Complete subvarieties of moduli spaces of algebraic curves

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Publication date
2005

Citation for published version (APA):
Chapter 2. Explicit complete subvarieties of dimension \(d\) in \(M_{2d+1}\)

- The degree of \(\lambda\) on \(B_2\) is also a multiple of \(g - 1\). Since \(\lambda\) is ample on \(M_2\) [40], the image of \(B_2\) in \(M_2\) is degenerate.

If we allow more ramification points, then the first argument fails, but the other two arguments still hold. In fact, the following much stronger statement holds true.

**Theorem 2.4** Let \(X \rightarrow B\) be family of smooth curves of genus \(g\) over a irreducible base \(B\). Suppose that every fiber is a double cover of a fixed elliptic curve \((E,0)\), ramified in \(2k\) points. Then \(X \rightarrow B\) is isotrivial.

**Proof.** After a base change, we may assume that \(X \rightarrow B\) is a double cover of the constant family \(B \times E \rightarrow B\). Let \(R \subset X\) be the ramification divisor of \(X \rightarrow B \times E\). Denote its image in \(B \times E\) by \(T\). The curve \(T\) is a multisection of \(B \times E \rightarrow B\).

After a base change, \(T\) is a union of sections \(s_1, \ldots, s_{2k} : B \rightarrow B \times E\).

To be more precise, let \(T_B^{2k}\) be the \(2k\)-fold fiber product of \(T\) over \(B\). Remove from this product all the diagonals (the locus of points where two or more coordinates are the same) and take the closure of the remainder inside \(T_B^{2k}\). This is a curve which parametrizes distinct \(2k\)-tuples in the fiber of \(T \rightarrow B\). After base changing via this curve, the ramification locus is a union of sections \(s_1, \ldots, s_{2k} : B \rightarrow B \times E\).

By assumption the sections \(s_1, \ldots, s_{2k} : B \rightarrow B \times E\) are disjoint. We claim:

the sections \(s_1, \ldots, s_{2k}\), considered as maps \(B \rightarrow E\), differ by a constant, i.e., \(s_i = s_j + t_{ij}\) for a fixed \(t_{ij} \in E\).

To prove the claim, consider the differences \(s_i - s_j : B \rightarrow E\). Since \(s_i\) and \(s_j\) are disjoint, these are maps \(B \rightarrow E \setminus \{0\}\). Since \(E \setminus \{0\}\) is affine, these differences must be constant.

Now we can finish the proof of Theorem 2.4. From the second claim it follows that the fibers of \(X \rightarrow B\) are double covers \(X_b \rightarrow E\), of which the branch loci differ only by a translation of \(E\), i.e., all branch loci are isomorphic.

In general, given a prescribed branch locus, there are precisely \(2^{2g}\) different isomorphism classes of double branched covers of a fixed curve \(C\) of genus \(g\). So the fibers of \(X \rightarrow B\) fit in at most four different isomorphism classes, and the image of \(B \rightarrow M_k\) is finite. Hence \(X \rightarrow B\) is isotrivial. \(\square\)