. Denote its image in $B \times E$ by T. The curve T is a multisection of $B \times E \to B$. After a base change, T is a unions of sections $s_1, \dots, s_{2k} : B \to B \times E$.

To be more precise, let T_B^{2k} be the $2k$-fold fiber product of T over B. Remove from this product all the diagonals (the locus of points where two or more coordinates are the same) and take the closure of the remainder inside T_B^{2k}. This is a curve which parametrizes distinct $2k$-tuples in the fiber of $T \to B$. After base changing via this curve, the ramification locus is a union of sections $s_1, \dots, s_{2k} : B \to B \times E$. By assumption the sections $s_1, \dots, s_{2k} : B \to B \times E$ are *disjoint*. We claim:

> the sections $s_1, \dots, s_{2k}$, considered as maps $B \to E$, differ by a constant, i.e., $s_i = s_j + t_{ij}$ for a fixed $t_{ij} \in E$.

To prove the claim, consider the differences $s_i - s_j : B \to E$. Since s_i and s_j are disjoint, these are maps $B \to E \setminus \{0\}$. Since $E \setminus \{0\}$ is affine, these differences must be constant.

Now we can finish the proof of Theorem 2.4. From the second claim it follows that the fibers of $X \to B$ are double covers $X_b \to E$, of which the branch loci differ only by a translation of E, i.e., all branch loci are isomorphic.

In general, given a prescibed branch locus, there are precisely 2^{2g} different isomorphism classes of double branched covers of a fixed curve C of genus g. So the fibers of $X \to B$ fit in at most four different isomorphism classes, and the image of $B \to M_k$ is finite. Hence $X \to B$ is isotrivial. □